Chapter 1

For fixed s, For Almost All m, $f(m,s) = \mathrm{FC}(m,s)$

1.1 Introduction

Recall the Floor-Ceiling Theorem:

Theorem 1.1. *Assume that* $m, s \in \mathbb{N}$, $s < m$, *and* $\frac{m}{s} \notin \mathbb{N}$. *Then*

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

We will show the following:

(1) For fixed s, for large enough m, $f(m,s) = \mathrm{FC}(m,s)$.
(2) For fixed s there is a nice formula for $\mathrm{FC}(m,s)$ that is similar to those in the book for $f(m,3)$, $f(m,4)$, and $f(m,5)$.

Lemma 1.2. *If* $m > 2s$ *then*

$$\frac{1}{3} < \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}$$

Proof. 1) We show $\frac{1}{3} < \frac{m}{s\lceil 2m/s\rceil}$ and $\frac{1}{3} < 1 - \frac{m}{s\lfloor 2m/s\rfloor}$
1a)

Note that:

$$\lceil 2m/s\rceil < 2m/s + 1 = \frac{2m+s}{s}$$

$$\frac{m}{s\lceil 2m/s\rceil} > \frac{m}{2m+s}$$

Hence we need

$$\frac{m}{2m+s} > \frac{1}{3}$$

$$3m > 2m+s$$

$$m > s$$

1b) We show $\frac{1}{3} < 1 - \frac{m}{s\lfloor 2m/s\rfloor}$

$$\lfloor 2m/s\rfloor > 2m/s - 1 = ((2m-s)/s)$$

$$\frac{m}{s\lfloor 2m/s\rfloor} < \frac{m}{2m-s}$$

So need

$$\frac{m}{2m-s} < \frac{2}{3}$$

$$3m < 4m - 2s$$

$$2s < m$$

□

Using Lemma 1.2 and some notation that will come in handy later we restate Theorem 1.1

Notation 1.3. Let $V = \lceil \frac{2m}{s} \rceil$.

Theorem 1.4. *Let m, s be relatively prime such that $m > 2s$. Note that $V \notin \mathbb{N}$ and hence $\lfloor \frac{2m}{s} \rfloor = V - 1$. Then*

$$f(m,s) \le \min\left\{ \frac{m}{sV}, 1 - \frac{m}{s(V-1)} \right\}$$

Notation 1.5. Henceforth we will assume $m > 2s$ and hence we take:

$$\mathrm{FC}(m,s) = \min\left\{ \frac{m}{sV}, 1 - \frac{m}{s(V-1)} \right\}.$$

Note 1.6. Since our goal is to show $f(m,s) \ge \mathrm{FC}(m,s)$, and with $m > 2s$ $\mathrm{FC}(m,s) > \frac{1}{3}$, our procedures will never cut a muffin into ≥ 3 pieces. We can assume every muffin will be cut into 2 pieces.

For the rest of this section:

- $s \ge 3$.
- $m > 2s$ and m, s are relatively prime.
- $V = \lceil \frac{2m}{s} \rceil$. (Each student will get either V or $V - 1$ pieces.)

Let s_V (s_{V-1}) be how many students get V ($V - 1$) pieces. Since every muffin is cut into 2 pieces there will be $2m$ total pieces. Hence

$$\begin{aligned} s_V + s_{V-1} &= s \\ V s_V + (V-1) s_{V-1} &= 2m \end{aligned}$$

Algebra shows that:

- $s_V = s + 2m - Vs$
- $s_{V-1} = Vs - 2m$

1.2 Case I: $s_{V-1} > s_V$

We show that if $s_{V-1} > s_V$ and m is large enough then $f(m,s) = \mathrm{FC}(m,s)$.

Let q, r be such that $Vs_V = qs_{V-1} + r$ with $0 \le r \le s_{V-1} - 1$.

Lemma 1.7. *If* $m > \frac{s^2+s}{4}$ *and* $s_{V-1} > s_V$, *then* $\frac{m}{sV} \le 1 - \frac{m}{s(V-1)}$.

