Chapter 1

For fixed \(s\), For Almost All \(m\),
\[ f(m, s) = \text{FC}(m, s) \]

1.1 Introduction

Recall the Floor-Ceiling Theorem:

Theorem 1.1. Assume that \(m, s \in \mathbb{N}\), \(s < m\), and \(\frac{m}{s} \notin \mathbb{N}\).

Then
\[ f(m, s) \leq \max \left\{ \frac{1}{3}, \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \right\}. \]

We will show the following:

1. For fixed \(s\), for large enough \(m\), \(f(m, s) = \text{FC}(m, s)\).
2. For fixed \(s\) there is a nice formula for \(\text{FC}(m, s)\) that is similar to those in the book for \(f(m, 3)\), \(f(m, 4)\), and \(f(m, 5)\).

Lemma 1.2. If \(m > 2s\) then
\[ \frac{1}{3} < \min \left\{ \frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor} \right\} \]

Proof. 1) We show \(\frac{1}{3} < \frac{m}{s \lceil 2m/s \rceil}\) and \(\frac{1}{3} < 1 - \frac{m}{s \lfloor 2m/s \rfloor}\)

1a) Note that:
\[ \lceil 2m/s \rceil < 2m/s + 1 = \frac{2m + s}{s} \]
Hence we need
\[
\frac{m}{2m + s} > \frac{1}{3}
\]
\[
3m > 2m + s
\]
\[
m > s
\]

1b) We show \( \frac{1}{3} < 1 - \frac{m}{s \lceil 2m/s \rceil} \)

\[
\lceil 2m/s \rceil > 2m/s - 1 = ((2m - s)/s)
\]
\[
\frac{m}{s \lceil 2m/s \rceil} < \frac{m}{2m - s}
\]
So need
\[
\frac{m}{2m - s} < \frac{2}{3}
\]
\[
3m < 4m - 2s
\]
\[
2s < m
\]

Using Lemma ?? and some notation that will come in handy later we restate Theorem ??

Notation 1.3. Let \( V = \lceil \frac{2m}{s} \rceil \).
For fixed $s$, For Almost All $m$, $f(m, s) = FC(m, s)$

**Theorem 1.4.** Let $m, s$ be relatively prime such that $m > 2s$. Note that $V \notin \mathbb{N}$ and hence $\left\lfloor \frac{2m}{s} \right\rfloor = V - 1$. Then

$$f(m, s) \leq \min \left\{ \frac{m}{sV}, 1 - \frac{m}{s(V-1)} \right\}$$

**Notation 1.5.** Henceforth we will assume $m > 2s$ and hence we take:

$$FC(m, s) = \min \left\{ \frac{m}{sV}, 1 - \frac{m}{s(V-1)} \right\}.$$ 

**Note 1.6.** Since our goal is to show $f(m, s) \geq FC(m, s)$, and with $m > 2s$ FC$(m, s) > \frac{1}{3}$, our procedures will never cut a muffin into $\geq 3$ pieces. We can assume every muffin will be cut into 2 pieces.

For the rest of this section:

- $s \geq 3$.
- $m > 2s$ and $m, s$ are relatively prime.
- $V = \left\lceil \frac{2m}{s} \right\rceil$. (Each student will get either $V$ or $V - 1$ pieces.)

Let $s_V, (s_{V-1})$ be how many students get $V (V - 1)$ pieces. Since every muffin is cut into 2 pieces there will be $2m$ total pieces. Hence

$$s_V + s_{V-1} = s$$
$$Vs_V + (V - 1)s_{V-1} = 2m$$

Algebra shows that:

- $s_V = s + 2m - Vs$
- $s_{V-1} = Vs - 2m$

**1.2 Case I: $s_{V-1} > s_V$**

We show that if $s_{V-1} > s_V$ and $m$ is large enough then $f(m, s) = FC(m, s)$. 
Let \( q, r \) be such that \( V s_V = q s_{V-1} + r \) with \( 0 \leq r \leq s_{V-1} - 1 \).

