The Muffin Problem

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How it Began

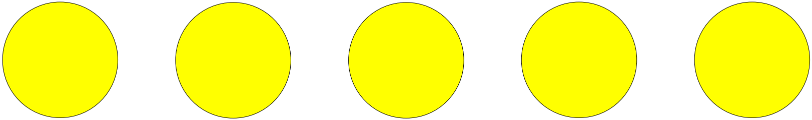
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

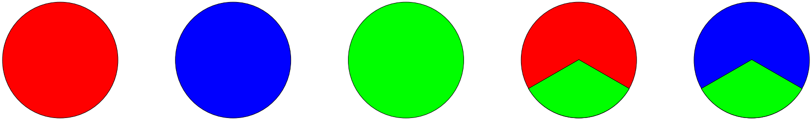
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



5 Muffins, 3 Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

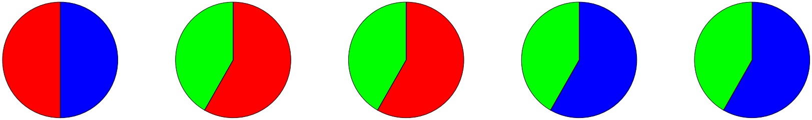
The smallest piece in the above solution is $\frac{1}{3}$.
Is there a procedure with a larger smallest piece?
Work on it with your neighbor

5 Muffins, 3 People–Proc by Picture

YES WE CAN!

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12}+\frac{7}{12}+\frac{7}{12}$
Carol	GREEN	$\frac{5}{12}+\frac{5}{12}+\frac{5}{12}+\frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor

5 Muffins, 3 People–Can't Do Better Than $\frac{5}{12}$

NO WE CAN'T!

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases. (**Henceforth:** All muffins cut into $\geq \mathbf{2}$ pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins cut into 2 pieces.)

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Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (**Henceforth:** All muffins cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others

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- Find a new technique **which was interesting**.
- Lather, Rinse, Repeat.

General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

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We showed $f(m,s)$ exists, rational, computable, via a Mixed Int Program.

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
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Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

7 Muffins, 3 Students

Work on $f(7,3)$ in groups in breakout rooms.
7 Muffins, 3 Students.
Get upper and lower bounds that match!

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3. 7 muffins, each one cut in two 2 pieces, so 14 pieces total.

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Now what?

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5. That student must get a piece $\geq \frac{7}{3} \times \frac{1}{4} = \frac{7}{12}$.
6. That piece came from a muffin. Other piece is $\leq 1 - \frac{7}{12} = \frac{5}{12}$.

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8. Can we show a protocol that gives $f(7,3) \geq \frac{5}{12}$?

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Want $f(7,3) \geq \frac{5}{12}$.

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Want $f(7,3) \geq \frac{5}{12}$.
Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.

7 Muffins, 3 Students: How to think about protocol

Want $f(7,3) \geq \frac{5}{12}$.
Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.
Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.

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Will be cutting some muffins $(\frac{5}{12}, \frac{7}{12})$.
Can also cut some muffins $(\frac{6}{12}, \frac{6}{12})$.
Need to know what combos of $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$ add to $\frac{7}{3} = \frac{28}{12}$.
Need to know what combos of $5, 6, 7$ add to 28.
$7 + 7 + 7 + 7 = 28$
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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.

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2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.

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1. Cut 4 muffins $(\frac{5}{12}, \frac{7}{12})$.
2. Cut 3 muffins $(\frac{6}{12}, \frac{6}{12})$.
3. Give 1 student 4 pieces of size $\frac{7}{12}$.
4. Give 2 students 2 pieces of size $\frac{5}{12}$ and 3 pieces of size $\frac{6}{12}$.

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2. If a muffin is cut in ≥ 3 pieces then some piece $\leq \frac{1}{3}$. Unlikely that thats a good idea.
3. 8 muffins, each one cut in two 2 pieces, so 16 pieces total.

