The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?
## 5 Muffins, 3 Students, Proc by Picture

<table>
<thead>
<tr>
<th>Person</th>
<th>Color</th>
<th>What they Get</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>RED</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Bob</td>
<td>BLUE</td>
<td>$1 + \frac{2}{3} = \frac{5}{3}$</td>
</tr>
<tr>
<td>Carol</td>
<td>GREEN</td>
<td>$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$</td>
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</tbody>
</table>

**Smallest Piece:** $\frac{1}{3}$
Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor
YES WE CAN!

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Smallest Piece: $\frac{5}{12}$
Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Work on it with your neighbor.
NO WE CAN’T!

There is a procedure for 5 muffins, 3 students where each student gets \( \frac{5}{3} \) muffins, smallest piece \( N \). We want \( N \leq \frac{5}{12} \).

**Case 0:** Some muffin is uncut. Cut it \((\frac{1}{2}, \frac{1}{2})\) and give both \(\frac{1}{2}\)-sized pieces to whoever got the uncut muffin. (Note \(\frac{1}{2} > \frac{5}{12}\).) Reduces to other cases. (**Henceforth:** All muffins cut into \( \geq 2 \) pieces.)
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Case 1: Some muffin is cut into $\geq 3$ pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$. (Henceforth: All muffins cut into 2 pieces.)
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**Case 0:** Some muffin is uncut. Cut it \( \left( \frac{1}{2}, \frac{1}{2} \right) \) and give both \( \frac{1}{2} \)-sized pieces to whoever got the uncut muffin. (Note \( \frac{1}{2} > \frac{5}{12} \).) Reduces to other cases. (**Henceforth:** All muffins cut into \( \geq 2 \) pieces.)

**Case 1:** Some muffin is cut into \( \geq 3 \) pieces. Then \( N \leq \frac{1}{3} < \frac{5}{12} \). (**Henceforth:** All muffins cut into \( 2 \) pieces.)

**Case 2:** All muffins are cut into \( 2 \) pieces. 10 pieces, 3 students: **Someone** gets \( \geq 4 \) pieces. He has some piece

\[
\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12}
\]

Great to see \( \frac{5}{12} \)
What Else Was in the Pamphlet?

The pamphlet also had asked about

1. 4 muffins, 7 students.
2. 12 muffins, 11 students.
3. a few others
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2. 12 muffins, 11 students.
3. a few others

This seemed like a nice exercise and it was.

There can’t be much more to this.
If there is not much more to this then how come

https://www.amazon.com/
Mathematical-Muffin-Morsels-Problem-Mathematics/dp/
9811215170
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- Find a technique that solves many problems (e.g., Floor-Ceiling).
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The following happened:

▸ Find a technique that solves many problems (e.g., Floor-Ceiling).
▸ Come across a problem where the techniques do not work.
▸ Find a new technique **which was interesting**.
▸ Lather, Rinse, Repeat.
General Problem

\( f(m, s) \) be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide \( m \) muffins among \( s \) students so that everyone gets \( \frac{m}{s} \).

We have shown \( f(5, 3) = \frac{5}{12} \) here.

We have shown \( f(m, s) \) exists, is rational, and is computable using a **Mixed Int Program**.
General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide $m$ muffins among $s$ students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program.

This was a case of a Theorem in Applied Math being used to prove a Theorem in Pure Math.
Amazing Results! / Amazing Theorems!

1. \( f(43, 33) = \frac{91}{264} \).
2. \( f(52, 11) = \frac{83}{176} \).
3. \( f(35, 13) = \frac{64}{143} \).
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Have **General Theorems** from which **upper bounds** follow. Have **General Procedures** from which **lower bounds** follow.
Conventions

**Duality Theorem:** \( f(m, s) = \frac{m}{s} f(s, m) \).
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4. If assuming \( f(m, s) > \alpha > \frac{1}{3} \), assume all muffin in \( \leq 2 \) pcs.
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4. If assuming \( f(m, s) > \alpha > \frac{1}{3} \), assume all muffin in \( \leq 2 \) pcs.
5. \( f(m, s) > \alpha > \frac{1}{3} \), so exactly 2 pcs, is common case.
FC Thm Generalizes \( f(5, 3) \leq \frac{5}{12} \)

\[
f(m, s) \leq FC(m, s) = \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}.
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Case 1: Some muffin is cut into \(\geq 3\) pieces. Some piece \(\leq \frac{1}{3}\).

Case 2: Every muffin is cut into 2 pieces, so \(2m\) pieces.
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**Someone** gets $\geq \left\lceil \frac{2m}{s} \right\rceil$ pieces. $\exists$ piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$. 

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The other piece from that muffin is of size \(\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor} \).
THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.
THREE Students

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\[ f(1, 3) = \frac{1}{3} \]
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\[ f(3k, 3) = 1. \]
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f(3k + 1, 3) = \frac{3k-1}{6k}, \ k \geq 1.
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**Note:** A Mod 3 Pattern.

**Theorem:** For all \( m \geq 3 \), \( f(m, 3) = FC(m, 3) \).
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\[ f(4k, 4) = 1 \text{ (easy)} \]
FOUR Students

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\[ f(4k + 3, 4) = \frac{4k+1}{8k+4}. \]

Note: A Mod 4 Pattern.

