The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:
The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman
which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
Five Muffins, Three Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
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</tbody>
</table>

Smallest Piece: $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor.
YES WE CAN!

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Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is \( \frac{5}{12} \).

**Is there a procedure with a larger smallest piece?**

Work on it with your neighbor.
NO WE CAN’T!

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases.
5 Muffins, 3 People–Can’t Do Better Than $\frac{5}{12}$

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There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece $N$. We want $N \leq \frac{5}{12}$.

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*(Henceforth: All muffins are cut into $\geq 2$ pieces.)*

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. 
NO WE CAN’T!

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

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Case 1: Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \).

(\textbf{Henceforth:} All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: \textbf{Someone} gets \( \geq 4 \) pieces. He has some piece

\[
\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}
\]

Great to see \( \frac{5}{12} \).
What Happened Next?
What Happened Next?

Yada Yada Yada- in 2020:
What Happened Next?

Yada Yada Yada- in 2020:

**MATHEMATICAL MUFFIN MORSELS: NOBODY WANTS A SMALL PIECE**

William Gasarch, Erik Metz, Jacob Prinz, Daniel Smolyak
*University of Maryland, USA*

In this book we consider THE MUFFIN PROBLEM: what is the best way to divide up m muffins for s students so that everyone gets m/s muffins, with the smallest pieces maximized.

This problem takes us through much mathematics of interest, for example, combinatorics and optimization theory.

228pp
978-981-121-597-1(pbk) US$28 / £25 / SGD41
978-981-121-517-9 US$58 / £50 / SGD86
978-981-121-519-3(mbook) US$22 / £20 / SGD33

Is there a way to divide five muffins for three students so that everyone gets 5/3, and all pieces are larger than 1/3? *Spoiler alert: Yes!*

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https://doi.org/10.1142/11689
General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide $m$ muffins among $s$ students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have two proofs that shown $f(m, s)$ exists, is rational, and is computable.
One use Linear Programming.
One use Integer Programming.
Amazing Results! / Amazing Theorems!

1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).

All done by hand, no use of a computer by Co-author Erik Metz is a muffin savant!
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Have General Theorems from which upper bounds follow. Have General Procedures from which lower bounds follow.
Clearly $f(3, 5) \geq \frac{1}{5}$.

Can we get $f(3, 5) > \frac{1}{5}$?

Work on it with your neighbor
$f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $\left(\frac{5}{20}, \frac{7}{20}\right)$
4. Give 1 students $\left(\frac{6}{20}, \frac{6}{20}\right)$

Can we do better? Work on it with your neighbor
\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
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Can we do better?
Work on it with your neighbor
NO WE CAN’T!

There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

**Case 0:** Alice gets 1 piece of size $\frac{3}{5}$. Look at the rest of that muffin which totals to $\frac{2}{5}$. (1) That piece is cut. Have piece $\leq \frac{2}{5} \times \frac{1}{2} = \frac{1}{5}$, OR (2) That piece uncut. So someone gets a $\frac{2}{5}$-piece. Must also get a $\frac{1}{5}$ piece.
Three Muffins, Five People–Can’t Do Better Than $\frac{1}{4}$

NO WE CAN’T!
There is a procedure for 3 muffins, 5 students where each student gets $\frac{3}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{1}{4}$.

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Case 1: Alice gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$. 
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*(Henceforth: All people get $\geq 2$ pieces.)*

**Case 1:** Alice gets $\geq 3$ pieces. Then $N \leq \frac{3}{5} \times \frac{1}{3} = \frac{1}{5}$.

*(Henceforth: Everyone gets 2 pieces.)*

**Case 2:** Everyone gets 2 pieces. 10 pieces, 3 muffins: Some muffin gets $\geq 4$ pieces. So some piece is $\leq \frac{1}{4}$. 

\( f(3, 5) \) and \( f(5, 3) \)

1. Divide 4 muffins \([\frac{5}{12}, \frac{7}{12}]\)
2. Divide 1 muffin \([\frac{6}{12}, \frac{6}{12}]\)
3. Give 2 students \((\frac{6}{12}, \frac{7}{12}, \frac{7}{12})\)
4. Give 1 students \((\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})\)
\( f(3, 5) \) and \( f(5, 3) \)

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\( f(3, 5) \geq \frac{1}{4} \)

1. Divide 2 muffin \( \left[ \frac{6}{20}, \frac{7}{20}, \frac{7}{20} \right] \)
2. Divide 1 muffin \( \left[ \frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20} \right] \)
3. Give 4 students \( \left( \frac{5}{20}, \frac{7}{20} \right) \)
4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

Duality Theorem:

\( f(m, s) = m \cdot s = f(s, m) \)
\( f(3, 5) \) and \( f(5, 3) \)

1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
3. Give 2 students \( \left( \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \right) \)
4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

\[ f(3, 5) \geq \frac{1}{4} \]

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4. Give 1 students \( \left( \frac{6}{20}, \frac{6}{20} \right) \)

\( f(3, 5) \) proc is \( f(5, 3) \) proc but swap Divide/Give and mult by 3/5.
1. Divide 4 muffins \( \left[ \frac{5}{12}, \frac{7}{12} \right] \)
2. Divide 1 muffin \( \left[ \frac{6}{12}, \frac{6}{12} \right] \)
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4. Give 1 students \( \left( \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \right) \)

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\[
f(3, 5) \text{ proc is } f(5, 3) \text{ proc but swap Divide/Give and mult by } \frac{3}{5}.
\]

**Duality Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
Conventions

We know and use the following:

1. By duality can assume \( m > s \)
2. If \( s \) divides \( m \) then \( f(m, s) = 1 \) so assume \( s \) does not divide \( m \).
3. All muffins are cut in \( \geq 2 \) pcs. Replace uncut muff with \( 2 \frac{1}{2} \)’s
4. If assuming \( f(m, s) > \alpha > \frac{1}{3} \), assume all muffin in \( \leq 2 \) pcs.
5. \( f(m, s) > \alpha > \frac{1}{3} \), so exactly 2 pcs, is common case.

