24 Muffins, 25 Students
By William Gasarch, Answering a Question Posed by James Propp

1 James Propp Asked The Following

Problem: A teacher has 24 muffins and 24 students. Great! She will give one to each student. But wait! She then thinks *I want some muffin!* So she will now give 24 muffins to 25 people (the 24 students and her). That sounds like a mess! Find a way to divide and distribute 25 muffins to 24 people so that its not that big a mess. More formally, try to maximize the smallest piece.

Work on it before turning the page.

2 A Procedure with Smallest Piece $\frac{8}{25}$

1. Divide 6 muffins $\{\frac{12}{25}, \frac{13}{25}\}$.
2. Divide 6 muffins $\{\frac{11}{25}, \frac{14}{25}\}$.
3. Divide 6 muffins $\{\frac{10}{25}, \frac{15}{25}\}$.
4. Divide 6 muffins $\{\frac{8}{25}, \frac{8}{25}, \frac{9}{25}\}$.
5. Give 3 students $\{\frac{12}{25}, \frac{12}{25}\}$.
6. Give 6 students $\{\frac{11}{25}, \frac{13}{25}\}$.
7. Give 6 students $\{\frac{10}{25}, \frac{14}{25}\}$.
8. Give 6 students $\{\frac{9}{25}, \frac{15}{25}\}$.
9. Give 4 students $\{\frac{8}{25}, \frac{8}{25}, \frac{8}{25}\}$.

Is there a procedure with smallest piece $> \frac{8}{25}$.
Think about it before turning the page.

3 There is no Procedure with Smallest Piece $> \frac{8}{25}$

Assume there is a procedure for 24 muffins, 25 people where everyone gets $\frac{24}{25}$.
Case 1: Some muffin is cut into ≥ 4 pieces. Then some piece is $\le \frac{1}{4} < \frac{8}{25}$.
Case 2: Some muffin is uncut. Whoever gets that muffin has $1 > \frac{24}{25}$ which is impossible.
Case 3: Some student gets ≥ 3 pieces. One of the pieces must be $\le \frac{24}{25} \times \frac{1}{3} = \frac{8}{25}$.
Case 4: Some student gets 1 piece. That piece must be of size $\frac{24}{25}$. Look at the muffin it came from. The other pieces add up to $1 - \frac{24}{25} = \frac{1}{25} < \frac{8}{25}$. Hence some piece is $< \frac{8}{25}$.
Case 5: All muffins are cut into either 2 or 3 pieces and all students get 2 pieces. We will call muffins cut into 2 pieces *2-muffins* and muffins cut into 3-pieces *3-muffins*. We will call the pieces from a 2-muffin *2-pieces* and the pieces from a 3-muffin *3-pieces*.
Case 5a: Some 3-piece is $> \frac{9}{25}$. Look at the muffin this piece came from. The other 2 pieces of it add up to $< 1 - \frac{9}{25} = \frac{16}{25}$. Hence some piece is $< \frac{9}{25}$.
Case 5b: Some 2-piece is $< \frac{9}{25}$. Look at the muffin this piece came from. The other piece of it is $> 1 - \frac{9}{25} = \frac{16}{25}$. Look at who gets that piece. The rest of what he gets is of size $< \frac{24}{25} - \frac{16}{25} = \frac{8}{25}$.
Case 5c: Some 2-piece is $> \frac{16}{25}$. Look at the person who has that piece. The other piece they have is of size $< \frac{24}{25} - \frac{16}{25} = \frac{8}{25}$.

OKAY, we now got rid of all the easy cases. Now what?

Think about it before turning the page.

The following diagram illustrates what is going on:

$$\begin{array}{ccccc} (& \text{3-pieces} &)(& \text{2-pieces} &) \\ \frac{8}{25} & & \frac{9}{25} & & \frac{16}{25} \end{array}$$

Let m_2 (m_3) be the number of 2-muffins (3-muffins). Since (1) every student gets 2 pieces there are 50 pieces, and (2) there are 24 muffins:

$$\begin{aligned} 2m_2 + 3m_3 &= 50 \\ m_2 + m_3 &= 24 \end{aligned}$$

Hence $m_2 = 22$ and $m_3 = 2$, so there are 44 2-pieces and 6 3-pieces

$$\begin{array}{ccccc} (& \text{6 3-pieces} &)(& \text{44 2-pieces} &) \\ \frac{8}{25} & & \frac{9}{25} & & \frac{16}{25} \end{array}$$