Theorem: For all \( m \geq 4, \ f(m, 4) = FC(m, 4). \)
FOUR Students

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FC Theorem for optimality.

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Theorem: For all \( m \geq 4 \), \( f(m, 4) = FC(m, 4) \).

FC-Conjecture: For all \( m, s \) with \( m \geq s \), \( f(m, s) = FC(m, s) \).
FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k,5) = 1$. 

Note: A Mod 5 Pattern. 

Theorem: For all $m \geq 5$ except $m = 11$, $f(m,5) = \text{FC}(m,5)$. 
CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For \( k \geq 1 \), \( f(5k, 5) = 1 \).

For \( k = 1 \) and \( k \geq 3 \), \( f(5k + 1, 5) = \frac{5k+1}{10k+5} \). \( f(11, 5) \)?

Note: A Mod 5 Pattern.
FIVE Students

**CLEVERNESS, COMP PROGS** for procedures.

**FC Theorem** for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$
FIVE Students

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FC Theorem for optimality.

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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$
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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

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For $k \geq 1$, $f(5k, 5) = 1$.

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For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

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Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ except $m=11$, $f(m, 5) = FC(m, 5)$. 
What About FIVE students, ELEVEN muffins?

\[
f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \left\lceil \frac{22}{5} \right\rceil}, 1 - \frac{11}{5 \left\lfloor \frac{22}{5} \right\rfloor} \right\} \right\} = \frac{11}{25}.
\]
What About FIVE students, ELEVEN muffins?

$$f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil 22/5 \rceil}, 1 - \frac{11}{5 \lfloor 22/5 \rfloor} \right\} \right\} = \frac{11}{25}.$$  

We tried to find a protocol to divide 11 muffins for 5 people, each gets $\frac{11}{5}$, and smallest piece is size $\frac{11}{25} = 0.44$. 
What About FIVE students, ELEVEN muffins?

\[ f(11, 5) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{11}{5 \lceil \frac{22}{5} \rceil}, 1 - \frac{11}{5 \lceil \frac{22}{5} \rceil} \right\} \right\} = \frac{11}{25}. \]

We tried to find a protocol to divide 11 muffins for 5 people, each gets \( \frac{11}{5} \), and smallest piece is size \( \frac{11}{25} = 0.44 \).

We found a protocol with smallest piece \( \frac{13}{30} = 0.4333 \ldots \)

1. Divide 1 muffin (\( \frac{15}{30}, \frac{15}{30} \)).
2. Divide 2 muffins (\( \frac{14}{30}, \frac{16}{30} \)).
3. Divide 8 muffins (\( \frac{13}{30}, \frac{17}{30} \)).
4. Give 2 students [\( \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{14}{30} \)]
5. Give 1 students [\( \frac{16}{30}, \frac{16}{30}, \frac{17}{30}, \frac{17}{30} \)]
6. Give 2 students [\( \frac{15}{30}, \frac{17}{30}, \frac{17}{30}, \frac{17}{30} \)]
So Now What?

We have:

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots
\]
So Now What?

We have:

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots
\]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

Vote
So Now What?

We have:

\[
\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\ldots
\]

Options:

1. \( f(11, 5) = \frac{11}{25} \). Need to find procedure.
2. \( f(11, 5) = \frac{13}{30} \). Need to find new technique for upper bounds.
3. \( f(11, 5) \) in between. Need to find both.
4. \( f(11, 5) \) unknown to science!

Vote \textbf{WE SHOW}: \( f(11, 5) = \frac{13}{30} \). \textbf{Exciting} new technique!
Assume that in some protocol every muffin is cut into two pieces.

Let $x$ be a piece from muffin $M$. The other piece from muffin $M$ is the buddy of $x$.

Note that the buddy of $x$ is of size $1 - x$. 
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it ($\frac{1}{2}$, $\frac{1}{2}$) and give both halves to whoever got the uncut muffin. Reduces to other cases.
There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece $N$. We want $N \leq \frac{13}{30}$.

**Case 0:** Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

**Case 1:** Some muffin is cut into $\geq 3$ pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

*(Negation of Case 0 and Case 1: All muffins cut into 2 pieces.)*
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$
Case 2: Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

Case 3: Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]
$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

**Case 2:** Some student gets $\geq 6$ pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$ 

**Case 3:** Some student gets $\leq 3$ pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$ 

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$ 

Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.
\( f(11, 5) = \frac{13}{30} \), Easy Case Based on Students

**Case 2:** Some student gets \( \geq 6 \) pieces.

\[
N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.
\]

**Case 3:** Some student gets \( \leq 3 \) pieces.

One of the pieces is

\[
\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.
\]

Look at the muffin it came from to find a piece that is

\[
\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.
\]

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note \( \leq 11 \) pieces are \( > \frac{1}{2} \).

- \( s_4 \) is number of students who get 4 pieces
- \( s_5 \) is number of students who get 5 pieces
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\sum_{i=1}^{3} s_i = 5
\]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4:** Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
s_4 + s_5 = 5
\]

$s_4 = 3$: There are 3 students who have 4 shares.  
$s_5 = 2$: There are 2 students who have 5 shares.
Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note $\leq 11$ pieces are $> \frac{1}{2}$.

- $s_4$ is number of students who get 4 pieces
- $s_5$ is number of students who get 5 pieces

\[
4s_4 + 5s_5 = 22 \\
\begin{align*}
s_4 + s_5 &= 5 
\end{align*}
\]

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a 4-share. We call a share that goes to a person who gets 5 shares a 5-share.
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.1:** Some 4-share is $\leq \frac{1}{2}$.

Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.

Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$
\[ f(11, 5) = \frac{13}{30}, \text{ Fun Cases} \]

**Case 4.1:** Some 4-share is \( \leq \frac{1}{2} \).

Alice gets \( w \leq x \leq y \leq z \) and \( w \leq \frac{1}{2} \).

Since \( w + x + y + z = \frac{11}{5} \) and \( w \leq \frac{1}{2} \)

\[
x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}
\]

\[
z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}
\]
$f(11, 5) = \frac{13}{30}$, Fun Cases

**Case 4.1:** Some 4-share is $\leq \frac{1}{2}$.
Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at **buddy** of $z$.

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$
Case 4.1: Some 4-share is $\leq \frac{1}{2}$.
Alice gets $w \leq x \leq y \leq z$ and $w \leq \frac{1}{2}$.
Since $w + x + y + z = \frac{11}{5}$ and $w \leq \frac{1}{2}$

$$x + y + z \geq \frac{11}{5} - \frac{1}{2} = \frac{17}{10}$$

$$z \geq \frac{17}{10} \times \frac{1}{3} = \frac{17}{30}$$

Look at buddy of $z$.

$$B(z) \leq 1 - z = 1 - \frac{17}{30} = \frac{13}{30}$$

GREAT! This is where $\frac{13}{30}$ comes from!
Case 4.2: All 4-shares are $\geq \frac{1}{2}$. There are $4s_4 = 12$ 4-shares. There are $\geq 12$ pieces $> \frac{1}{2}$. Can’t occur.
INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.
INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$. 
Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$.

Then we found a case where neither FC nor HALF worked.
INT Method

Proof that $f(11, 5) \leq \frac{13}{30}$ was an example of the HALF method.

FC or HALF worked on everything with $s = 3, 4, 5, \ldots, 23$.

Then we found a case where neither FC nor HALF worked.

We found a new method: INT.
More Sophisticated INT: \( f(24, 11) \leq \frac{19}{44} \)

Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).
More Sophisticated INT: \( f(24, 11) \leq \frac{19}{44} \)

Assume \((24, 11)\)-procedure with smallest piece \( > \frac{19}{44} \).
Can assume all muffin cut in two and all student gets \( \geq 2 \) shares.
We show that there is a piece \( \leq \frac{19}{44} \).

**Case 1:** A student gets \( \geq 6 \) shares. Some piece \( \leq \frac{24}{11 \times 6} < \frac{19}{44} \).
More Sophisticated INT: $f(24, 11) \leq \frac{19}{44}$

Assume $(24, 11)$-procedure with smallest piece $> \frac{19}{44}$. Can assume all muffin cut in two and all student gets $\geq 2$ shares. We show that there is a piece $\leq \frac{19}{44}$.

**Case 1:** A student gets $\geq 6$ shares. Some piece $\leq \frac{24}{11 \times 6} < \frac{19}{44}$.

**Case 2:** A student gets $\leq 3$ shares. Some piece $\geq \frac{24}{11 \times 3} = \frac{8}{11}$. Buddy of that piece $\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}$.
Assume \((24, 11)\)-procedure with smallest piece \(> \frac{19}{44}\).
Can assume all muffin cut in two and all student gets \(\geq 2\) shares.
We show that there is a piece \(\leq \frac{19}{44}\).

**Case 1:** A student gets \(\geq 6\) shares. Some piece \(\leq \frac{24}{11 \times 6} < \frac{19}{44}\).

**Case 2:** A student gets \(\leq 3\) shares. Some piece \(\geq \frac{24}{11 \times 3} = \frac{8}{11}\).
Buddy of that piece \(\leq 1 - \frac{8}{11} \leq \frac{3}{11} < \frac{19}{44}\).

**Case 3:** Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. \( s_4 \) is the number of them.
5-students: a student who gets 5 shares. \( s_5 \) is the number of them.
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.

5-students: a student who gets 5 shares. $s_5$ is the number of them.

4-share: a share that a 4-student who gets.

5-share: a share that a 5-student who gets.
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. $s_4$ is the number of them.
5-students: a student who gets 5 shares. $s_5$ is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

$$4s_4 + 5s_5 = 48$$
$$s_4 + s_5 = 11$$
How many students get 4? 5? Where are Shares?

4-students: a student who gets 4 shares. \( s_4 \) is the number of them.
5-students: a student who gets 5 shares. \( s_5 \) is the number of them.

4-share: a share that a 4-student who gets.
5-share: a share that a 5-student who gets.

\[
4s_4 + 5s_5 = 48 \\
\]
\[
s_4 + s_5 = 11 \\
\]

\( s_4 = 7 \). Hence there are \( 4s_4 = 4 \times 7 = 28 \) 4-shares.
\( s_5 = 4 \). Hence there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.
Case 3.1 and 3.2: Too Big or Too Small

Case 3.1:
There is a share $\geq \frac{25}{44}$. Then its buddy is $\leq 1 - \frac{25}{44} = \frac{19}{44}$.

Case 3.2:
There is a share $\leq \frac{19}{44}$. Duh.

