ANY TWO-COLORING OF THE PLANE CONTAINS MONOCHROMATIC 3-TERM ARITHMETIC PROGRESSIONS

GABRIEL CURRIER, KENNETH MOORE, CHI HOI YIP

Presenter: Chaewoon Kyoung

Notation

$\mathbb{E}^n$ = n-dimensional Euclidean space

ℓ_m = m collinear points with distance one apart

I.e. m-term arithmetic progression with common difference of 1

Theorem 1.1.

In any two-coloring of $\mathbb{E}^2$,

there exists a monochromatic congruent copy of ℓ_3.

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By scaling, there exists a monochromatic 3-term arithmetic progression

with any common difference.

Theorem 1.2. (Erdős et. al.)

Let $n \geq 2$. A given 2-coloring of $\mathbb{E}^n$ admits a monochromatic (a, b, c) triangle if and only if it admits a monochromatic equilateral triangle of side a, b, or c.

Theorem 1.2. (Erdős et. al.)

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An L3 is a (1, 1, 2) monochromatic (degenerate) triangle.

Thus, if there is no L3 in a given coloring of $\mathbb{E}^n$,

there are no monochromatic triangle of side length 1 or 2 in the same coloring

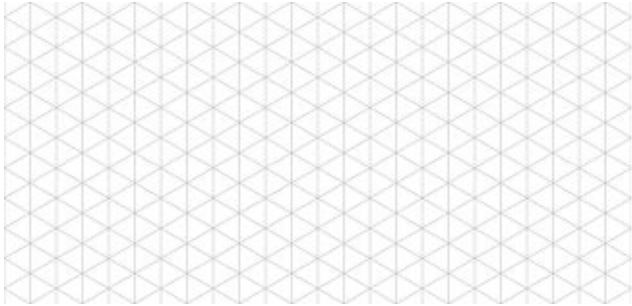
Lemmas we will use

Theorem 1.2. (Erdős et. al.) Will Not Prove

Lemma 2.1. If E^2 is 2-colored without a monochromatic L_3, any unit eq. triangle colored R-R-B has B centroid Will Prove

Lemma 2.2. If E^2 is 2-colored without a monochromatic L_3, a $\frac{1}{\sqrt{3}}$ scaled

Hexagonal grid has one valid coloring Will Prove



Proof for Lemma 2.1:

If E2 is 2-colored without a monochromatic L3, any unit eq. triangle colored R-R-B has B centroid

For the Sake of Contradiction…

Assume there is no monochromatic L_3 in an arbitrary 2-coloring of E^2.

Construct a point set that includes many L_3s:

[a,b,c,d]:= $\left(\frac{a\sqrt{3}+b\sqrt{11}}{12}, \frac{c+d\sqrt{33}}{12}\right)$,

Let X={ [a,b,c,d] : a,b,c,d $\in \mathbb{Z}$ }

Assume there is a 2-coloring of $\mathbb{E}^2$ without a mono L3.

For the sake of contradiction,

assume that a unit equilateral triangle of color R-B-B with the 3 points and its centroid in X has a centroid of color R.

Set of 56 points that admits no possible coloring for counterclaim

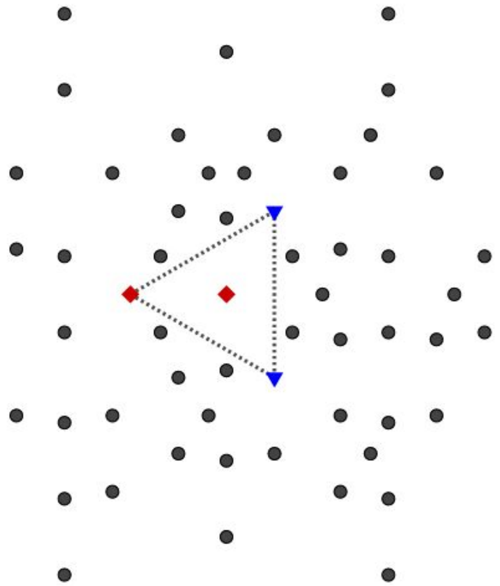


FIGURE 1. A set of 56 points admitting no coloring compatible with the initial coloring of the unit triangle and its center

A subset of the 56 point set

$$p_1 = [-4, 0, 0, 0], \quad p_2 = [0, 0, 0, 0], \quad p_3 = [2, 0, -6, 0], \quad p_4 = [2, 0, 6, 0],$$
$$q_1 = [-1, -3, 3, -1], \quad q_2 = [-1, -3, -3, 1], \quad q_3 = [2, 0, 0, 2], \quad q_3' = [2, 0, 0, -2],$$
$$q_4 = [-3, -3, -3, -1], \quad q_5 = [-3, -3, 3, 1].$$