Proof. By definition, $s_{V-1} > s_V \implies Vs - 2m > s + 2m - Vs$, which can be simplified to $\frac{2m}{s} < V - \frac{1}{2}$. Letting $\left\{\frac{2m}{s}\right\} = \frac{2m}{s} - \left\lfloor\frac{2m}{s}\right\rfloor$, $\left\{\frac{2m}{s}\right\} < \frac{1}{2}$ (this follows from the definition of V). Since $\left\{\frac{2m}{s}\right\}$ is a fraction with integer numerator and denominator s, it can be at most $\frac{s-1}{2s}$. We have

$$\begin{aligned}
m > \frac{s^2+s}{4} &\implies \frac{2m}{s} - 1 > \frac{s-1}{2} \\
&\implies V - 1 = \left\lfloor \frac{2m}{s} \right\rfloor > \frac{s-1}{2} \\
&\implies \frac{s-1}{2s} < \frac{V-1}{2V-1} \\
&\implies \left\{\frac{2m}{s}\right\} < \frac{V-1}{2V-1} \\
&\implies \frac{m}{s} < \frac{\frac{2m}{s} - \left\{\frac{2m}{s}\right\}}{2} + \frac{V-1}{4V-2} \\
&\implies \frac{m}{s} < \frac{V-1}{2} + \frac{V-1}{4V-2} \\
&\implies \frac{m}{s}\left(\frac{1}{V} + \frac{1}{V-1}\right) \le 1 \\
&\implies \boxed{\frac{m}{sV} \le 1 - \frac{m}{s(V-1)}}
\end{aligned}$$

□

The third implication follows because $\frac{x}{2x+1}$ is increasing for positive x. We present a procedure that will, if m, s satisfy conditions to be named later (though they will include the premise of Lemma 1.7) yield $f(m, s) = \mathrm{FC}(m, s)$.

(1) Divide Vs_V muffins

$$\left\{\frac{m}{sV}, 1 - \frac{m}{sV}\right\}.$$

(Need $\frac{m}{sV} \le 1 - \frac{m}{s(V-1)}$.)

(2) Divide $(s_{V-1} - r)r$ muffins

$$\left\{ \frac{1}{2} - \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right), \frac{1}{2} + \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right) \right\}.$$

(3) Divide

$\frac{1}{2}(s_{V-1}(V-1-q-2r) + 2r^2 - r) = m - Vs_V - (s_{V-1} - r)r$

muffins

$$\left\{ \frac{1}{2}, \frac{1}{2} \right\}.$$

(We will later see that this equality holds and does not need a condition on m, s.)

(4) Give s_V students $\{V : \frac{m}{sV}\}$. (These students have $\frac{m}{s}$ muffins.)

(5) Give $s_{V-1} - r$ students

$$\left\{ q : 1 - \frac{m}{sV}, r : \frac{1}{2} + \frac{1}{s_{V-1}}\left(\frac{1}{2} - \frac{m}{sV}\right), V - 1 - q - r : \frac{1}{2} \right\}.$$

(Need $V - 1 - q - r \ge 0$.)

(6) Give r students

$$\left\{ q+1 : 1-\frac{m}{sV}, s_{V-1}-r : \frac{1}{2}-\frac{1}{s_{V-1}}\left(\frac{1}{2}-\frac{m}{sV}\right), V-q-2-s_{V-1}+r : \frac{1}{2} \right\}.$$

(Need $V - q - 2 - s_{V-1} + r \ge 0$.)

Claim 1: $\frac{1}{2}(s_{V-1}(V-1-q-2r)+2r^2-r) = m - Vs_V - (s_{V-1} - r)r$.

Proof:

We give two proofs.

Proof 1: A Conceptual Approach

Consider steps 1,2,3 with step 3 dividing

$$m - Vs_V - (s_{V-1} - r)r$$

muffins into $(\frac{1}{2}, \frac{1}{2})$. Step three creates

$$2(m - Vs_V - (s_{V-1} - r)r)$$

pieces of size $\frac{1}{2}$.

Distribute all of the pieces as in steps 4, 5, and 6, except do not distribute the $\frac{1}{2}$ pieces yet. We can compute that the students in the s_{V-1} group still need

$$s_{V-1}(V - 1 - q - 2r) + 2r^2 - r$$

pieces of muffin, and nobody else needs any more pieces. After step 3, we have cut every muffin into 2 pieces. Thus, we have exactly enough pieces to give s_{V-1} students $V - 1$ pieces and s_V students V pieces. We have computed already that we have $2(m - Vs_V - (s_{V-1} - r)r)$ pieces left to give out, and that the students still need to receive

$$s_{V-1}(V - 1 - q - 2r) + 2r^2 - r$$

pieces, so those values must be equal. Dividing by two yields the desired result.