Henceforth assume that all shares are in $\left( \frac{19}{44}, \frac{25}{44} \right)$.
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

**Case 3.2:** There is a share $\leq \frac{19}{44}$. Duh.
Case 3.1 and 3.2: Too Big or Too Small

**Case 3.1:** There is a share $\geq \frac{25}{44}$. Then its buddy is

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

**Case 3.2:** There is a share $\leq \frac{19}{44}$. Duh.

Henceforth assume that all shares are in

$$\left( \frac{19}{44}, \frac{25}{44} \right)$$

$$\left( \frac{19}{44}, \frac{25}{44} \right)$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Since $v + w + x + y + z = \frac{2411}{44}$ and $z \geq \frac{20}{44}$

$v + w + x + y \leq \frac{2411}{44} - \frac{20}{44} = \frac{76}{44} = \frac{19}{44}$.

Henceforth we assume all 5-shares are in $\frac{19}{44}$, $\frac{20}{44}$.

Recall: there are 5 $\frac{20}{44} = 5 \times 4 = 20$ 5-shares.
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$.
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

**Claim:** If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

**Proof:** Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$. Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

\[
v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}
\]

\[
v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}
\]
Case 3.3: Some 5-shares \( \geq \frac{20}{44} \)

5-share: a share that a 5-student who gets.

**Claim:** If some 5-shares is \( \geq \frac{20}{44} \) then some share \( \leq \frac{19}{44} \).

**Proof:** Assume Alice has \( v \leq w \leq x \leq y \leq z \) and \( z \geq \frac{20}{44} \).

Since \( v + w + x + y + z = \frac{24}{11} \) and \( z \geq \frac{20}{44} \):

\[
v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}
\]

\[
v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}
\]

Henceforth we assume all 5-shares are in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).
Case 3.3: Some 5-shares $\geq \frac{20}{44}$

5-share: a share that a 5-student who gets.

Claim: If some 5-shares is $\geq \frac{20}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $v \leq w \leq x \leq y \leq z$ and $z \geq \frac{20}{44}$.

Since $v + w + x + y + z = \frac{24}{11}$ and $z \geq \frac{20}{44}$

$$v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44}$$

$$v \leq \frac{76}{44} \times \frac{1}{4} = \frac{19}{44}$$

Henceforth we assume all 5-shares are in $\left(\frac{19}{44}, \frac{20}{44}\right)$.

Recall: there are $5s_5 = 5 \times 4 = 20$ 5-shares.
Case 3.3: Some 5-shares \( \geq \frac{20}{44} \)

*5-share*: a share that a 5-student who gets.

**Claim**: If some 5-shares is \( \geq \frac{20}{44} \) then some share \( \leq \frac{19}{44} \).

**Proof**: Assume Alice has \( v \leq w \leq x \leq y \leq z \) and \( z \geq \frac{20}{44} \). Since \( v + w + x + y + z = \frac{24}{11} \) and \( z \geq \frac{20}{44} \):

\[
v + w + x + y \leq \frac{24}{11} - \frac{20}{44} = \frac{76}{44} \]

Henceforth we assume all 5-shares are in \( \left( \frac{19}{44}, \frac{20}{44} \right) \).

**Recall**: there are \( 5s_5 = 5 \times 4 = 20 \) 5-shares.

\[
\begin{pmatrix}
\frac{19}{44} & \frac{20}{44} & \frac{25}{44}
\end{pmatrix}
\begin{pmatrix}
20 \text{ 5-shs}
\end{pmatrix}
\]
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{21}{44}$ and $w \leq \frac{21}{44}$,

$x + y + z \geq \frac{24}{44} - \frac{21}{44} = \frac{75}{44}$.

$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$.

The buddy of $z$ is of size $\leq 1 - \frac{25}{44} = \frac{19}{44}$.

Henceforth we assume all 4-shares are in $[\frac{21}{44}, \frac{25}{44}]$. 

Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = 24\frac{11}{44}$ and $w \leq \frac{21}{44} x + y + z \geq 24\frac{11}{44} - \frac{21}{44} = 75\frac{44}{44}$

$z \geq 75\frac{44}{44} \times \frac{1}{3} = 25\frac{44}{44}$

Henceforth we assume all 4-shares are in $\frac{21}{44}, \frac{25}{44}$. 
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.
Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$. Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$
Case 3.4: Some 4-shares \( \leq \frac{21}{44} \)

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is \( \leq \frac{21}{44} \) then some share \( \leq \frac{19}{44} \).

Proof: Assume Alice has \( w \leq x \leq y \leq z \leq \) and \( w \leq \frac{21}{44} \).

Since \( w + x + y + z = \frac{24}{11} \) and \( w \leq \frac{21}{44} \)

\[
x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}
\]

\[
z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}
\]

The buddy of \( z \) is of size

\[
\leq 1 - \frac{25}{44} = \frac{19}{44}
\]
Case 3.4: Some 4-shares $\leq \frac{21}{44}$

4-share: a share that a 4-student who gets.

Claim: If some 4-shares is $\leq \frac{21}{44}$ then some share $\leq \frac{19}{44}$.

Proof: Assume Alice has $w \leq x \leq y \leq z \leq$ and $w \leq \frac{21}{44}$.