$$\left(\frac{8}{25}, \frac{9}{25}\right) \text{ has 6 pieces.}$$

There are 6 muffins in $(\frac{8}{25}, \frac{9}{25})$. We have a bijection from these 6 pieces to the pieces in $(\frac{15}{25}, \frac{16}{25})$ as follows: if $x \in (\frac{8}{25}, \frac{9}{25})$ then let Alice have x. Alice has only two pieces. Map x to y, the other piece Alice has. Since $x + y = \frac{24}{25}$ and $x \in (\frac{8}{25}, \frac{9}{25})$, $y \in (\frac{15}{25}, \frac{16}{25})$. The map is invertible. Hence

$$\left(\frac{15}{25}, \frac{16}{25}\right) \text{ has 6 pieces.}$$

We have a bijection from the pieces in $(\frac{15}{25}, \frac{16}{25})$ to the pieces in $(\frac{9}{25}, \frac{10}{25})$. Let x be a piece in $(\frac{15}{25}, \frac{16}{25})$. Look at the muffin that x came from. Since x is a 2-piece there is only one other piece from that muffin. Call its size y. Then $x + y = 1$. Since $x \in (\frac{15}{25}, \frac{16}{25})$ and $x + y = 1$, $y \in \frac{9}{25}, \frac{10}{25})$. The map is invertible. Hence

$$\left(\frac{9}{25}, \frac{10}{25}\right) \text{ has 6 pieces.}$$

By similar methods to the two above we can show that the following open intervals have 6 pieces:

$$\left(\frac{14}{25}, \frac{15}{25}\right), \left(\frac{10}{25}, \frac{11}{25}\right) \left(\frac{13}{25}, \frac{14}{25}\right)$$

$$\left(\frac{11}{25}, \frac{12}{25}\right), \left(\frac{12}{25}, \frac{13}{25}\right)$$

There are no pieces of size the endpoint of any of these intervals. Since these 7 intervals are all distinct and have 6 pieces in them, there are 42 pieces. This contradicts that we have 50 pieces. Hence this case cannot occur.

What about s muffins, $s + 1$ students?

Think about this before turning the page.

4 s muffins, $s+1$ students

This breaks down into three cases.

4.1 $f(3k, 3k+1)$

Theorem 4.1 $f(3k, 3k+1) = \frac{k}{3k+1}$

Proof:
$f(3k, 3k+1) \geq \frac{k}{3k+1}$, with two cases.
Case 1: $f(6k, 6k+1) = \frac{2k}{6k+1}$

1. Divide 6 muffins $\{\frac{2k}{6k+1}, \frac{2k}{6k+1}, \frac{2k+1}{6k+1}\}$.
2. For $2 \leq i \leq k$, divide 6 muffins $\{\frac{2k+i}{6k+1}, \frac{4k-i+1}{6k+1}\}$.
3. Give 4 students $\{\frac{2k}{6k+1}, \frac{2k}{6k+1}, \frac{2k}{6k+1}\}$.
4. For $1 \leq i \leq k-1$, give 6 students $\{\frac{2k+i}{6k+1}, \frac{4k-i}{6k+1}\}$.
5. Give 3 students $\{\frac{3k}{6k+1}, \frac{3k}{6k+1}\}$.

Case 2: $f(6k+3, 6k+4) = \frac{2k+1}{6k+4}$

1. Divide 6 muffins $\{\frac{2k+1}{6k+4}, \frac{2k+1}{6k+4}, \frac{2k+2}{6k+4}\}$.
2. For $3 \leq i \leq k+1$, divide 6 muffins $\{\frac{2k+i}{6k+4}, \frac{4k-i+4}{6k+4}\}$.
3. Divide 3 muffins $\{\frac{3k+2}{6k+4}, \frac{3k+2}{6k+4}\}$.
4. Give 4 students $\{\frac{2k+1}{6k+4}, \frac{2k+1}{6k+4}, \frac{2k+1}{6k+4}\}$.
5. For $2 \leq i \leq k+1$, give 6 students $\{\frac{2k+i}{6k+4}, \frac{4k-i+3}{6k+4}\}$.

Assume, by way of contradiction, that there is a procedure for $(3k, 3k+1)$ where the smallest piece is $> \frac{k}{3k+1}$.
Case 1: Alice gets 1 piece; its $\frac{3k}{3k+1}$. Its buddy is $1 - \frac{3k}{3k+1} = \frac{1}{3k+1} \leq \frac{k}{3k+1}$.
Case 2: Alice gets ≥ 3 pieces. Then she has some piece of size $\leq \frac{3k}{3k+1} \times \frac{1}{3} = \frac{k}{3k+1}$.
Case 3: Some muffin is uncut. Say Alice gets that uncut muffin. Then she has $1 > \frac{6k+2}{3k+1}$ which cannot happen.
Case 4: Some muffin is cut into 4 pieces. Then some piece is $\leq \frac{1}{4} < \frac{k}{3k+1}$.
Case 5: Every muffin is cut into 2 or 3 pieces, and everyone gets 2 pieces. Hence there are $6k+2$ pieces.