Proof 2: An Algebraic Approach

It is clear by algebra that

$$(V - 1)(s + (V - 1)s - 2m) - 2m + (V)(2m - (V - 1)s) = 0$$

By definition of s_{V-1} and s_V,

$$\implies (V - 1)s_{V-1} - 2m + Vs_V = 0$$

Since $qs_{V-1} + r = Vs_V$,

$$\implies (V - 1)s_{V-1} - qs_{V-1} - r = 2m - 2Vs_V$$

$$\implies (V-1)s_{V-1} - qs_{V-1} - r - 2rs_{V-1} + 2r^2 = 2m - 2Vs_V - 2rs_{V-1} + 2r^2$$

$$\implies \frac{1}{2}(s_{V-1}(V - 1 - q - 2r) + 2r^2 - r) = m - Vs_V - (s_{V-1} - r)r$$

End of Proof of Claim 1

Claim 2: Every student gets $\frac{m}{s}$.

Proof:

Clearly the s_V students will receive $\frac{m}{s}$ muffins. Thus if we distribute the remaining muffin evenly among the s_{V-1} students, they will each receive $\frac{m}{s}$ muffin also. We may compute

$$\begin{aligned}
&q\left(1-\frac{m}{sV}\right)+r\left(\frac{1}{2}+\frac{1}{s_{V-1}}\left(\frac{1}{2}-\frac{m}{sV}\right)\right)+\frac{1}{2}(V-1-q-r)\\
&-\left((q+1)\left(1-\frac{m}{sV}\right)+(s_{V-1}-r)\left(\frac{1}{2}-\frac{1}{s_{V-1}}\left(\frac{1}{2}-\frac{m}{sV}\right)\right)\right)\\
&-\left(\frac{1}{2}(V-2-q-s_{V-1}+r)\right)\\
&=\frac{m}{sV}-1+\left(\frac{1}{2}-\frac{m}{sV}\right)+\frac{1}{2}\\
&=0
\end{aligned}$$

So each student receives $\frac{m}{s}$.
End of Proof of Claim 2

Lemma 1.8. *If $m \ge \frac{s^3+2s^2+s}{2}$ and $s_{V-1} > s_V$, then $V-1-q-r \ge 0$ and $V-q-2-s+r \ge 0$ are satisfied.*

Proof.

$$\begin{aligned}
&s_{V-1}-1 \ge s_V \text{ and } Vs_V = qs_{V-1}+r\\
&\implies V(s_{V-1}-1) \ge qs_{V-1}+r\\
&\implies V-1-q \ge \frac{r+V}{s_{V-1}}-1
\end{aligned}$$

Also,

$$\begin{aligned} m \geq \frac{s^3+2s^2+s}{2} \geq \frac{s^3+s^2+s}{2} &\implies \frac{2m}{s} - 1 \geq s^2+s \\ &\implies V-1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2+s \\ &\implies V-1 \geq s_{V-1}r + s_{V-1} \\ &\implies V-1 \geq s_{V-1}r + s_{V-1} - r - 1 \\ &\implies \frac{r+V}{s_{V-1}} - 1 \geq r \end{aligned}$$

The two inequalities give us $\boxed{V-1-q-r \geq 0}$

Also,

$$\begin{aligned} m \geq \frac{s^3+2s^2+s}{2} &\implies \frac{2m}{s} - 1 \geq s^2+2s \\ &\implies V-1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2+2s \\ &\implies V-1 \geq s s_{V-1} + 2s_{V-1} \\ &\implies V-1 \geq s s_{V-1} + 2s_{V-1} - r s_{V-1} - r - 1 \\ &\implies \frac{r+V}{s_{V-1}} - 2 - s + r \geq 0 \end{aligned}$$

Thus, $\boxed{V-q-2-s+r \geq 0}$ so we are done. □

Putting this all together we have the following theorem:

Theorem 1.9. *If* $s_{V-1} > s_V$ *and* $m \geq \frac{s^3+2s^2+s}{2}$ *then* $f(m,s) = \mathrm{FC}(m,s)$.

1.3 Case II: $s_{V-1} < s_V$

We show that if $s_{V-1} < s_V$ and m is large enough then $f(m,s) = \mathrm{FC}(m,s)$.

Let q,r be such that $(V-1)s_{V-1} = qs_V + r$ with $0 \leq r \leq s_V - 1$.

Lemma 1.10. *If* $s_{V-1} < s_V$ *then* $\frac{m}{sV} \geq 1 - \frac{m}{s(V-1)}$.