Since $w + x + y + z = \frac{24}{11}$ and $w \leq \frac{21}{44}$

$$x + y + z \geq \frac{24}{11} - \frac{21}{44} = \frac{75}{44}$$

$$z \geq \frac{75}{44} \times \frac{1}{3} = \frac{25}{44}$$

The buddy of $z$ is of size

$$\leq 1 - \frac{25}{44} = \frac{19}{44}$$

Henceforth we assume all 4-shares are in

$$\left( \frac{21}{44}, \frac{25}{44} \right).$$
Case 3.5: All Shares in Their Proper Intervals

Case 3.5: 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).
Case 3.5: 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

Recall: there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

Recall: there are \(5s_5 = 5 \times 4 = 20\) 5-shares.
Case 3.5: All Shares in Their Proper Intervals

**Case 3.5:** 4-shares in \((\frac{21}{44}, \frac{25}{44})\), 5-shares in \((\frac{19}{44}, \frac{20}{44})\).

**Recall:** there are \(4s_4 = 4 \times 7 = 28\) 4-shares.

**Recall:** there are \(5s_5 = 5 \times 4 = 20\) 5-shares.

\[
\begin{pmatrix}
\frac{19}{44} & 20 & 5\text{-shs} \\
\frac{20}{44} & 0 & \text{shs} \\
\frac{21}{44} & 28 & 4\text{-shs}
\end{pmatrix}
\]
More Refined Picture of What is Going On

Claim 1: There are no shares $x \in [23, 24]$. If there was such a share then buddy is in $[20, 21]$. QED.

The following picture captures what we know so far.

\[
\begin{pmatrix}
\frac{19}{44} & 20 & \text{5-shs} & \frac{20}{44} & 0 & \text{shs} & \frac{21}{44} & 28 & \text{4-shs} & \frac{25}{44}
\end{pmatrix}
\]
More Refined Picture of What is Going On

\[
\begin{pmatrix}
20 & 0 \\
\frac{19}{44} & \frac{20}{44}
\end{pmatrix}
\begin{pmatrix}
28 & 0 \\
\frac{21}{44} & \frac{25}{44}
\end{pmatrix}
\]

**Claim 1:** There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).
More Refined Picture of What is Going On

\[
\left( \begin{array}{c}
20 \text{ 5-shs} \\
19 \\
\end{array} \right) \left[ \begin{array}{c}
0 \text{ shs} \\
20 \\
\end{array} \right] \left( \begin{array}{c}
28 \text{ 4-shs} \\
21 \\
\end{array} \right) \left[ \begin{array}{c}
\frac{25}{44} \\
\frac{23}{44} \\
\frac{24}{44} \\
\end{array} \right]
\]

Claim 1: There are no shares \( x \in \left[ \frac{23}{44}, \frac{24}{44} \right] \).

If there was such a share then buddy is in \( \left[ \frac{20}{44}, \frac{21}{44} \right] \). QED.
Claim 1: There are no shares $x \in \left[ \frac{23}{44}, \frac{24}{44} \right]$. If there was such a share then buddy is in $\left[ \frac{20}{44}, \frac{21}{44} \right]$. QED.

The following picture captures what we know so far.
More Refined Picture of What is Going On

\[
\begin{pmatrix}
20 & 5\text{-shs} \\
19 & 44
\end{pmatrix}
\begin{pmatrix}
0 & \text{shs} \\
20 & 44
\end{pmatrix}
\begin{pmatrix}
28 & 4\text{-shs} \\
21 & 44
\end{pmatrix}
\begin{pmatrix}
25 & 44
\end{pmatrix}
\]

Claim 1: There are no shares \( x \in \left[\frac{23}{44}, \frac{24}{44}\right] \).

If there was such a share then buddy is in \( \left[\frac{20}{44}, \frac{21}{44}\right] \). QED.

The following picture captures what we know so far.

\[
\begin{pmatrix}
20 & 5\text{-shs} \\
19 & 44
\end{pmatrix}
\begin{pmatrix}
0 & \\ \\
20 & 44
\end{pmatrix}
\begin{pmatrix}
8 & \text{S4-shs} \\
21 & 44
\end{pmatrix}
\begin{pmatrix}
0 & \\ \\
23 & 44
\end{pmatrix}
\begin{pmatrix}
20 & \text{L4-shs} \\
24 & 44
\end{pmatrix}
\begin{pmatrix}
25 & 44
\end{pmatrix}
\]

S4 = Small 4-shares

L4 = Large 4-shares. L4 shares, 5-share: buddies, so \(|L4| = 20\).
Claim 2: Every 4-student has at least 3 L4 shares. If a 4-student had $\leq 2$ L4 shares then he has $< 2 \times \frac{23}{44} + 2 \times \frac{25}{44} = \frac{24}{11}$.

Contradiction: Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares. But there are only 20.
Claim 2: Every 4-student has at least 3 L4 shares.
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$ 

Contradiction: Each 4-student gets $\geq 3$ L4 shares.
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

\[ < 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}. \]

Contradiction: Each 4-student gets ≥ 3 L4 shares.
There are \( s_4 = 7 \) 4-students.
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had $\leq 2$ L4 shares then he has

$$< 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}.$$ 

Contradiction: Each 4-student gets $\geq 3$ L4 shares. There are $s_4 = 7$ 4-students. Hence there are $\geq 21$ L4-shares.
Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

\[ < 2 \times \left( \frac{23}{44} \right) + 2 \times \left( \frac{25}{44} \right) = \frac{24}{11}. \]

Contradiction: Each 4-student gets ≥ 3 L4 shares.
There are \( s_4 = 7 \) 4-students.
Hence there are ≥ 21 L4-shares. But there are only 20.
GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.
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FC or HALF or INT worked on everything with $s = 3, 4, 5, \ldots, 30$. 
Proof that \( f(24, 11) \leq \frac{19}{44} \) was an example of the INT method.