We do not know this but still use: \( f(m, s) \) only depends on \( \frac{m}{s} \). All of our techniques that hold for \( (m, s) \) hold for \( (Am, As) \). For particular numbers, we only look at \( m, s \) rel prime.
What if $m < s$?

Duality Theorem:

$$f(m, s) = m s f(s, m).$$

Hence we will just look at $m > s$. 

What if $m < s$?

**Duality Theorem:** $f(m, s) = \frac{m}{s} f(s, m)$. 
What if \( m < s \)?

**Duality Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).

Hence we will just look at \( m > s \).
Floor-Ceiling Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$
Floor-Ceiling Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq \text{FC}(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.$$ 

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}, \frac{1}{2}$) and give both halves to whoever got the uncut muffin, so reduces to other cases.
Floor-Ceiling Thm Generalizes \( f(5, 3) \leq \frac{5}{12} \)

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**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. Some piece $\leq \frac{1}{3}$.

**Case 2:** Every muffin is cut into 2 pieces, so $2m$ pieces.
Floor-Ceiling Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

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Someone gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\left\lfloor 2m/s \right\rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.
Floor-Ceiling Thm Generalizes $f(5, 3) \leq \frac{5}{12}$

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**Someone** gets $\geq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

**Someone** gets $\leq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.
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Someone gets $\leq \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\geq \frac{m}{s} \left\lfloor \frac{1}{2m/s} \right\rfloor = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$. 
FC Gives Upper Bound

Give \( m, s \):

\[
FC(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}
\]

Gives an upper bound. So we know

\[(\forall m, s)[f(m, s) \leq FC(m, s)].\]

Is the following true?

\[(\forall m, s)[f(m, s) = FC(m, s)]\]
Give $m, s$:

$$FC(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil2m/s\rceil}, 1 - \frac{m}{s\lfloor2m/s\rfloor}\right\}\right\}$$

Gives an upper bound. So we know

$$(\forall m, s)[f(m, s) \leq FC(m, s)].$$

Is the following true?

$$(\forall m, s)[f(m, s) = FC(m, s)]$$

**No:** If so my book would be about 20 pages.
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

\[ f(1, 3) = \frac{1}{3} \]

\[ f(3k, 3) = 1. \]

\[ f(3k + 1, 3) = \frac{3k-1}{6k}, \ k \geq 1. \]

\[ f(3k + 2, 3) = \frac{3k+2}{6k+6}. \]

Note: A Mod 3 Pattern.

Theorem: For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

\[ f(4k, 4) = 1 \text{ (easy)} \]

\[ f(1, 4) = \frac{1}{4} \text{ (easy)} \]

\[ f(4k + 1, 4) = \frac{4k-1}{8k}, \quad k \geq 1. \]

\[ f(4k + 2, 4) = \frac{1}{2}. \]

\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Note: A Mod 4 Pattern.

Theorem: For all \( m \geq 4, \ f(m, 4) = FC(m, 4). \)
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \text{FC}(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = \text{FC}(m, 5)$. 
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows \( f(11, 5) \geq \frac{13}{30} \).
2. \( f(11, 5) \leq \max\left\{ \frac{1}{3}, \min\left\{ \frac{11}{\lceil 22/5 \rceil}, 1 - \frac{11}{\lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25} \).

So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25}
\]
\( \text{Diff} = 0.0066666\ldots \)

Options:
1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

Vote
What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows \( f(11, 5) \geq \frac{13}{30} \).
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So
\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff= 0.006666...}
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Options:
1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
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3. \( f(11, 5) \) in between. Need to find both.
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Vote WE SHOW \( f(11, 5) = \frac{13}{30} \). Exciting new technique!
Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$. The *other piece* from muffin $M$ is the **buddy of $x$**.

Note that the **buddy** of $x$ is of size $1 - x$. 
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}$, $\frac{1}{2}$) and give both halves to whoever got the uncut muffin. Reduces to other cases.
There is a procedure for 11 muffins, 5 students where each student gets \( \frac{11}{5} \) muffins, smallest piece \( N \). We want \( N \leq \frac{13}{30} \).

**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. \( N \leq \frac{1}{3} < \frac{13}{30} \).
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
Case 2: Some student gets $\geq 6$ pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

Case 3: Some student gets $\leq 3$ pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]
\[ f(11, 5) = \frac{13}{30}, \text{ Easy Case Based on Students} \]

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[ N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}. \]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[ \geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}. \]

That pieces **buddy** is of size:

\[ \leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}. \]
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

That piece buddy is of size:

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$. 