Proof. In fact, we will prove that $1 - \frac{m}{s(V-1)} \le \frac{m}{sV}$ if and only if $(V-1)s_{V-1} \le Vs_V$. Since $V-1 < V$, the lemma follows. Note that $((V-1)s_{V-1})\left(\frac{m}{s(V-1)}\right) + (Vs_V)\left(\frac{m}{sV}\right) = m = \frac{1}{2}((V-1)s_{V-1}) + \frac{1}{2}(Vs_V)$. Let $x = \frac{m}{s(V-1)} - \frac{1}{2}$ and let $y = \frac{1}{2} - \frac{m}{sV}$. Then we have $((V-1)s_{V-1})\left(\frac{1}{2}+x\right) + (Vs_V)\left(\frac{1}{2}-y\right) = \frac{1}{2}((V-1)s_{V-1}) + \frac{1}{2}(Vs_V)$, so $(x)((V-1)s_{V-1}) = (y)(Vs_V)$, so $\frac{x}{y} = \frac{Vs_V}{(V-1)s_{V-1}}$. The lemma follows. □

We present a procedure that will, if m, s satisfy conditions to be named later (though they will include the premise of Lemma 1.10) yield $f(m,s) = \mathrm{FC}(m,s)$.

(1) Divide $s_{V-1}(V-1)$ muffins $\{\frac{m}{s(V-1)}, 1 - \frac{m}{s(V-1)}\}$.

(2) Divide $(s_V - r)r$ muffins
$$\left\{\frac{1}{2} - \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V-1)}\right), \frac{1}{2} + \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V-1)}\right)\right\}.$$

(3) Divide
$\frac{1}{2}(s_V(V-1-q-2r)+2r^2-s_V+r) = m-s_{V-1}(V-1)-(s_V-r)r$
muffins
$$\left\{\frac{1}{2}, \frac{1}{2}\right\}.$$

(4) Give s_{V-1} students $\{V-1 : \frac{m}{s(V-1)}\}$. (These students have $\frac{m}{s}$ muffins.)

(5) Give $s_V - r$ students
$$\left\{q : 1 - \frac{m}{s(V-1)}, r : \frac{1}{2} + \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V-1)}\right), V-q-r : \frac{1}{2}\right\}$$
(Need $V - q - r \ge 0$.)

(6) Give r students
$$\left\{q+1 : 1 - \frac{m}{s(V-1)}, s_V - r : \frac{1}{2} - \frac{1}{s_V}\left(\frac{1}{2} - \frac{m}{s(V-1)}\right), V-1-q-s_V+r : \frac{1}{2}\right\}.$$
(Need $V - 1 - q - s_V + r \ge 0$.)

Claims 1 and 2 below have proofs very similar to Claims 1 and 2 in Section 1.2.

Claim 1: $\frac{1}{2}(s_V(V-1-q-2r)+2r^2-s_V+r) = m - s_{V-1}(V-1)-(s_V-r)r$ is identical.

Claim 2: Every student gets $\frac{m}{s}$.

Theorem 1.11. *If $m \geq \frac{s^3+s}{2}$ and $s_{V-1} < s_V$, then $V-q-r \geq 0$ and $V-1-q-s_V+r \geq 0$ are satisfied.*

Proof. From Lemma 1.10, we know that $s_{V-1} < s_V$ implies Case 2. $s_{V-1} < s_V$ and $(V-1)s_{V-1} = qs_V + r$

$$\implies (V-1)(s_V-1) \geq qs_V + r$$
$$\implies V-1-q \geq \frac{V-1+r}{s_V}$$

Also,

$$\begin{aligned} m \geq \frac{s^3+s}{2} &\implies \frac{2m}{s} - 1 \geq s^2 \\ &\implies V-1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \\ &\implies V-1 \geq rs_V \\ &\implies V-1 \geq rs_V - s_V - r \\ &\implies \frac{V-1+r}{s_V} \geq r-1 \end{aligned}$$

Thus, $\boxed{V-q-r \geq 0}$

Also,

$$\begin{aligned} m \geq \frac{s^3+s}{2} &\implies \frac{2m}{s} - 1 \geq s^2 \\ &\implies V-1 = \left\lfloor \frac{2m}{s} \right\rfloor \geq s^2 \\ &\implies V-1 \geq (s_V)^2 \\ &\implies V-1 \geq (s_V)^2 - rs_V - r \\ &\implies \frac{V-1+r}{s_V} \geq s_V - r \end{aligned}$$

The two inequalities give us $\boxed{V - q - s_{V+1} + r \geq 0}$ so we are done. □

Theorem 1.12. *If* $s_{V-1} < s_V$ *and* $m \geq \frac{s^3+s}{2}$ *then* $f(m,s) = \mathrm{FC}(m,s)$.