A *2-muffin* is a muffin that is cut into 2-pieces. Ditto for 3-muffin. A *2-piece* is a piece that comes from a 2-muffin. Ditto for 3-piece. Let m_2 (m_3) be the number of 2-muffins (3-muffins). Note that

$$\begin{aligned} 2m_2 + 3m_3 = &\ 6k+2 \\ m_2 + m_3 = &\ 3k \end{aligned}$$

hence $m_2 = 3k-2$ and $m_3 = 2$.

Case 5a: There is a 3-muffin with pieces $x \le y \le z$ where $z \ge \frac{k+1}{3k+1}$.

$$x + y + z = 1$$

$$x + y = 1 - z \le 1 - \frac{k+1}{3k+1} = \frac{2k}{3k+1}$$

Hence

$$x \le \frac{2k}{3k+1} \times \frac{1}{2} = \frac{k}{3k+1}.$$

Case 5b: There is a 2-muffin with pieces $x \le y$ where $x \le \frac{k+1}{3k+1}$.

$$y = 1 - x \ge 1 - \frac{k+1}{3k+1} = \frac{2k}{3k+1}$$

The student who gets y also gets a piece of size $\frac{3k}{3k+1} - y$. Note that

$$\frac{3k}{3k+1} - y \le \frac{3k}{3k+1} - \frac{2k}{3k+1} = \frac{k}{3k+1}$$

Case 5c: Let

1. $I_1 = (\frac{k}{3k+1}, \frac{k+1}{3k+1})$
2. $I_2 = (\frac{k+1}{3k+1}, \frac{2k}{3k+1})$

All of the 3-pieces are in I_1 and all of the 2-pieces are in I_2. Hence I_1 has $2m_2 = 6k-4$ pieces and I_2 has $3m_3 = 6$ pieces.

We leave the rest of the proof as an exercise but with a hint: Look at

$A_1 = I_1$

The match of I_1, call it A_2. A_2 will be within the 2-pieces. Hence you can look at its buddy.

The buddy of A_2, call it A_3.

The set $A_1, A_2, \ldots$ (stop when you are asked to buddy an interval that has 3-pieces in it, so you cannot) will eventually cover the entire interval with disjoint intervals. Each A_i has 6 pieces. The number of pieces will be a multiple of 6, where as the number of pieces is $6k+2$. This will be the contradiction. ∎

4.2 $f(3k+1,3k+2)=\frac{2k+1}{6k+4}$

Theorem 4.2 $f(3k+1,3k+2)=\frac{2k+1}{6k+4}$

Proof: $f(3k+1,3k+2)\ge\frac{2k+1}{6k+4}$ with two cases.
Case 1: $f(6k+1,6k+2)\ge\frac{4k+1}{12k+4}$.

1. Divide 2 muffins $\{\frac{4k+1}{12k+4},\frac{4k+1}{12k+4},\frac{4k+2}{12k+4}\}$.
2. For $i\equiv 1$, $3\le i\le 2k+1$, divide 4 muffins $\{\frac{4k+i}{12k+4},\frac{8k-i+4}{12k+4}\}$.
3. For $i\equiv 0$, $4\le i\le 2k$, divide 2 muffins $\{\frac{4k+i}{12k+4},\frac{8k-i+4}{12k+4}\}$.
4. Divide 1 muffin $\{\frac{6k+2}{12k+4},\frac{6k+2}{12k+4}\}$.
5. For $i\equiv 1$, $1\le i\le 2k-1$, give 4 students $\{\frac{4k+i}{12k+4},\frac{8k-i+2}{12k+4}\}$.
6. For $i\equiv 0$, $2\le i\le 2k$, give 2 students $\{\frac{4k+i}{12k+4},\frac{8k-i+2}{12k+4}\}$.
7. Give 2 students $\{\frac{6k+1}{12k+4},\frac{6k+1}{12k+4}\}$.