$s_4$ is number of students who get 4 pieces

$s_5$ is number of students who get 5 pieces

$s_4 + s_5 = 22$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.
\( f(11, 5) = \frac{13}{30} \), Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).
- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22
\]
\[
s_4 + s_5 = 5
\]
\( f(11, 5) = \frac{13}{30} \), Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares.
\( s_5 = 2 \): There are 2 students who have 5 shares.
\( f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \)

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\quad s_4 + s_5 = 5
\]

\( s_4 = 3 \): There are 3 students who have 4 shares.
\( s_5 = 2 \): There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.
Case 4.1: Some 4-share is \( \leq \frac{1}{2} \).
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

**Case 4.1:** Some 4-share is \( \leq \frac{1}{2}. \)

Alice gets \( w, x, y, z \) and \( w \leq \frac{1}{2}. \)

Since \( w + x + y + z = \frac{11}{5} \) and \( w \leq \frac{1}{2} \)

\[
x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}
\]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.1:** Some 4-share is $\leq \frac{1}{2}$.
Alice gets $w, x, y, z$ and $w \leq \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let $x$ be the largest of $x, y, z$

$$x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.1:** Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w, x, y, z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let $x$ be the largest of $x, y, z$

$$x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

The **buddy** of $x$ is of size

$$\leq 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$
**Case 4.1:** Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w, x, y, z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let $x$ be the largest of $x, y, z$

$$x \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

The **buddy** of $x$ is of size

$$\leq 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!
\( f(11, 5) = \frac{13}{30}, \) Fun Cases

**Case 4.2:** All 4-shares are \( > \frac{1}{2}. \) There are \( 4s_4 = 12 \) 4-shares. There are \( \geq 12 \) pieces \( > \frac{1}{2}. \) Can’t occur.
HALF Method

The above reasoning can be used to verify that \( f(11, 5) \leq \frac{13}{30} \) but could not generate the upper bound \( \frac{13}{30} \).
HALF Method

The above reasoning can be used to verify that $f(11, 5) \leq \frac{13}{30}$ but could not generate the upper bound $\frac{13}{30}$.

Can modify the method so that we have an easily computable function $\text{HALF}(m, s)$ such that

$$(\forall m, s)[f(m, s) \leq \min\{\text{FC}(m, s), \text{HALF}(m, s)\}]$$

Is the following true?

$$(\forall m, s)[f(m, s) = \min\{\text{FC}(m, s), \text{HALF}(m, s)\}]$$

No:

If so my book would be about 40 pages.

For $f(24, 11)$ it fails!
HALF Method

The above reasoning can be used to verify that \( f(11, 5) \leq \frac{13}{30} \) but could not generate the upper bound \( \frac{13}{30} \).

Can modify the method so that we have an easily computable function \( \text{HALF}(m, s) \) such that

\[
(\forall m, s)[f(m, s) \leq \min\{\text{FC}(m, s), \text{HALF}(m, s)\}]
\]

Is the following true?

\[
(\forall m, s)[f(m, s) = \min\{\text{FC}(m, s), \text{HALF}(m, s)\}]
\]

No: If so my book would be about 40 pages. For \( f(24, 11) \) it fails!
HALF Method

The above reasoning can be used to verify that \( f(11, 5) \leq \frac{13}{30} \) but could not generate the upper bound \( \frac{13}{30} \).

Can modify the method so that we have an easily computable function HALF\((m, s)\) such that

\[
(\forall m, s)[f(m, s) \leq \min\{FC(m, s), \text{HALF}(m, s)\}]
\]

Is the following true?

\[
(\forall m, s)[f(m, s) = \min\{FC(m, s), \text{HALF}(m, s)\}]
\]

No: If so my book would be about 40 pages.
HALF Method

The above reasoning can be used to verify that $f(11, 5) \leq \frac{13}{30}$ but could not generate the upper bound $\frac{13}{30}$.

Can modify the method so that we have an easily computable function HALF($m, s$) such that

$$(\forall m, s) [f(m, s) \leq \min\{FC(m, s), \text{HALF}(m, s)\}]$$

Is the following true?

$$(\forall m, s) [f(m, s) = \min\{FC(m, s), \text{HALF}(m, s)\}]$$

No: If so my book would be about 40 pages.
For $f(24, 11)$ it fails!
Assume \((24, 11)\)-procedure with smallest piece > \(\frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).
Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).

**Case 1:** A student gets \(\geq 6\) shares. Some piece \(\leq \frac{24}{11 \times 6} < \frac{19}{44}\).
$f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

**Case 2:** A student gets $\leq 3$ shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$. Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$. 
\[ f(24, 11) \leq \frac{19}{44} \]

Assume \((24, 11)\)-procedure with smallest piece \( > \frac{19}{44} \).
Can assume all muffin cut in two and all student gets \( \geq 2 \) shares.
We show that there is a piece \( \leq \frac{19}{44} \).

**Case 1:** A student gets \( \geq 6 \) shares. Some piece \( \leq \frac{24}{11 \times 6} < \frac{19}{44} \).

**Case 2:** A student gets \( \leq 3 \) shares. Some piece \( \geq \frac{24}{11 \times 3} = \frac{8}{11} \).
Buddy of that piece \( \leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44} \).

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.

5-students: a student who gets 5 shares. $s_5$ is the number of them.

$4s_4 + 5s_5 = 48$ 
$s_4 + s_5 = 11$

$s_4 = 7$. Hence there are 4 $s_4 = 4 \times 7 = 28$ 4-shares.

$s_5 = 4$. Hence there are 5 $s_5 = 5 \times 4 = 20$ 5-shares.
How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.