**Lemma 1.7.** If \( m > \frac{s^2 + s}{4} \) and \( s_{V-1} > s_V \), then \( \frac{m}{s_V} \leq 1 - \frac{m}{s(V-1)} \).

**Proof.** By definition, \( s_{V-1} > s_V \implies V s - 2m > s + 2m - V s \), which can be simplified to \( \frac{2m}{s} < V - \frac{1}{2} \). Letting \( \{ \frac{2m}{s} \} = \frac{2m}{s} - \left\lfloor \frac{2m}{s} \right\rfloor \), \( \{ \frac{2m}{s} \} < \frac{1}{2} \) (this follows from the definition of \( V \)). Since \( \{ \frac{2m}{s} \} \) is a fraction with integer numerator and denominator \( s \), it can be at most \( \frac{s-1}{2s} \). We have

\[
m > \frac{s^2 + s}{4} \implies \frac{2m}{s} - 1 > \frac{s-1}{2}
\]

\[
\implies V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor > \frac{s-1}{2}
\]

\[
\implies \frac{s-1}{2s} < \frac{V-1}{2V-1}
\]

\[
\implies \left\{ \frac{2m}{s} \right\} < \frac{V-1}{2V-1}
\]

\[
\implies m \cdot \frac{V - \left\{ \frac{2m}{s} \right\}}{2} + \frac{V-1}{4V-2}
\]

\[
\implies m \cdot \frac{V - \left\{ \frac{2m}{s} \right\}}{2} + \frac{V-1}{4V-2}
\]

\[
\implies m \cdot \left( \frac{1}{V} + \frac{1}{V-1} \right) \leq 1
\]

\[
\implies \frac{m}{sV} \leq 1 - \frac{m}{s(V-1)}
\]

The third implication follows because \( \frac{x}{2x+1} \) is increasing for positive \( x \). We present a procedure that will, if \( m, s \) satisfy conditions to be named later (though they will include the premise of Lemma 1.7) yield \( f(m, s) = FC(m, s) \).

(1) Divide \( V s_V \) muffins

\[
\left\{ \frac{m}{sV}, 1 - \frac{m}{sV} \right\}.
\]
For fixed \( s \), For Almost All \( m \), \( f(m, s) = FC(m, s) \)

(1) Divide \((s_{V-1} - r)r\) muffins

\[
\left\{ \frac{1}{2} - \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right), \frac{1}{2} + \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right) \right\}.
\]

(2) Divide \((s_{V-1} - r)r\) muffins

\[
\frac{1}{2}(s_{V-1}(V - 1 - q - 2r) + 2r^2 - r) = m - V s_V - (s_{V-1} - r)r
\]

muffins

\[
\left\{ \frac{1}{2}, \frac{1}{2} \right\}.
\]

(We will later see that this equality holds and does not need a condition on \( m, s \).)

(3) Divide

\[
\frac{1}{2}(s_{V-1}(V - 1 - q - 2r) + 2r^2 - r) = m - V s_V - (s_{V-1} - r)r
\]

muffins

\[
\left\{ \frac{1}{2}, \frac{1}{2} \right\}.
\]

(We will later see that this equality holds and does not need a condition on \( m, s \).)

(4) Give \( s_V \) students \( \{V : \frac{m}{sV}\} \). (These students have \( \frac{m}{s} \) muffins.)

(5) Give \( s_{V-1} - r \) students

\[
\left\{ q : 1 - \frac{m}{sV}, r : \frac{1}{2} + \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right), V - 1 - q - r : \frac{1}{2} \right\}.
\]

(Need \( V - 1 - q - r \geq 0 \).)

(6) Give \( r \) students

\[
\left\{ q+1 : 1 - \frac{m}{sV}, s_{V-1} - r : \frac{1}{2} - \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right), V - q - 2 - s_{V-1} + r : \frac{1}{2} \right\}.
\]

(Need \( V - q - 2 - s_{V-1} + r \geq 0 \).