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Then we found a case where neither FC nor HALF nor INT worked.
GAPS Method

Proof that $f(24, 11) \leq \frac{19}{44}$ was an example of the INT method.

FC or HALF or INT worked on everything with $s = 3, 4, 5, \ldots, 30$.

Then we found a case where neither FC nor HALF nor INT worked.

We found a new method: GAP.
Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$.
Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$. 
Example of GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

We show \( f(31, 19) \leq \frac{54}{133} \).
Assume \((31, 19)\)-procedure with smallest piece \(>\) \(\frac{54}{133}\).
By INT-technique methods obtain:
\( s_3 = 14, s_4 = 5. \)

\[
\begin{pmatrix}
54 & 20 & 4 \text{-shs} \\
55 & 55 & 133
\end{pmatrix}
\begin{pmatrix}
0 & 22 & \text{S3 shs} \\
59 & 59 & 133
\end{pmatrix}
\begin{pmatrix}
74 & 0 & 20 & \text{L3-shs} \\
78 & 78 & 133 & 79 & 133
\end{pmatrix}
\]

We just look at the 3-shares:
Example of GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

We show $f(31, 19) \leq \frac{54}{133}$. Assume $(31, 19)$-procedure with smallest piece $> \frac{54}{133}$. By INT-technique methods obtain:

$s_3 = 14$, $s_4 = 5$.

$$
\begin{pmatrix}
\frac{54}{133} & 20 \text{ 4-shs} & [ & 0 & ] & \begin{pmatrix}
\frac{55}{133} & 22 \text{ S3 shs} & [ & 0 & ] & \begin{pmatrix}
\frac{59}{133} & 20 \text{ L3-shs} & [ & 0 & ] & \frac{74}{133} & \frac{78}{133} & \frac{79}{133}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
$$

We just look at the 3-shares:

$$
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & [ & 0 & ] & \begin{pmatrix}
\frac{74}{133} & 20 \text{ L3-shs} & [ & 0 & ] & \frac{78}{133} & \frac{79}{133}
\end{pmatrix}
\end{pmatrix}
$$
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\[
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 & \frac{74}{133} & 20 \text{ L3-shs} & \frac{79}{133}
\end{pmatrix}
\]
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

\[
\left( \begin{array}{ccc}
\frac{59}{133} & 22 \text{ S3 shs} & 0 \\
\frac{74}{133} & \frac{78}{133} & \frac{79}{133}
\end{array} \right)
\]

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \) \((|J_1| = |J_2|)\)
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \) \((|J_3| = 20)\)
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

\[
\begin{pmatrix}
\frac{59}{133} & 22 & \text{S3 shs} & \frac{74}{133} & 0 & \frac{78}{133} & 20 & \text{L3-shs} & \frac{79}{133}
\end{pmatrix}
\]

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3. $J_3 = \left( \frac{78}{133}, \frac{79}{133} \right)$ ($|J_3| = 20$)

**Notation:** An $e(1, 1, 3)$ students is a student who has a $J_1$-share, a $J_1$-share, and a $J_3$-share.

Generalize to $e(i, j, k)$ easily.
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

$$
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 \\
\frac{74}{133} & 0 & \frac{78}{133} \\
\frac{79}{133} & 20 \text{ L3-shs} & 0
\end{pmatrix}
$$

1. $J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right)$
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Generalize to $e(i, j, k)$ easily.

I"LL STOP THE PROOF HERE.
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

\[
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & 0 \\
\frac{74}{133} & \frac{78}{133} & \frac{79}{133}
\end{pmatrix}
\]

1. $J_1 = (\frac{59}{133}, \frac{66.5}{133})$
2. $J_2 = (\frac{66.5}{133}, \frac{74}{133})$ ($|J_1| = |J_2|$)
3. $J_3 = (\frac{78}{133}, \frac{79}{133})$ ($|J_3| = 20$)

**Notation:** An $e(1, 1, 3)$ students is a student who has a $J_1$-share, a $J_1$-share, and a $J_3$-share. Generalize to $e(i, j, k)$ easily.

I’ll stop the proof here. I’ve made the point that the arguments are complicated.
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

\[
\begin{pmatrix}
\frac{59}{133} & 22 \text{ S3 shs} & \frac{74}{133} & 0 \\
\frac{78}{133} & \frac{79}{133} & 20 \text{ L3-shs}
\end{pmatrix}
\]

1. $J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right)$ ($|J_1| = |J_2|$)
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**Notation:** An $e(1, 1, 3)$ students is a student who has a $J_1$-share, a $J_1$-share, and a $J_3$-share.

Generalize to $e(i, j, k)$ easily.

I”LL STOP THE PROOF HERE. I”VE MADE THE POINT THAT THE ARGUMENTS ARE COMPLICATED. THE SLIDES HAVE THE REST OF THE PROOF, BUT I WILL SKIP THAT.
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) (|J_1| = |J_2|) \)
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) (|J_3| = 20) \)
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \ (|J_1| = |J_2|)$
3. $J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \ (|J_3| = 20)$

1) Only students allowed: $e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3)$. All others have either $\frac{31}{19}$ or $\frac{31}{19}$. 
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

1. $J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right)$
2. $J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right)$ ($|J_1| = |J_2|$)
3. $J_3 = \left( \frac{78}{133}, \frac{79}{133} \right)$ ($|J_3| = 20$)

1) Only students allowed: $e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3)$. All others have either $< \frac{31}{19}$ or $> \frac{31}{19}$.