5-students: a student who gets 5 shares. $s_5$ is the number of them.

4-share: a share that a 4-student who gets.

5-share: a share that a 5-student who gets.
How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.  
5-students: a student who gets 5 shares. $s_5$ is the number of them.  

4-share: a share that a 4-student who gets.  
5-share: a share that a 5-student who gets.  

$$4s_4 + 5s_5 = 48$$  
$$s_4 + s_5 = 11$$
How many students get 4? 5? Where are the Shares?

4-students: a student who gets 4 shares. \( s_4 \) is the number of them.
5-students: a student who gets 5 shares. \( s_5 \) is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

\[
4s_4 + 5s_5 = 48 \\
s_4 + s_5 = 11
\]

\( s_4 = 7 \). Hence there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.
\( s_5 = 4 \). Hence there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** \( \exists \) a share \( \geq \frac{25}{44} \). Its **buddy** is

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: \( \exists \) a share \( \geq \frac{25}{44} \). Its buddy is

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]

Case 3.2: There is a share \( \leq \frac{19}{44} \). Duh.
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1: \( \exists \) a share \( \geq \frac{25}{44} \). Its buddy is

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]

Case 3.2: There is a share \( \leq \frac{19}{44} \). Duh.

Henceforth assume that all shares are in

\[
\left( \frac{19}{44}, \frac{25}{44} \right)
\]
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.

Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$.
Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces $A, B, C, D, E$ and $E \geq \frac{20}{44}$. Since $A + B + C + D + E = \frac{24}{11}$ and $E > \frac{20}{44}$

$$A + B + C + D < \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume $A$ is the smallest of $A, B, C, D$.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in

$$\left( \frac{19}{44}, \frac{20}{44} \right).$$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Proof: Assume that Alice gets 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$,

$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$

Assume $A$ is the largest of $A, B, C$.

$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$

The buddy of $A$ is of size $\leq 1 - \frac{25}{44} = \frac{19}{44}$

Henceforth we assume all 4-shares are in $(\frac{21}{44}, \frac{25}{44})$. 

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice has A, B, C, D and D $\leq \frac{21}{44}$.

Since $A + B + C + D = 24\frac{11}{44}$ and $D \leq \frac{21}{44}$, $A + B + C > 24\frac{11}{44} - \frac{21}{44} = 25\frac{3}{44}$.

Assume A is the largest of A, B, C.

$A \geq 25\frac{3}{44} \times \frac{1}{3} = 25\frac{1}{44}$.

The buddy of A is of size $\leq 2\frac{1}{44}$.

Henceforth we assume all 4-shares are in $(\frac{21}{44}, \frac{25}{44})$.
Case 3.4: Some 4-shares \( \leq \frac{21}{44} \)

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is \( \leq \frac{21}{44} \) then some share \( \leq \frac{19}{44} \).

Proof: Assume that Alice 4 pieces \( A, B, C, D \) and \( D \leq \frac{21}{44} \). Since \( A + B + C + D = \frac{24}{11} \) and \( D \leq \frac{21}{44} \),

\[
A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}
\]
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$. Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 4 pieces $A, B, C, D$ and $D \leq \frac{21}{44}$.
Since $A + B + C + D = \frac{24}{11}$ and $D \leq \frac{21}{44}$

$$A + B + C > \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume $A$ is the largest of $A, B, C$.

$$A \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $A$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44}\right)$$
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \( \left( \frac{21}{44}, \frac{25}{44} \right) \), 5-shares in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).
Case 3.5: 4-shares in \(( \frac{21}{44}, \frac{25}{44} )\), 5-shares in \(( \frac{19}{44}, \frac{20}{44} )\).

\[
\begin{align*}
\left( \frac{19}{44}, \frac{20}{44} \right] & \quad \text{0 shs} & \quad \left( \frac{21}{44}, \frac{25}{44} \right)
\end{align*}
\]
Case 3.5: 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

\[
\begin{pmatrix}
?? & 5\text{-shs} & 0 \text{ shs} & ?? & 4\text{-shs} \\
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\]

Recall: there are \(4s_4 = 4 \times 7 = 28\) 4-shares.
Recall: there are \(5s_5 = 5 \times 4 = 20\) 5-shares.
**Case 3.5: All Shares in Their Proper Intervals**

**Case 3.5:** 4-shares in \( \left( \frac{21}{44}, \frac{25}{44} \right) \), 5-shares in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).

\[
\begin{align*}
&\left( \begin{array}{c}
?? & 5\text{-shs} \\
19 & 20 \\
\end{array} \right) \left[ 
\begin{array}{c}
0 \text{ shs} \\
\frac{21}{44} & \frac{25}{44} \\
\end{array} \right] \left( 
\begin{array}{c}
?? & 4\text{-shs} \\
\end{array} \right)
\end{align*}
\]