Claim 1: \( \frac{1}{2}(s_{V-1}(V - 1 - q - 2r) + 2r^2 - r) = m - V s_V - (s_{V-1} - r)r \).

Proof:

We give two proofs.

Proof 1: A Conceptual Approach

Consider steps 1,2,3 with step 3 dividing

\[
m - V s_V - (s_{V-1} - r)r
\]
muffins into \( \left( \frac{1}{2}, \frac{1}{2} \right) \). Step three creates
\[
2(m - V s_V - (s_{V-1} - r)r)
\]
pieces of size \( \frac{1}{2} \).

Distribute all of the pieces as in steps 4, 5, and 6, except do not distribute the \( \frac{1}{2} \) pieces yet. We can compute that the students in the \( s_{V-1} \) group still need
\[
s_{V-1}(V - 1 - q - 2r) + 2r^2 - r
\]
pieces of muffin, and nobody else needs any more pieces. After step 3, we have cut every muffin into 2 pieces. Thus, we have exactly enough pieces to give \( s_{V-1} \) students \( V - 1 \) pieces and \( s_V \) students \( V \) pieces. We have computed already that we have
\[
2(m - V s_V - (s_{V-1} - r)r)
\]
pieces left to give out, and that the students still need to receive
\[
s_{V-1}(V - 1 - q - 2r) + 2r^2 - r
\]
pieces, so those values must be equal. Dividing by two yields the desired result.

**Proof 2: An Algebraic Approach**

It is clear by algebra that
\[
(V - 1)(s + (V - 1)s - 2m) - 2m + (V)(2m - (V - 1)s) = 0
\]
By definition of \( s_{V-1} \) and \( s_V \),
\[
\implies (V - 1)s_{V-1} - 2m + V s_V = 0
\]
Since \( q s_{V-1} + r = V s_V \),
\[
\implies (V - 1)s_{V-1} - q s_{V-1} - r = 2m - 2V s_V
\]
\[
\implies (V - 1)s_{V-1} - q s_{V-1} - r - 2r s_{V-1} + 2r^2 = 2m - 2V s_V - 2r s_{V-1} + 2r^2
\]
\[
\implies \frac{1}{2}(s_{V-1}(V - 1 - q - 2r) + 2r^2 - r) = m - V s_V - (s_{V-1} - r)r
\]
*End of Proof of Claim 1*

**Claim 2:** Every student gets \( \frac{m}{s} \).
For fixed $s$, For Almost All $m$, $f(m, s) = FC(m, s)$

Proof:
Clearly the $s_V$ students will receive $\frac{m}{s}$ muffins. Thus if we distribute the remaining muffin evenly among the $s_{V-1}$ students, they will each receive $\frac{m}{s}$ muffin also. We may compute

\[
q \left(1 - \frac{m}{sV}\right) + r \left(\frac{1}{2} + \frac{1}{s_{V-1}} \left(\frac{1}{2} - \frac{m}{sV}\right)\right) + \frac{1}{2}(V - 1 - q - r)
- \left((q + 1) \left(1 - \frac{m}{sV}\right) + (s_{V-1} - r) \left(\frac{1}{2} - \frac{1}{s_{V-1}} \left(\frac{1}{2} - \frac{m}{sV}\right)\right)\right)
- \left(\frac{1}{2}(V - 2 - q - s_{V-1} + r)\right)
= \frac{m}{sV} - 1 + \left(\frac{1}{2} - \frac{m}{sV}\right) + \frac{1}{2}
= 0
\]

So each student receives $\frac{m}{s}$.

End of Proof of Claim 2

Lemma 1.8. If $m \geq \frac{s^3 + 2s^2 + s}{2}$ and $s_{V-1} > s_V$, then $V - 1 - q - r \geq 0$ and $V - q - 2 - s + r \geq 0$ are satisfied.