1.4 $s_{V-1} = s_V$

Lemma 1.13. $s_{V-1} = s_V \implies s = 4$.

Proof. Assume $s_{V-1} = s_V$. Then $s + (V-1)s - 2m = 2m - (V-1)s$ so:

$$s + 2(V-1)s = 4m \implies \frac{2m}{s} = V - \frac{1}{2}$$

So $\{\frac{2m}{s}\} = \frac{1}{2}$. But since $2m$ is even, s must be a multiple of 4. Letting $s = 4k$, $2m = 4k\left(V - \frac{1}{2}\right) = 2k(2V-1)$ so $m = k(2V-1)$. Therefore, (m,s) is of the form $(k[2V-1], 4k)$, and m, s relatively prime implies that $k = 1$ and $s = 4$, which we have solved in the book. □

1.5 For almost all m, $f(m,s) = \mathrm{FC}(m,s)$ and Has a Nice Form

Recall that we are assuming:

- $s \geq 3$.
- $m > 2s$ and m, s are relatively prime.
- $V = \lceil \frac{2m}{s} \rceil$. (Each student will get either V or $V-1$ pieces.)

Combining these assumptions with Theorem's 1.9 and 1.12 we get:

Theorem 1.14. *If* $s \geq 3$, m, s *are relatively prime, and* $m \geq \frac{s^3+2s^2+s}{2}$ *then* $f(m,s) = \mathrm{FC}(m,s)$.

For large m, $\mathrm{FC}(m,s)$ has a very nice form.

Theorem 1.15. *Let $s \ge 3$.*

(1) *There exists $\{a_i\}_{i=0}^{s-1}$, $\{b_i\}_{i=0}^{s-1}$, $\{c_i\}_{i=0}^{s-1}$, $\{d_i\}_{i=0}^{s-1}$ such that, for all $m \ge \frac{s^2+s}{4}$ if $m = ks+i$ with $0 \le i \le s-1$ then*

$$\mathrm{FC}(m,s) = \frac{a_i k + b_i}{c_i k + d_i}$$

(2) *For all $m \ge \frac{s^3+2s^2+s}{2}$ $f(m,s)$ follows the formula in Part 1. (this follows from Part 1 and Theorem 1.14).*

(3) *Fix s. Then $f(m,s)$ can be computed in $O(s^3 M(L))$ time where L is the length of $\lfloor m/s \rfloor$ and $M(L)$ is the time to multiply two L-bit numbers. Hence $f(m,s)$ is fixed parameter tractable. (By Part 1 $f(m,s)$ can be computed with a mod, 2 multiplications by constants, 2 additions, 1 division, with all number of magnitude $O(m/s)$. The Newton-Raphson division algorithm takes $O(M(L))$ time.*

Proof. Given $m \ge \frac{s^2+s}{4}$ Lemma 1.7 and Lemma 1.10 show which of $\{\frac{m}{sV}, 1 - \frac{m}{s(V-1)}\}$ is smaller. It is easy to see whether $\{\frac{2m}{s}\} < \frac{1}{2}$, or whether equivalently $s_{V-1} > s_V$ (see proof of Lemma 1.6), for each i, and substituting $m = ks+i$ gives the following result:

Case 1: $1 \le i \le \lceil \frac{s}{4} \rceil - 1$. $\mathrm{FC}(m,s) = \frac{sk+i}{2sk+s}$.
Case 2: $\lceil \frac{s}{4} \rceil \le i \le \lceil \frac{s}{2} \rceil - 1$. $\mathrm{FC}(m,s) = \frac{sk-i}{2sk}$.
Case 3: $\lceil \frac{s}{2} \rceil \le i \le \lceil \frac{3s}{4} \rceil - 1$. $\mathrm{FC}(m,s) = \frac{sk+i}{2sk+2s}$.
Case 4: $\lceil \frac{3s}{4} \rceil \le i \le s-1$. $\mathrm{FC}(m,s) = \frac{sk+s-i}{2sk+s}$. □