**Recall:** there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.

**Recall:** there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.

\[
\begin{align*}
&\left( \begin{array}{c}
20 & 5\text{-shs} \\
19 & 20 \\
\end{array} \right) \left[ 
\begin{array}{c}
0 \text{ shs} \\
\frac{21}{44} & \frac{25}{44} \\
\end{array} \right] \left( 
\begin{array}{c}
28 & 4\text{-shs} \\
\end{array} \right)
\end{align*}
\]
More Refined Picture of What is Going On

\[
\begin{pmatrix}
19/44 & 20/44 & 0/44 & 21/44 & 25/44 \\
20/44 & 0/44 & 28/44
\end{pmatrix}
\]
More Refined Picture of What is Going On

\[
\begin{align*}
&\left( \begin{array}{c}
5 \text{-shs} \\
\frac{19}{44}
\end{array} \right)
\left[ \begin{array}{c}
0 \text{ shs} \\
\frac{20}{44}
\end{array} \right]
\left( \begin{array}{c}
4 \text{-shs} \\
\frac{21}{44}
\end{array} \right)
\frac{28}{44}
\frac{25}{44}
\end{align*}
\]

Claim 1: There are no shares \( x \in [\frac{23}{44}, \frac{24}{44}] \).
More Refined Picture of What is Going On

\[
\begin{pmatrix}
(20 \text{ 5-shs}) [0 \text{ shs}] (28 \text{ 4-shs}) \\
\frac{19}{44} & \frac{20}{44} & \frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\]

**Claim 1:** There are no shares \( x \in \left[\frac{23}{44}, \frac{24}{44}\right]. \)

If there was such a share then its **buddy** is in \( \left[\frac{20}{44}, \frac{21}{44}\right]. \)
More Refined Picture of What is Going On

Claim 1: There are no shares $x \in \left[ \frac{23}{44}, \frac{24}{44} \right]$.

If there was such a share then its buddy is in $\left[ \frac{20}{44}, \frac{21}{44} \right]$.

The following picture captures what we know so far.

S4 $=$ Small 4-shares
L4 $=$ Large 4-shares. L4 shares, 5-share: buddies, so $|L4|=20$. 
Claim 2:

Every 4-student has at least 3 L4 shares. If a 4-student had \( \leq 2 \) L4 shares then he has \(< 2 \times (\frac{23}{44}) + 2 \times (\frac{25}{44}) = \frac{24}{44} \). Contradiction: Each 4-student gets \( \geq 3 \) L4 shares. There are \( s_4 = 7 \) 4-students. Hence there are \( \geq 21 \) L4-shares. But there are only 20.

<table>
<thead>
<tr>
<th>20 5-shs</th>
<th>0</th>
<th>8 S4-shs</th>
<th>0</th>
<th>20 L4-shs</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{19}{44} )</td>
<td>( \frac{20}{44} )</td>
<td>( \frac{21}{44} )</td>
<td>( \frac{23}{44} )</td>
<td>( \frac{24}{44} )</td>
</tr>
</tbody>
</table>
Claim 2: Every 4-student has at least 3 L4 shares.
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If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had \( \leq 2 \) L4 shares then he has

\[
< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.
\]
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares. But there are only 20.
The above reasoning can be used to verify that \( f(24, 11) \leq \frac{19}{44} \) but could not generate the upper bound \( \frac{19}{44} \).

Is the following true?\[
(\forall m, s)\left[ f(m, s) = \min\left\{ FC(m, s), \text{HALF}(m, s), \text{INT}(m, s) \right\} \right]
\]

No: If so my book would be about 60 pages. For \( f(31, 19) \) it fails!
The above reasoning can be used to verify that $f(24, 11) \leq \frac{19}{44}$ but could not generate the upper bound $\frac{19}{44}$.

Can modify the method so that we have an easily computable function $\text{INT}(m, s)$ such that

$$(\forall m, s)[f(m, s) \leq \min\{\text{FC}(m, s), \text{HALF}(m, s), \text{INT}(m, s)\}]$$
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Is the following true?

$$(\forall m, s)[f(m, s) = \min\{\text{FC}(m, s), \text{HALF}(m, s), \text{INT}(m, s)\}]$$
INT Technique

The above reasoning can be used to verify that $f(24, 11) \leq \frac{19}{44}$ but could not generate the upper bound $\frac{19}{44}$.

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No: If so my book would be about 60 pages. For $f(31,19)$ it fails!
We show $f(31, 19) \leq \frac{54}{133}$.

Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$. 

By INT-technique methods obtain:

$s_3 = 14,$  
$s_4 = 5.$
We show $f(31, 19) \leq \frac{54}{133}$.

Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$.

By INT-technique methods obtain:

$s_3 = 14$, $s_4 = 5$.

\[
\begin{pmatrix}
\frac{54}{133} & 20 \text{ 4-shs} & \frac{55}{133} & 0 & \frac{59}{133} & \text{(S3 shs)} & \frac{74}{133} & 0 & \frac{78}{133} & \text{(20 L3-shs)} & \frac{79}{133}
\end{pmatrix}
\]
We show $f(31, 19) \leq \frac{54}{133}$.

Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$.