Case 2: $f(6k+4,6k+5)\ge\frac{4k+3}{12k+10}$

1. Divide 2 muffins $\{\frac{4k+3}{12k+10},\frac{4k+3}{12k+10},\frac{4k+4}{12k+10}\}$.
2. For $i\equiv 1$, $5\le i\le 2k+3$, divide 4 muffins $\{\frac{4k+i}{12k+10},\frac{8k+10-i}{12k+10}\}$.
3. For $i\equiv 0$, $6\le i\le 2k+4$, divide 2 muffins $\{\frac{4k+i}{12k+10},\frac{8k+10-i}{12k+10}\}$.
4. Divide 2 muffins $\{\frac{6k+5}{12k+10},\frac{6k+5}{12k+10}\}$.
5. For $i\equiv 1$, $3\le i\le 2k+3$, give 4 students $\{\frac{4k+i}{12k+10},\frac{8k+8-i}{12k+10}\}$.
6. For $i\equiv 0$, $4\le i\le 2k+2$, give 2 students $\{\frac{4k+i}{12k+10},\frac{8k+8-i}{12k+10}\}$.
7. Give 1 student $\{\frac{6k+4}{12k+10},\frac{6k+4}{12k+10}\}$.

Assume, by way of contradiction, that there is a procedure for $(3k+1,3k+2)$ where the smallest piece is $>\frac{2k+1}{6k+4}$. Everyone gets a $\frac{6k+2}{6k+4}$.
Case 1: Alice gets 1 piece; its $\frac{6k+2}{6k+4}$. Its buddy is $1-\frac{6k+2}{6k+4}=\frac{2}{6k+4}\le\frac{2k+1}{6k+4}$.
Case 2: Alice gets ≥ 3 pieces. One of her pieces is $\le\frac{6k+2}{6k+4}\times\frac{1}{3}<\frac{2k+1}{6k+4}$.
Case 3: Some muffin is uncut. Say Alice gets that uncut muffin. Then she has $1>\frac{6k+2}{6k+4}$ which cannot happen.
Case 4: Some muffin is cut into 4 pieces. Then some piece is $\le\frac{1}{4}<\frac{2k+1}{6k+4}$.
Case 5: Every muffin is cut into 2 or 3 pieces and everyone gets 2 pieces. Hence there are $6k+4$ pieces.

We leave the rest to the reader; however, it is similar to the proof of the upper bound in Theorem 4.1.

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4.3 $f(3k+2, 3k+3) = \frac{k+1}{3k+3}$

Theorem 4.3 $f(3k+2, 3k+3) = \frac{k+1}{3k+3}$.

Proof: $f(3k+2, 3k+3) \ge \frac{k+1}{3k+3}$ with 2 cases.
Case 1: $f(6k+2, 6k+3) \ge \frac{2k+1}{6k+3}$.

1. Divide 2 muffins $\{\frac{2k+1}{6k+3}, \frac{2k+1}{6k+3}, \frac{2k+1}{6k+3}\}$.
2. For $2 \le i \le k+1$, divide 6 muffins $\{\frac{2k+i}{6k+3}, \frac{4k-i+3}{6k+3}\}$.
3. For $1 \le i \le k$, give 6 students $\{\frac{2k+i}{6k+3}, \frac{4k-i+2}{6k+3}\}$.
4. Give 3 students $\{\frac{3k+1}{6k+3}, \frac{3k+1}{6k+3}\}$.

Case 2: $f(6k+5, 6k+6) \ge \frac{2k+2}{6k+6}$.
We leave this to the reader. It is similar in spirit to all of the prior cases.

Assume, by way of contradiction, that there is a procedure for $(3k+2, 3k+3)$ where the smallest piece is $> \frac{k+1}{3k+3}$.
Case 1: Alice gets 1 piece; its $\frac{3k+2}{3k+3}$. It's buddy is $1 - \frac{3k+2}{3k+3} = \frac{1}{3k+3} \le \frac{k+1}{3k+3}$.
Case 2: Alice gets ≥ 3 pieces. One of those pieces is of size $\frac{3k+2}{3k+3} \times \frac{1}{3} \le \frac{k+1}{3k+3}$.
Case 3: Some muffin is uncut. Say Alice gets that uncut muffin. Then she has $1 > \frac{6k+2}{3k+3}$ which cannot happen.
Case 4: Some muffin is cut into 4 pieces. Then some piece is $\le \frac{1}{4} < \frac{k+1}{3k+3}$.
Case 5: Every muffin is cut into 2 or 3 pieces, and everyone gets 2 pieces. Hence there are $6k+6$ pieces.

We leave the rest to the reader; however, it is similar to the proof of the upper bound in Theorem 4.1.

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