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4. 3 students, so some student gets $\geq \lceil \frac{16}{3} \rceil = 6$ pieces. That student must get a piece $\leq \frac{8}{3} \times \frac{1}{6} = \frac{4}{9}$.

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6. Great! We know $f(8,3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.

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6. Great! We know $f(8,3) \leq \min\{\frac{4}{9}, \frac{7}{15}\} = \frac{4}{9}$.
7. Can we show a protocol that gives $f(8,3) \geq \frac{4}{9}$?

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Will be cutting some muffins $(\frac{4}{9}, \frac{5}{9})$.
$\frac{1}{2}$ was helpful last time so lets also include $\frac{4.5}{9}$.
Need to know what combos of $\frac{4}{9}, \frac{4.5}{9}, \frac{5}{9}$ add to $\frac{8}{3} = \frac{24}{9}$.

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Need to know what combos of $4, 4.5, 5$ add to 24

$4+4+4+4+4+4=24$

$4.5+4.5+5+5+5=24$

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$4+4+4+4+4+4=24$

$4.5+4.5+5+5+5=24$

1. Cut 6 muffins $(\frac{4}{9}, \frac{5}{9})$.
2. Cut 2 muffins $(\frac{4.5}{9}, \frac{4.5}{9})$.
3. Give 1 student six $\frac{4}{9}$ pieces.
4. Give 2 students two $\frac{4.5}{9}$ pieces and four $\frac{5}{9}$ pieces.

Conventions

We know and use the following:

1. Known: $f(m,s) = \frac{m}{s} f(s,m)$. Hence we assume $m > s$.
2. If s divides m then $f(m,s) = 1$ so assume s does not divide m.
3. All muffins are cut in ≥ 2 pcs. Replace uncut muff with 2 $\frac{1}{2}$'s
4. If assuming $f(m,s) > \alpha > \frac{1}{3}$, assume all muffin in ≤ 2 pcs.
5. $f(m,s) > \alpha > \frac{1}{3}$, so exactly 2 pcs, is common case.

We do not know this but still use: $f(m,s)$ only depends on $\frac{m}{s}$.
All of our techniques that hold for (m,s) hold for (Am, As).
For particular numbers, we only look at m, s rel prime.

FC Thm Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le \mathrm{FC}(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{m}{s\lceil 2m/s\rceil}$.

Someone gets $\le \left\lfloor \frac{2m}{s} \right\rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1$.

$f(3k+1,3) = \frac{3k-1}{6k}$, $k \geq 1$.

$f(3k+2,3) = \frac{3k+2}{6k+6}$.

Note: A Mod 3 Pattern.
Theorem: For all $m \geq 3$, $f(m,3) = \mathrm{FC}(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k,4) = 1$ (easy)

$f(1,4) = \frac{1}{4}$ (easy)

$f(4k+1,4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2,4) = \frac{1}{2}$.

$f(4k+3,4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m,4) = \mathrm{FC}(m,4)$.
FC-Conjecture: For all m, s with $m \geq s$, $f(m,s) = \mathrm{FC}(m,s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \ge 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \ge 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \ge 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = \mathrm{FC}(7, 5) = \frac{1}{3}$

For $k \ge 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \ge 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \ge 5$ **except m=11**, $f(m, 5) = \mathrm{FC}(m, 5)$.

What About FIVE students, ELEVEN muffins?

$$f(11,5) \le \max\left\{\frac{1}{3}, \min\left\{\frac{11}{5\lceil 22/5\rceil}, 1-\frac{11}{5\lfloor 22/5\rfloor}\right\}\right\} = \frac{11}{25}.$$

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.

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We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$.
We found a protocol with smallest piece $\frac{13}{30} = 0.4333\ldots$.