Proof.

\[
s_{V-1} - 1 \geq s_V \text{ and } vs_V = qs_{V-1} + r
\implies V(s_{V-1} - 1) \geq qs_{V-1} + r
\implies V - 1 - q \geq \frac{r + V}{s_{V-1}} - 1
\]
Also,
\[ m \geq \frac{s^3 + 2s^2 + s}{2} \geq \frac{s^3 + s^2 + s}{2} \implies \frac{2m}{s} - 1 \geq s^2 + s \]
\[ \implies V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 + s \]
\[ \implies V - 1 \geq s_{V-1}r + s_{V-1} \]
\[ \implies V - 1 \geq s_{V-1}r + s_{V-1} - r - 1 \]
\[ \implies \frac{r + V}{s_{V-1}} - 1 \geq r \]

The two inequalities give us \[ V - 1 - q - r \geq 0 \]

Also,
\[ m \geq \frac{s^3 + 2s^2 + s}{2} \implies \frac{2m}{s} - 1 \geq s^2 + 2s \]
\[ \implies V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 + 2s \]
\[ \implies V - 1 \geq ss_{V-1} + 2s_{V-1} \]
\[ \implies V - 1 \geq ss_{V-1} + 2s_{V-1} - rs_{V-1} - r - 1 \]
\[ \implies \frac{r + V}{s_{V-1}} - 2 - s + r \geq 0 \]

Thus, \[ V - q - 2 - s + r \geq 0 \] so we are done. \( \square \)

Putting this all together we have the following theorem:

**Theorem 1.9.** If \( s_{V-1} > s_V \) and \( m \geq \frac{s^3 + 2s^2 + s}{2} \) then \( f(m, s) = FC(m, s) \).

### 1.3 Case II: \( s_{V-1} < s_V \)

We show that if \( s_{V-1} < s_V \) and \( m \) is large enough then \( f(m, s) = FC(m, s) \).

Let \( q, r \) be such that \( (V - 1)s_{V-1} = qs_V + r \) with \( 0 \leq r \leq s_V - 1 \).

**Lemma 1.10.** If \( s_{V-1} < s_V \) then \( \frac{m}{s_V} \geq 1 - \frac{m}{s_{V-1}} \).
For fixed $s$, For Almost All $m$, $f(m, s) = FC(m, s)$

**Proof.** In fact, we will prove that $1 - \frac{m}{s(V - 1)} \leq \frac{m}{sV}$ if and only if $(V - 1)s_{V-1} \leq Vs_V$. Since $V - 1 < V$, the lemma follows.

Note that $((V - 1)s_{V-1}) \left(\frac{m}{s(V - 1)}\right) + (Vs_V) \left(\frac{m}{sV}\right) = m = \frac{1}{2}((V - 1)s_{V-1}) + \frac{1}{2}(Vs_V)$. Let $x = \frac{m}{s(V - 1)} - \frac{1}{2}$ and let $y = \frac{1}{2} - \frac{m}{sV}$.

Then we have $((V - 1)s_{V-1}) \left(\frac{1}{2} + x\right) + (Vs_V) \left(\frac{1}{2} - y\right) = \frac{1}{2}((V - 1)s_{V-1}) + \frac{1}{2}(Vs_V)$, so $(x)(((V - 1)s_{V-1}) = (y)(Vs_V)$, so $\frac{x}{y} = \frac{Vs_V}{(V-1)s_{V-1}}$. The lemma follows.

We present a procedure that will, if $m, s$ satisfy conditions to be named later (though they will include the premise of Lemma ??) yield $f(m, s) = FC(m, s)$.