2) No shares in $\left[ \frac{61}{133}, \frac{64}{133} \right]$. Look at $J_1$-shares:
   An $e(1, 2, 3)$-student has $J_1$-share $> \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133}$.
   An $e(1, 3, 3)$-student has $J_1$-share $< \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133}$. 
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = \left( \frac{59}{133}, \frac{66.5}{133} \right) \)
2. \( J_2 = \left( \frac{66.5}{133}, \frac{74}{133} \right) \) (\(|J_1| = |J_2|\))
3. \( J_3 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\(|J_3| = 20\))

1) Only students allowed: \( e(1, 2, 3), e(1, 3, 3), e(2, 2, 2), e(2, 2, 3) \). All others have either < \( \frac{31}{19} \) or > \( \frac{31}{19} \).

2) No shares in \( \left[ \frac{61}{133}, \frac{64}{133} \right] \). Look at \( J_1 \)-shares:
   An \( e(1, 2, 3) \)-student has \( J_1 \)-share > \( \frac{31}{19} - \frac{74}{133} - \frac{79}{133} = \frac{64}{133} \).
   An \( e(1, 3, 3) \)-student has \( J_1 \)-share < \( \frac{31}{19} - 2 \times \frac{78}{133} = \frac{61}{133} \).

3) No shares in \( \left[ \frac{69}{133}, \frac{72}{133} \right] \): \( x \in \left[ \frac{69}{133}, \frac{72}{133} \right] \implies 1 - x \in \left[ \frac{61}{133}, \frac{64}{133} \right] \).
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = \left( \frac{59}{133}, \frac{61}{133} \right) \)
2. \( J_2 = \left( \frac{64}{133}, \frac{66.5}{133} \right) \)
3. \( J_3 = \left( \frac{66.5}{133}, \frac{69}{133} \right) \) (\(|J_2| = |J_3|\))
4. \( J_4 = \left( \frac{72}{133}, \frac{74}{133} \right) \) (\(|J_1| = |J_4|\))
5. \( J_5 = \left( \frac{78}{133}, \frac{79}{133} \right) \) (\(|J_5| = 20\))
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

1. \( J_1 = (\frac{59}{133}, \frac{61}{133}) \)
2. \( J_2 = (\frac{64}{133}, \frac{66.5}{133}) \)
3. \( J_3 = (\frac{66.5}{133}, \frac{69}{133}) \) \( (|J_2| = |J_3|) \)
4. \( J_4 = (\frac{72}{133}, \frac{74}{133}) \) \( (|J_1| = |J_4|) \)
5. \( J_5 = (\frac{78}{133}, \frac{79}{133}) \) \( (|J_5| = 20) \)

The following are the only students who are allowed.

\( e(1, 5, 5). \)
\( e(2, 4, 5). \)
\( e(3, 4, 5). \)
\( e(4, 4, 4). \)
GAPS Technique: \( f(31, 19) \leq \frac{54}{133} \)

e\((1, 5, 5)\). Let the number of such students be \( x \)
e\((2, 4, 5)\). Let the number of such students be \( y_1 \)
e\((3, 4, 5)\). Let the number of such students be \( y_2 \).
e\((4, 4, 4)\). Let the number of such students be \( z \).
**GAPS Technique:** $f(31, 19) \leq \frac{54}{133}$

$e(1, 5, 5)$. Let the number of such students be $x$.

$e(2, 4, 5)$. Let the number of such students be $y_1$.

$e(3, 4, 5)$. Let the number of such students be $y_2$.

$e(4, 4, 4)$. Let the number of such students be $z$.

1) $|J_2| = |J_3|$, only students using $J_2$ are $e(2, 4, 5)$ – they use one share each, only students using $J_3$ are $e(3, 4, 5)$ – they use one share each. Hence $y_1 = y_2$. We call them both $y$. 
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

e(1, 5, 5). Let the number of such students be $x$
e(2, 4, 5). Let the number of such students be $y_1$
e(3, 4, 5). Let the number of such students be $y_2$.\n$e(4, 4, 4)$. Let the number of such students be $z$.

1) $|J_2| = |J_3|$, only students using $J_2$ are $e(2, 4, 5)$ – they use one share each, only students using $J_3$ are $e(3, 4, 5)$ – they use one share each. Hence $y_1 = y_2$. We call them both $y$.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$. 
GAPS Technique: $f(31, 19) \leq \frac{54}{133}$

e(1, 5, 5). Let the number of such students be $x$
e(2, 4, 5). Let the number of such students be $y_1$
e(3, 4, 5). Let the number of such students be $y_2$.
e(4, 4, 4). Let the number of such students be $z$.
1) $|J_2| = |J_3|$, only students using $J_2$ are $e(2, 4, 5)$ – they use one share each, only students using $J_3$ are $e(3, 4, 5)$ – they use one share each. Hence $y_1 = y_2$. We call them both $y$.

2) Since $|J_1| = |J_4|$, $x = 2y + 3z$.