1. Divide 1 muffin $(\frac{15}{30}, \frac{15}{30})$.
2. Divide 2 muffins $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 8 muffins $(\frac{13}{30}, \frac{17}{30})$.
4. Give 2 students $[\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30}]$
5. Give 1 students $[\frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30}]$
6. Give 2 students $[\frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30}]$

So Now What?

We have:

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

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$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote

So Now What?

We have:

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

Options:

1. $f(11,5) = \frac{11}{25}$. Need to find procedure.
2. $f(11,5) = \frac{13}{30}$. Need to find new technique for upper bounds.
3. $f(11,5)$ in between. Need to find both.
4. $f(11,5)$ unknown to science!

Vote WE SHOW: $\mathbf{f(11,5) = \frac{13}{30}}$. **Exciting** new technique!

Terminology: Buddy

Assume that in some protocol every muffin is cut into two pieces.

Let x be a piece from muffin M.
The *other piece* from muffin M is the *buddy of* x.

Note that the buddy of x is of size

$$1 - x.$$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

$f(11,5)=\frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.1: Some 4-share is $\le \frac{1}{2}$.
Alice gets w, x, y, z and $w \le \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \le \frac{1}{2}$

$$x + y + z \ge \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

Let x be the largest of x, y, z

$$x \ge \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of x.

$$B(x) \le 1 - x = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All 4-shares are $> \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

INT Method

Proof that $f(11,5) \le \frac{13}{30}$ was an example of the INT method.
We give a more sophisticated example

More Sophisticated INT: $f(24, 11) \le \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11\times 6} < \frac{19}{44}$.

More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$.
Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

More Sophisticated INT: $f(24,11) \le \frac{19}{44}$

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\leq \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\leq \frac{24}{11\times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\geq \frac{24}{11\times 3} = \frac{8}{11}$.
Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. s_4 is the number of them.
5-students: a student who gets 5 shares. s_5 is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \end{aligned}$$

$s_4 = 7$. Hence there are $4s_4 = 4 \times 7 = 28$ 4-shares.
$s_5 = 4$. Hence there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.1 and 3.2: Too Big or Too Small

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Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

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Case 3.1: There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Case 3.2: There is a share $\leq \frac{19}{44}$. Duh.
Henceforth assume that all shares are in

$$\left(\frac{19}{44}, \frac{25}{44}\right)$$

$$\underset{\frac{19}{44}}{(} \qquad\qquad\qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.
Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.
Proof: Assume that Alice 5 pieces A, B, C, D, E and $E \geq \frac{20}{44}$.
Since $A+B+C+D+E = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$A+B+C+D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume A is the smallest of A, B, C, D.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.3: Some 5-shares $\geq \frac{20}{44}$

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Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume that Alice 5 pieces A, B, C, D, E and $E \geq \frac{20}{44}$.

Since $A+B+C+D+E = \frac{24}{11}$ and $E \geq \frac{20}{44}$

$$A+B+C+D \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

Assume A is the smallest of A, B, C, D.

$$A \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \qquad\qquad \underset{\frac{25}{44}}{)}$$

Case 3.4: Some 4-shares $\le \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\le \frac{21}{44}$ then some share $\le \frac{19}{44}$.

Proof: Assume that Alice 4 pieces A, B, C, D and $D \le \frac{21}{44}$.

Since $A + B + C + D = \frac{24}{11}$ and $D \le \frac{21}{44}$

$$A + B + C \ge \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

Assume A is the largest of A, B, C.

$$A \ge \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of A is of size

$$\le 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left(\frac{21}{44}, \frac{25}{44}\right).$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad ??\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\underset{\frac{19}{44}}{(} \quad \text{?? 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{?? 4-shs} \quad \underset{\frac{25}{44}}{)}$$

Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\underset{\frac{19}{44}}{(} \quad ??\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad ??\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

Recall: there are $4s_4 = 4 \times 7 = 28$ 4-shares.
Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.

$$\underset{\frac{19}{44}}{(} \quad 20\ \text{5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0\ \text{shs} \quad \underset{\frac{21}{44}}{](} \quad 28\ \text{4-shs} \quad \underset{\frac{25}{44}}{)}$$

More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$.