By INT-technique methods obtain:

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\[
\begin{pmatrix}
20 \text{ } 4\text{-shs} \\
\frac{54}{133}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{55}{133}
\end{pmatrix}
\begin{pmatrix}
\text{S3 shs} \\
\frac{59}{133}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{74}{133}
\end{pmatrix}
\begin{pmatrix}
20 \text{ } \text{L3-shs} \\
\frac{78}{133}
\end{pmatrix}
\begin{pmatrix}
\text{ } \\
\frac{79}{133}
\end{pmatrix}
\]

We just look at the 3-shares:

\[
\begin{pmatrix}
\text{S3 shs} \\
\frac{59}{133}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{74}{133}
\end{pmatrix}
\begin{pmatrix}
20 \text{ } \text{L3-shs} \\
\frac{78}{133}
\end{pmatrix}
\begin{pmatrix}
\text{ } \\
\frac{79}{133}
\end{pmatrix}
\]
\( f(31, 19) \leq \frac{54}{133} \)

\[
\begin{pmatrix}
\frac{59}{133} & \text{S3 shs} & 0 & \frac{79}{133} \\
\frac{74}{133} & \frac{78}{133} & 20 \text{ L3-shs} & \frac{79}{133}
\end{pmatrix}
\]
\[ f(31, 19) \leq \frac{54}{133} \]

\[
\begin{pmatrix}
\frac{59}{133} & S3\text{ shs} \\
\frac{74}{133} & 0 \\
\frac{78}{133} & 20 \text{ L3-shs}
\end{pmatrix}
\]

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \) (\(|J_1| = |J_2|\))
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\(|J_3| = 20\))

**Note:** Split the shares of size 66.5 between \( J_1 \) and \( J_2 \).

**Notation:** An \( e(1, 1, 3) \) students is a student who has a \( J_1 \)-share, a \( J_1 \)-share, and a \( J_3 \)-share.

Generalize to \( e(i, j, k) \) easily.
\( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \) (\(|J_1| = |J_2|\))
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\(|J_3| = 20\))
$f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) (|J_1| = |J_2|)$
3. $J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) (|J_3| = 20)$

1) Only students allowed: $e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3)$. All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$. 
\( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) (\(|J_1| = |J_2|\))
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) (\(|J_3| = 20\))

1) Only students allowed: \( e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3) \). All others have either \(< \frac{31}{19}\) or \(> \frac{31}{19}\).

2) No shares in \([\frac{61}{133}, \frac{64}{133}]\). Look at \( J_1 \)-shares:
An \( e(1, 2, 3) \)-student has \( J_1 \)-share \( \geq \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133} \).
An \( e(1, 3, 3) \)-student has \( J_1 \)-share \( < \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133} \).
\[ f(31, 19) \leq \frac{54}{133} \]

1. \( J_1 = (\frac{59}{133}, \frac{66.5}{133}) \)
2. \( J_2 = (\frac{66.5}{133}, \frac{74}{133}) \) (\(|J_1| = |J_2|\))
3. \( J_3 = (\frac{78}{133}, \frac{79}{133}) \) (\(|J_3| = 20\))

1) Only students allowed: \( e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3) \). All others have either \(< \frac{31}{19}\) or \(> \frac{31}{19}\).

2) No shares in \([\frac{61}{133}, \frac{64}{133}]\). Look at \(J_1\)-shares:
   An \(e(1, 2, 3)\)-student has \(J_1\)-share \(> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}\).
   An \(e(1, 3, 3)\)-student has \(J_1\)-share \(< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}\).

3) No shares in \([\frac{69}{133}, \frac{72}{133}]\): \(x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]\).
\( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = (\frac{59}{133}, \frac{61}{133}) \)
2. \( J_2 = (\frac{64}{133}, \frac{66.5}{133}) \)
3. \( J_3 = (\frac{66.5}{133}, \frac{69}{133}) \) (\(|J_2| = |J_3|\))
4. \( J_4 = (\frac{72}{133}, \frac{74}{133}) \) (\(|J_1| = |J_4|\))
5. \( J_5 = (\frac{78}{133}, \frac{79}{133}) \) (\(|J_5| = 20\))
\[
f(31, 19) \leq \frac{54}{133}
\]

1. \( J_1 = (\frac{59}{133}, \frac{61}{133}) \)
2. \( J_2 = (\frac{64}{133}, \frac{66.5}{133}) \)
3. \( J_3 = (\frac{66.5}{133}, \frac{69}{133}) \) \( |J_2| = |J_3| \)
4. \( J_4 = (\frac{72}{133}, \frac{74}{133}) \) \( |J_1| = |J_4| \)
5. \( J_5 = (\frac{78}{133}, \frac{79}{133}) \) \( |J_5| = 20 \)

The following are the only students who are allowed.
e(1, 5, 5).
e(2, 4, 5),
e(3, 4, 5).
e(4, 4, 4).
\[ f(31, 19) \leq \frac{54}{133} \]

\( e(1, 5, 5) \). Let the number of such students be \( x \)
\( e(2, 4, 5) \). Let the number of such students be \( y_1 \)
\( e(3, 4, 5) \). Let the number of such students be \( y_2 \).
\( e(4, 4, 4) \). Let the number of such students be \( z \).
\( f(31, 19) \leq \frac{54}{133} \)

\( e(1, 5, 5) \). Let the number of such students be \( x \)
\( e(2, 4, 5) \). Let the number of such students be \( y_1 \)
\( e(3, 4, 5) \). Let the number of such students be \( y_2 \).
\( e(4, 4, 4) \). Let the number of such students be \( z \).

1) \(|J_2| = |J_3|\),
only students using \( J_2 \) are \( e(2, 4, 5) \) – they use one share each,
only students using \( J_3 \) are \( e(3, 4, 5) \) – they use one share each.
Hence \( y_1 = y_2 \). We call them both \( y \).
\( f(31, 19) \leq \frac{54}{133} \)

\( e(1, 5, 5) \). Let the number of such students be \( x \)
\( e(2, 4, 5) \). Let the number of such students be \( y_1 \)
\( e(3, 4, 5) \). Let the number of such students be \( y_2 \).
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Hence \( y_1 = y_2 \). We call them both \( y \).