3) Since $s_3 = 14$, $x + 2y + z = 14$.

$(2y + 3z) + 2y + z = 14 \implies 4(y + z) = 14 \implies y + z = \frac{7}{2}$. Contradiction.
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.
Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$
MATRIX Technique: $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin** = pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors $\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, $m_1$ muffins of this type.
$\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, $m_2$ muffins of this type.
**MATRIX Technique:** $f(5, 3) \geq \frac{5}{12}$

Want proc for $f(5, 3) \geq \frac{5}{12}$.

1) **Guess** that the only piece sizes are $\frac{5}{12}, \frac{6}{12}, \frac{7}{12}$

2) **Muffin** = pieces add to 1: $\{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\}$. Vectors $\{\frac{6}{12}, \frac{6}{12}\}$ is $(0, 2, 0)$, $m_1$ muffins of this type. $\{\frac{5}{12}, \frac{7}{12}\}$ is $(1, 0, 1)$, $m_2$ muffins of this type.

3) **Student** = pieces add to $\frac{5}{3}$
   $\{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\}$ is $(0, 1, 2)$, $s_1$ students of this type. $\{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\}$ is $(4, 0, 0)$, $s_2$ students of this type.
MATRIX Technique:  \( f(5, 3) \geq \frac{5}{12} \)

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \)

2) **Muffin** = pieces add to 1: \( \{\frac{6}{12}, \frac{6}{12}\}, \{\frac{5}{12}, \frac{7}{12}\} \). Vectors \( \{\frac{6}{12}, \frac{6}{12}\} \) is \( (0, 2, 0) \), \( m_1 \) muffins of this type.  
\( \{\frac{5}{12}, \frac{7}{12}\} \) is \( (1, 0, 1) \), \( m_2 \) muffins of this type.

3) **Student** = pieces add to \( \frac{5}{3} \)  
\( \{\frac{6}{12}, \frac{7}{12}, \frac{7}{12}\} \) is \( (0, 1, 2) \), \( s_1 \) students of this type.  
\( \{\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\} \) is \( (4, 0, 0) \), \( s_2 \) students of this type.

4) **Set up equations:**  
\( m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0) \)  
\( m_1 + m_2 = 5 \)  
\( s_1 + s_2 = 3 \)
**MATRIX Technique: \( f(5, 3) \geq \frac{5}{12} \)**

Want proc for \( f(5, 3) \geq \frac{5}{12} \).

1) **Guess** that the only piece sizes are \( \frac{5}{12}, \frac{6}{12}, \frac{7}{12} \).

2) **Muffin**=pieces add to 1: \( \{ \frac{6}{12}, \frac{6}{12} \}, \{ \frac{5}{12}, \frac{7}{12} \} \). Vectors \( \{ \frac{6}{12}, \frac{6}{12} \} \) is \((0, 2, 0)\), \( m_1 \) muffins of this type. \( \{ \frac{5}{12}, \frac{7}{12} \} \) is \((1, 0, 1)\), \( m_2 \) muffins of this type.

3) **Student**=pieces add to \( \frac{5}{3} \). \( \{ \frac{6}{12}, \frac{7}{12}, \frac{7}{12} \} \) is \((0, 1, 2)\), \( s_1 \) students of this type. \( \{ \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12} \} \) is \((4, 0, 0)\), \( s_2 \) students of this type.

4) **Set up equations:**
\[
m_1(0, 2, 0) + m_2(1, 0, 1) = s_1(0, 1, 2) + s_2(4, 0, 0)
\]
\[
m_1 + m_2 = 5
\]
\[
s_1 + s_2 = 3
\]

**Natural Number Solution:** \( m_1 = 1, m_2 = 4, s_1 = 2, s_2 = 1 \)
MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$. 
Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$.
Want proc for \( f(m, s) \geq \frac{a}{b} \).

1) **Guess** that the only piece sizes are \( \frac{a}{b}, \ldots, \frac{b-a}{b} \)

2) **Muffin** = pieces add to 1: Vectors \( \vec{v}_i \). \( x \) types. \( m_i \) muffins of type \( \vec{v}_i \)
MATRIX Technique

Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

2) **Muffin** = pieces add to 1: Vectors $\vec{v}_i$. $x$ types.
   $m_i$ muffins of type $\vec{v}_i$

3) **Student** = pieces add to $\frac{m}{s}$: Vectors $\vec{u}_j$. $y$ types.
   $s_j$ students of type $\vec{u}_j$
Want proc for $f(m, s) \geq \frac{a}{b}$.

1) **Guess** that the only piece sizes are $\frac{a}{b}, \ldots, \frac{b-a}{b}$

2) **Muffin** = pieces add to 1: Vectors $\vec{v}_i$. $x$ types. $m_i$ muffins of type $\vec{v}_i$

3) **Student** = pieces add to $\frac{m}{s}$: Vectors $\vec{u}_j$. $y$ types. $s_j$ students of type $\vec{u}_j$

4) **Set up equations:**
   
   $m_1 \vec{v}_1 + \cdots + m_x \vec{v}_x = s_1 \vec{u}_1 + \cdots + s_y \vec{u}_y$
   
   $m_1 + \cdots + m_x = m$
   
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5) **Look for Nat Numb sol.** If find can translate into procedure.
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First Year Royalties: $\$40.00$. The break-even point!
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