More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then buddy is in $[\frac{20}{44}, \frac{21}{44}]$.
The following picture captures what we know so far.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

S4= Small 4-shares
L4= Large 4-shares. L4 shares, 5-share: **buddies**, so $|L4|=20$.

Diagram

(	20 5-shs	)[	0	](	8 S4-shs	)[	0	](	20 L4-shs	)
$\frac{19}{44}$		$\frac{20}{44}$		$\frac{21}{44}$		$\frac{23}{44}$		$\frac{24}{44}$		$\frac{25}{44}$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

Diagram

$$\underset{\frac{19}{44}}{(} \quad 20\text{ 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad 8\text{ S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad 20\text{ L4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

Contradiction: Each 4-student gets ≥ 3 L4 shares. There are $s_4 = 7$ 4-students. Hence there are ≥ 21 L4-shares. But there are only 20.

INT **Technique**

INT is generalization of $f(24,11) \le \frac{19}{44}$ proof.
Definition: Let $\mathrm{INT}(m,s)$ be the bound obtained.

1. INT proofs can get more complicated than this one.
2. $\mathrm{INT}(m,s)$ can be computed in $O(\frac{2m\log m}{s})$. Note: do not need to know the answer ahead of time.
3. For $1 \le s \le 60$, $s < m \le 70$, m, s rel prime:
   3.1 There are 1360 cases total.
   3.2 For 927 of the (m,s), $f(m,s) = \mathrm{FC}(m,s)$. $\sim 68\%$
   3.3 For 268 of the (m,s), $f(m,s) = \mathrm{INT}(m,s)$. $\sim 20\%$
   3.4 The cases not covered use **interesting** new techniques!

Example of GAPS Technique: $f(31,19) \le \frac{54}{133}$

We show $f(31,19) \le \frac{54}{133}$.
Assume $(31,19)$-procedure with smallest piece $> \frac{54}{133}$.
By INT-technique methods obtain:
$s_3 = 14$, $s_4 = 5$.

$$\underset{\frac{54}{133}}{(} \quad \text{20 4-shs} \quad \underset{\frac{55}{133}}{)[} \quad 0 \quad \underset{\frac{59}{133}}{](} \quad \text{22 S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

We just look at the 3-shares:

$$\underset{\frac{59}{133}}{(} \quad \text{22 S3 shs} \quad \underset{\frac{74}{133}}{)[} \quad 0 \quad \underset{\frac{78}{133}}{](} \quad \text{20 L3-shs} \quad \underset{\frac{79}{133}}{)}$$

GAPS Technique: $f(31,19) \le \frac{54}{133}$

(	22 S3 shs	)[	0	](	20 L3-shs	)
$\frac{59}{133}$		$\frac{74}{133}$		$\frac{78}{133}$		$\frac{79}{133}$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

Note: Split the shares of size 66.5 between J_1 and J_2.

Notation: An $e(1,1,3)$ students is a student who has a J_1-share, a J_1-share, and a J_3-share.

Generalize to $e(i,j,k)$ easily.

GAPS Technique: $f(31,19) \le \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ $(|J_1| = |J_2|)$
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ $(|J_3| = 20)$

1) Only students allowed: $e(1,2,3)$, $e(1,3,3)$, $e(2,2,2)$, $e(2,2,3)$.
All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $[\frac{61}{133}, \frac{64}{133}]$. Look at J_1-shares:
An $e(1,2,3)$-student has J_1-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
An $e(1,3,3)$-student has J_1-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$.

3) No shares in $[\frac{69}{133}, \frac{72}{133}]$: $x \in [\frac{69}{133}, \frac{72}{133}] \implies 1 - x \in [\frac{61}{133}, \frac{64}{133}]$.