1. Divide $s_{V-1}(V - 1)$ muffins $\{\frac{m}{s(V-1)}, 1 - \frac{m}{s(V-1)}\}$.

2. Divide $(s_V - r)r$ muffins

   \[
   \left\{\frac{1}{2} - \frac{1}{s_V} \left(\frac{1}{2} - \frac{m}{s(V - 1)}\right), \frac{1}{2} + \frac{1}{s_V} \left(\frac{1}{2} - \frac{m}{s(V - 1)}\right)\right\}.
   \]

3. Divide

   \[
   \frac{1}{2}(s_V(V-1-q-2r)+2r^2-s_V+r) = m-s_{V-1}(V-1)-(s_V-r)r
   \]

   muffins

   \[
   \left\{\frac{1}{2}, \frac{1}{2}\right\}.
   \]

4. Give $s_{V-1}$ students $\{V - 1 : \frac{m}{s(V-1)}\}$. (These students have $\frac{m}{s}$ muffins.)

5. Give $s_V - r$ students

   \[
   \left\{q : 1 - \frac{m}{s(V - 1)}, r : \frac{1}{2} + \frac{1}{s_V} \left(\frac{1}{2} - \frac{m}{s(V - 1)}\right), V - q - r : \frac{1}{2}\right\}
   \]

   (Need $V - q - r \geq 0$.)

6. Give $r$ students

   \[
   \left\{q+1 : 1 - \frac{m}{s(V - 1)}, s_V - r : \frac{1}{2} - \frac{1}{s_V} \left(\frac{1}{2} - \frac{m}{s(V - 1)}\right), V - 1 - q - s_V + r : \frac{1}{2}\right\},
   \]

   (Need $V - 1 - q - s_V + r \geq 0$.)
Claims 1 and 2 below have proofs very similar to Claims 1 and 2 in Section ??.

**Claim 1:** \( \frac{1}{2}(s_V(V - 1 - q - 2r) + 2r^2 - s_V + r) = m - s_{V-1}(V - 1) - (s_V - r)r \) is identical.

**Claim 2:** Every student gets \( \frac{m}{s} \).

**Theorem 1.11.** If \( m \geq \frac{s^3 + s}{2} \) and \( s_{V-1} < s_V \), then \( V - q - r \geq 0 \) and \( V - 1 - q - s_V + r \geq 0 \) are satisfied.

**Proof.** From Lemma ??, we know that \( s_{V-1} < s_V \) implies Case 2. \( s_{V-1} < s_V \) and \( (V - 1)s_{V-1} = qs_V + r \)

\[ \Rightarrow (V - 1)(s_V - 1) \geq qs_V + r \]

\[ \Rightarrow V - 1 - q \geq \frac{V - 1 + r}{s_V} \]

Also,

\[ m \geq \frac{s^3 + s}{2} \Rightarrow \frac{2m}{s} - 1 \geq s^2 \]

\[ \Rightarrow V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \]

\[ \Rightarrow V - 1 \geq rs_V \]

\[ \Rightarrow V - 1 \geq rs_V - s_V - r \]

\[ \Rightarrow \frac{V - 1 + r}{s_V} \geq r - 1 \]

Thus, \( V - q - r \geq 0 \)

Also,

\[ m \geq \frac{s^3 + s}{2} \Rightarrow \frac{2m}{s} - 1 \geq s^2 \]

\[ \Rightarrow V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \]

\[ \Rightarrow V - 1 \geq (s_V)^2 \]

\[ \Rightarrow V - 1 \geq (s_V)^2 - rs_V - r \]

\[ \Rightarrow \frac{V - 1 + r}{s_V} \geq s_V - r \]
For fixed $s$, For Almost All $m$, $f(m, s) = FC(m, s)$