2) Since \(|J_1| = |J_4|\), \( x = 2y + 3z \).
\( f(31, 19) \leq \frac{54}{133} \)

\( e(1, 5, 5) \). Let the number of such students be \( x \)
\( e(2, 4, 5) \). Let the number of such students be \( y_1 \)
\( e(3, 4, 5) \). Let the number of such students be \( y_2 \).
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1) \(|J_2| = |J_3|\),
only students using \( J_2 \) are \( e(2, 4, 5) \) – they use one share each,
only students using \( J_3 \) are \( e(3, 4, 5) \) – they use one share each.
Hence \( y_1 = y_2 \). We call them both \( y \).

2) Since \(|J_1| = |J_4|\), \( x = 2y + 3z \).

3) Since \( s_3 = 14 \), \( x + 2y + z = 14 \).
\( f(31, 19) \leq \frac{54}{133} \)

e(1, 5, 5). Let the number of such students be \( x \)
e(2, 4, 5). Let the number of such students be \( y_1 \)
e(3, 4, 5). Let the number of such students be \( y_2 \).
e(4, 4, 4). Let the number of such students be \( z \).

1) \(|J_2| = |J_3|\),
only students using \( J_2 \) are \( e(2, 4, 5) \) – they use one share each,
only students using \( J_3 \) are \( e(3, 4, 5) \) – they use one share each.
Hence \( y_1 = y_2 \). We call them both \( y \).

2) Since \(|J_1| = |J_4|\), \( x = 2y + 3z \).

3) Since \( s_3 = 14 \), \( x + 2y + z = 14 \).

\[
(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}.
\]
Contradiction.
The above reasoning can be used to verify that $f(31, 19) \leq \frac{54}{133}$ but could not generate the upper bound $\frac{54}{133}$.
GAP Technique

The above reasoning can be used to verify that \( f(31, 19) \leq \frac{54}{133} \) but could not generate the upper bound \( \frac{54}{133} \).

Cannot quite modify the method, but we can use this method and a method we have to find procedure and to a binary search to zero-in on answer. We call this GAP\((m, s)\). So we have

\[
(\forall m, s)[f(m, s) \leq \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
\]

Is the following true?

\[
(\forall m, s)[f(m, s) = \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
\]

No: If so my book would be about 80 pages. For \( f(67, 21) \) it fails!
The above reasoning can be used to verify that \( f(31, 19) \leq \frac{54}{133} \) but could not generate the upper bound \( \frac{54}{133} \).

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\[
(\forall m, s)[f(m, s) \leq \min\{\text{FC}(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
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Is the following true?

\[
(\forall m, s)[f(m, s) = \min\{\text{FC}(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
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GAP Technique

The above reasoning can be used to verify that \( f(31, 19) \leq \frac{54}{133} \) but could not generate the upper bound \( \frac{54}{133} \).

Cannot quite modify the method, but we can use this method and a method we have to find procedure and to a binary search to zero-in on answer. We call this GAP\((m, s)\). So we have

\[
(\forall m, s)[f(m, s) \leq \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
\]

Is the following true?

\[
(\forall m, s)[f(m, s) = \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]
\]

No: If so my book would be about 80 pages.
GAP Technique

The above reasoning can be used to verify that $f(31, 19) \leq \frac{54}{133}$ but could not generate the upper bound $\frac{54}{133}$.

Cannot quite modify the method, but we can use this method and a method we have to find procedure and to a binary search to zero-in on answer. We call this GAP($m, s$). So we have

$$(\forall m, s)[f(m, s) \leq \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]$$

Is the following true?

$$(\forall m, s)[f(m, s) = \min\{FC(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{GAP}(m, s)\}]$$

No: If so my book would be about 80 pages. For $f(67, 21)$ it fails!
The Train Method

We developed the Train Method which showed settled $f(67, 21)$ and 13 other problems we could not do.
Upshot

Let

\[ A = \{(m, s) \mid 2 \leq s \leq 100 \text{ and } s < m \leq 110 \text{ and } m, s \text{ rel prime}\} \]

There are 3520 pairs \( (m, s) \) in \( A \). We solved all of them!

- For 2301 of them \( f(m, s) = FC(m, s) \). That is \( \sim 65.37\% \).
- For 329 of them \( f(m, s) = HALF(m, s) \). That is \( \sim 9.35\% \).
- For 186 of them \( f(m, s) = INT(m, s) \). That is \( \sim 5.28\% \).
- For 111 of them \( f(m, s) = MID(m, s) \). That is \( \sim 3.15\% \).
- For 240 of them \( f(m, s) = EBM(m, s) \). That is \( \sim 6.28\% \).
- For 89 of them \( f(m, s) = HBM(m, s) \). That is \( \sim 2.53\% \).
- For 250 of them \( f(m, s) = GAP(m, s) \). That is \( \sim 7.10\% \).
- For 13 of them \( f(m, s) = TRAIN(m, s) \). That is \( \sim 0.40\% \).
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$. 

1) Guess that the only piece sizes are $\frac{5}{12}$,

2) Muffin = pieces add to 1:

$\{\frac{6}{12}, \frac{6}{12}\}$,

$\{\frac{5}{12}, \frac{7}{12}\}$.

Vectors $\{\frac{6}{12}, \frac{6}{12}\}$ is (0, 2, 0), $m_1$ muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is (1, 0, 1), $m_2$ muffins of this type.