GAPS Technique: $f(31,19) \le \frac{54}{133}$

1. $J_1 = (\frac{59}{133}, \frac{61}{133})$
2. $J_2 = (\frac{64}{133}, \frac{66.5}{133})$
3. $J_3 = (\frac{66.5}{133}, \frac{69}{133})$ $(|J_2| = |J_3|)$
4. $J_4 = (\frac{72}{133}, \frac{74}{133})$ $(|J_1| = |J_4|)$
5. $J_5 = (\frac{78}{133}, \frac{79}{133})$ $(|J_5| = 20)$

The following are the only students who are allowed.

$e(1,5,5)$.

$e(2,4,5)$,

$e(3,4,5)$.

$e(4,4,4)$.

GAPS Technique: $f(31, 19) \le \frac{54}{133}$

$e(1,5,5)$. Let the number of such students be x
$e(2,4,5)$. Let the number of such students be y_1
$e(3,4,5)$. Let the number of such students be y_2.
$e(4,4,4)$. Let the number of such students be z.
1) $|J_2| = |J_3|$,
only students using J_2 are $e(2,4,5)$ – they use one share each,
only students using J_3 are $e(3,4,5)$ – they use one share each.
Hence $y_1 = y_2$. We call them both y.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$.
Contradiction.

MATRIX Technique: $f(5,3) \geq \frac{5}{12}$

Want proc for $f(5,3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin**=pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}$, $\{\frac{5}{12}, \frac{7}{12}\}$. Vectors
$\{\frac{6}{12}, \frac{6}{12}\}$ is $(0,2,0)$, m_1 muffins of this type.
$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1,0,1)$, m_2 muffins of this type.

3) **Student**=pieces add to $\frac{5}{3}$
$\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0,1,2)$, s_1 students of this type.
$\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4,0,0)$, s_2 students of this type.

4) **Set up equations:**

$$m_1(0,2,0) + m_2(1,0,1) = s_1(0,1,2) + s_2(4,0,0)$$

$$m_1 + m_2 = 5$$

$$s_1 + s_2 = 3$$

Natural Number Solution: $m_1 = 1$, $m_2 = 4$, $s_1 = 2$, $s_2 = 1$

MATRIX Technique

Want proc for $f(m,s) \ge \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

2) **Muffin**=pieces add to 1: Vectors $\vec{v_i}$. x types.
m_i muffins of type $\vec{v_i}$

3) **Student**=pieces add to $\frac{m}{s}$: Vectors $\vec{u_j}$. y types.
s_j students of type $\vec{u_j}$

4) **Set up equations:**
$m_1\vec{v_1} + \cdots + m_x\vec{v_x} = s_1\vec{u_1} + \cdots + s_y\vec{u_y}$
$m_1 + \cdots + m_x = m$
$s_1 + \cdots + s_y = s$

5) **Look for Nat Numb sol.** If find can translate into procedure.

Other Techniques

Here are the list of the Techniques we came up with:

1. Floor-Ceiling
2. Half
3. INT
4. GAP
5. Easy buddy-match
6. Hard buddy-match
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Time to say we are NOT going to find a finite set of techniques that covers all cases and take what we got and write a book.

Later Results by Other People

1. In Fall 2018 Scott Huddleston had code for an algorithm that, on input m, s, found $f(m, s)$ and the procedure REALLY FAST.
2. Jacob and Erik Understand WHAT his algorithm does and Jacob coded it up to make sure he understood it. Jacob's code is also REALLY FAST.
3. Neither Scott, Bill, Jacob, or Erik had a proof that Scott's algorithm was fast (poly in m, s).
4. Richard Chatwin independently came up with the same algorithm; however, he also has a proof that it works. Its on arXiv.
5. One corollary of the work: $f(m, s)$ only depends on m/s.

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  Second Year Royalties: \$40.00
  Third Year Royalties: Not known yet but I suspect it will be $<$ \$40.00

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   - Bill is a brilliant visionary and writer.