The two inequalities give us $V - q - s_{V+1} + r \geq 0$ so we are done. $\Box$

**Theorem 1.12.** If $s_{V-1} < s_V$ and $m \geq \frac{s^3 + s}{2}$ then $f(m, s) = FC(m, s)$.

### 1.4 $s_{V-1} = s_V$

**Lemma 1.13.** $s_{V-1} = s_V \implies s = 4$.

**Proof.** Assume $s_{V-1} = s_V$. Then $s + (V - 1)s - 2m = 2m - (V - 1)s$ so:

$$s + 2(V - 1)s = 4m \implies \frac{2m}{s} = V - \frac{1}{2}$$

So $\left\{\frac{2m}{s}\right\} = \frac{1}{2}$. But since $2m$ is even, $s$ must be a multiple of 4. Letting $s = 4k$, $2m = 4k(V - \frac{1}{2}) = 2k(2V - 1)$ so $m = k(2V - 1)$. Therefore, $(m, s)$ is of the form $(k[2V - 1], 4k)$, and $m, s$ relatively prime implies that $k = 1$ and $s = 4$, which we have solved in the book. $\Box$

### 1.5 For almost all $m$, $f(m, s) = FC(m, s)$ and Has a Nice Form

Recall that we are assuming:

- $s \geq 3$.
- $m > 2s$ and $m, s$ are relatively prime.
- $V = \lceil \frac{2m}{s} \rceil$. (Each student will get either $V$ or $V - 1$ pieces.)

Combining these assumptions with Theorem’s ?? and ?? we get:

**Theorem 1.14.** If $s \geq 3$, $m, s$ are relatively prime, and $m \geq \frac{s^3 + 2s^2 + s}{2}$ then $f(m, s) = FC(m, s)$. 
For large $m$, FC($m, s$) has a very nice form.

**Theorem 1.15.** Let $s \geq 3$.

1. There exists $\{a_i\}_{i=0}^{s-1}$, $\{b_i\}_{i=0}^{s-1}$, $\{c_i\}_{i=0}^{s-1}$, $\{d_i\}_{i=0}^{s-1}$ such that, for all $m \geq \frac{s^2 + s}{4}$ if $m = ks + i$ with $0 \leq i \leq s - 1$ then
   \[
   \text{FC}(m, s) = \frac{a_i k + b_i}{c_i k + d_i}
   \]

2. For all $m \geq \frac{s^3 + 2s^2 + s}{2}$ $f(m, s)$ follows the formula in Part 1. (this follows from Part 1 and Theorem ??).

3. Fix $s$. Then $f(m, s)$ can be computed in $O(s^3 M(L))$ time where $L$ is the length of $\lfloor m/s \rfloor$ and $M(L)$ is the time to multiply two $L$-bit numbers. Hence $f(m, s)$ is fixed parameter tractable. (By Part 1 $f(m, s)$ can be computed with a mod, 2 multiplications by constants, 2 additions, 1 division, with all number of magnitude $O(m/s)$. The Newton-Raphson division algorithm takes $O(M(L))$ time.

**Proof.** Given $m \geq \frac{s^2 + s}{4}$ Lemma ?? and Lemma ?? show which of $\{ \frac{m}{s^2}, 1 - \frac{m}{s(V - 1)} \}$ is smaller. It is easy to see whether $\{ \frac{2m}{s} \} < \frac{1}{2}$, or whether equivalently $sV - 1 > sV$ (see proof of Lemma 1.6), for each $i$, and substituting $m = ks + i$ gives the following result:

**Case 1:** $1 \leq i \leq \lceil \frac{s}{4} \rceil - 1$. FC($m, s$) = $\frac{sk + i}{2sk + s}$.

**Case 2:** $\lceil \frac{s}{4} \rceil \leq i \leq \lceil \frac{s}{2} \rceil - 1$. FC($m, s$) = $\frac{sk - i}{2sk}$.

**Case 3:** $\lceil \frac{s}{2} \rceil \leq i \leq \lceil \frac{s}{V} \rceil - 1$. FC($m, s$) = $\frac{sk + i}{2sk + 2s}$.

**Case 4:** $\lceil \frac{3s}{4} \rceil \leq i \leq s - 1$. FC($m, s$) = $\frac{sk + s - i}{2sk + s}$. \hfill $\square$