3) Student = pieces add to $\frac{5}{3}$

$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is (0, 1, 2), $s_1$ students of this type.

$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is (4, 0, 0), $s_2$ students of this type.

4) Set up equations:

$m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)$

$m_1 + m_2 = 5$

$s_1 + s_2 = 3$

Natural Number Solution:

$m_1 = 1, m_2 = 4, s_1 = 2, s_2 = 1$
MATRIX Technique: \( f(5, 3) \geq \frac{5}{12} \)

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \).
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**= pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors $\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, $m_1$ muffins of this type. $\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, $m_2$ muffins of this type.
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors

$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, $m_1$ muffins of this type.

$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, $m_2$ muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$

$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, $s_1$ students of this type.

$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, $s_2$ students of this type.
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin** = pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}$, $\{\frac{5}{12}, \frac{7}{12}\}$. Vectors $\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, $m_1$ muffins of this type.  
$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, $m_2$ muffins of this type.

3) **Student** = pieces add to $\frac{5}{3}$
$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, $s_1$ students of this type.  
$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, $s_2$ students of this type.

4) **Set up equations:**
$m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)$
$m_1 + m_2 = 5$
$s_1 + s_2 = 3$
MATRIX Technique: \( f(5, 3) \geq \frac{5}{12} \)

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \)

2) **Muffin** = pieces add to 1: \( \{ \frac{6}{12}, \frac{6}{12} \}, \{ \frac{5}{12}, \frac{7}{12} \} \). Vectors \( \{ \frac{6}{12}, \frac{6}{12} \} \) is \((0, 2, 0)\), \( m_1 \) muffins of this type. \( \{ \frac{5}{12}, \frac{7}{12} \} \) is \((1, 0, 1)\), \( m_2 \) muffins of this type.

3) **Student** = pieces add to \( \frac{5}{3} \)
   \( \{ \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \} \) is \((0, 1, 2)\), \( s_1 \) students of this type.
   \( \{ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \} \) is \((4, 0, 0)\), \( s_2 \) students of this type.

4) **Set up equations:**
   \( m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0) \)
   \( m_1 + m_2 = 5 \)
   \( s_1 + s_2 = 3 \)

**Natural Number Solution:** \( m_1 = 1, m_2 = 4, s_1 = 2, s_2 = 1 \)
Want proc for \( f(m, s) \geq \frac{a}{b} \).
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1) **Guess** that the only piece sizes are \( \frac{a}{b}, \ldots, \frac{b-a}{b} \).
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2) **Muffin**—pieces add to 1: Vectors $\vec{v}_i$. $x$ types. $m_i$ muffins of type $\vec{v}_i$
Want proc for $f(m, s) \geq \frac{a}{b}$.

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2) **Muffin** = pieces add to 1: Vectors $\vec{v}_i$. $x$ types.
   $m_i$ muffins of type $\vec{v}_i$

3) **Student** = pieces add to $\frac{m}{s}$: Vectors $\vec{u}_j$. $y$ types.
   $s_j$ students of type $\vec{u}_j$
Want proc for \( f(m, s) \geq \frac{a}{b} \).

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3) **Student**=pieces add to \( \frac{m}{s} \): Vectors \( \vec{u}_j \). \( y \) types. \( s_j \) students of type \( \vec{u}_j \)

4) **Set up equations:**

\[
\begin{align*}
 m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x &= s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y \\
 m_1 + \cdots + m_x &= m \\
 s_1 + \cdots + s_y &= s 
\end{align*}
\]
Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

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4) **Set up equations:**
   
   \[ m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x = s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y \]
   
   \[ m_1 + \cdots + m_x = m \]
   
   \[ s_1 + \cdots + s_y = s \]

5) **Look for Nat Numb sol.** If find can translate into procedure.
So Where Are We Now?

Is the following true: For all \( m, s \), \( f(m, s) \) is the min of
Is the following true: For all $m, s$, $f(m, s)$ is the min of

$FC(m, s), HALF(m, s), INT(m, s), MID(m, s), EBM(m, s), HBM(m, s), GAP(m, s), TRAIN(m, s)$
Is the following true: For all $m, s$, $f(m, s)$ is the min of $\text{FC}(m, s), \text{HALF}(m, s), \text{INT}(m, s), \text{MID}(m, s),$ $\text{EBM}(m, s), \text{HBM}(m, s), \text{GAP}(m, s), \text{TRAIN}(m, s)$

No. Did not work on

- $f(205, 178)$
- $f(226, 135)$
- $f(233, 141)$
Scott Huddleston has an algorithm that is REALLY FAST and seems to ALWAYS WORK. Erik and Jacob understand it, nobody else does. They have replicated his results and think that YES it solves ALL problems.

Richard Chatwin independently came up with the same algorithm and proved it worked, but never coded it up. He tells me its poly time and I believe this can be proved, though it's not in his paper. His paper is here: https://arxiv.org/abs/1907.08726
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▶ Important or NOT Important.

These are not well defined terms but

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**Important** Either useful OR about deep mathematical objects (hard to define that).

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4. Not Interesting and Not Important: So far $R(5)$ has not lead to any math of interest and is also not important to find. Same for most Ramsey-type Numbers.
Lessons Learned

I found this problem in a pamphlet at a Recreational math Conference.
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Math is all around you! Pursue your curiosity!
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You never know where the next big project will come from!