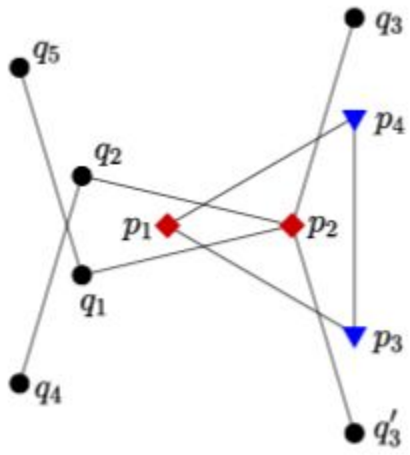


Figure 1: 56 point contradiction

$[-4,0,0,0]\, p_1,\ [0,0,0,0]\, p_2,\ [2,0,-6,0]\, p_3,\ [2,0,6,0]\, p_4,\ [-1,-3,3,-1]\, q_1,$
$[-1,-3,-3,1]\, q_2,\ [2,0,0,2]\, q_3,\ [2,0,0,-2]\, q_3', v[-3,-3,-3,-1]\, q_4,\ [-3,-3,3,1]\, q_5,$
$[-1,-3,-3,-3],\ [-1,3,3,1],\ [-1,3,-9,1],\ [3,3,-3,-1],\ [-2,0,-6,0],\ [1,3,-9,-1],$
$[-2,0,0,-2],\ [1,-3,-3,-1],\ [4,0,0,0],\ [-1,3,9,-1],\ [0,0,-6,-2],\ [-1,-3,9,1],$
$[1,3,-15,1],\ [1,-3,3,1],\ [1,3,-3,1],\ [-2,0,6,0],\ [-3,-3,9,-1],\ [1,3,-3,-3],$
$[6,0,0,2],\ [5,-3,-3,-1],\ [-1,-3,3,3],\ [3,3,-9,1],\ [-5,3,3,1],\ [6,0,0,-2],$
$[-2,6,0,0],\ [1,3,3,-1],\ [5,-3,3,1],\ [-1,-3,-15,1],\ [0,0,6,2],\ [3,-3,-3,1],$
$[0,0,6,-2],\ [5,3,-3,1],\ [-1,3,-3,-1],\ [0,0,-6,2],\ [0,0,-12,0],\ [-3,3,3,-1],$
$[-1,-3,-9,-1],\ [3,-3,3,-1],\ [-2,0,0,2],\ [-1,3,3,-3],\ [-3,3,-3,1],\ [3,3,3,1],$
$[5,3,3,-1],\ [1,-3,3,-3],\ [1,3,9,1],\ [1,3,3,3].$

A subset of the 56 point set

$$p_1 = [-4, 0, 0, 0], \quad p_2 = [0, 0, 0, 0], \quad p_3 = [2, 0, -6, 0], \quad p_4 = [2, 0, 6, 0],$$
$$q_1 = [-1, -3, 3, -1], \quad q_2 = [-1, -3, -3, 1], \quad q_3 = [2, 0, 0, 2], \quad q_3' = [2, 0, 0, -2],$$
$$q_4 = [-3, -3, -3, -1], \quad q_5 = [-3, -3, 3, 1].$$

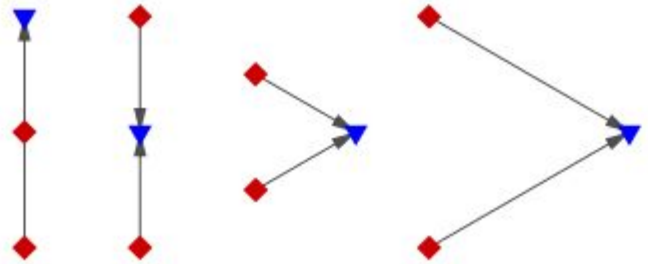
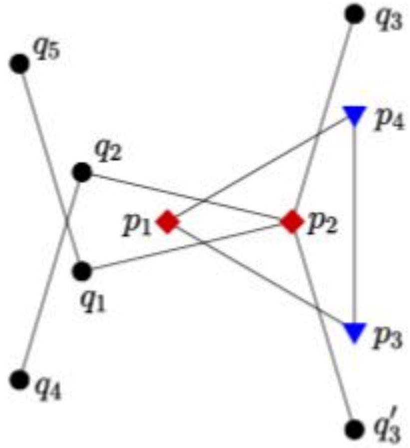
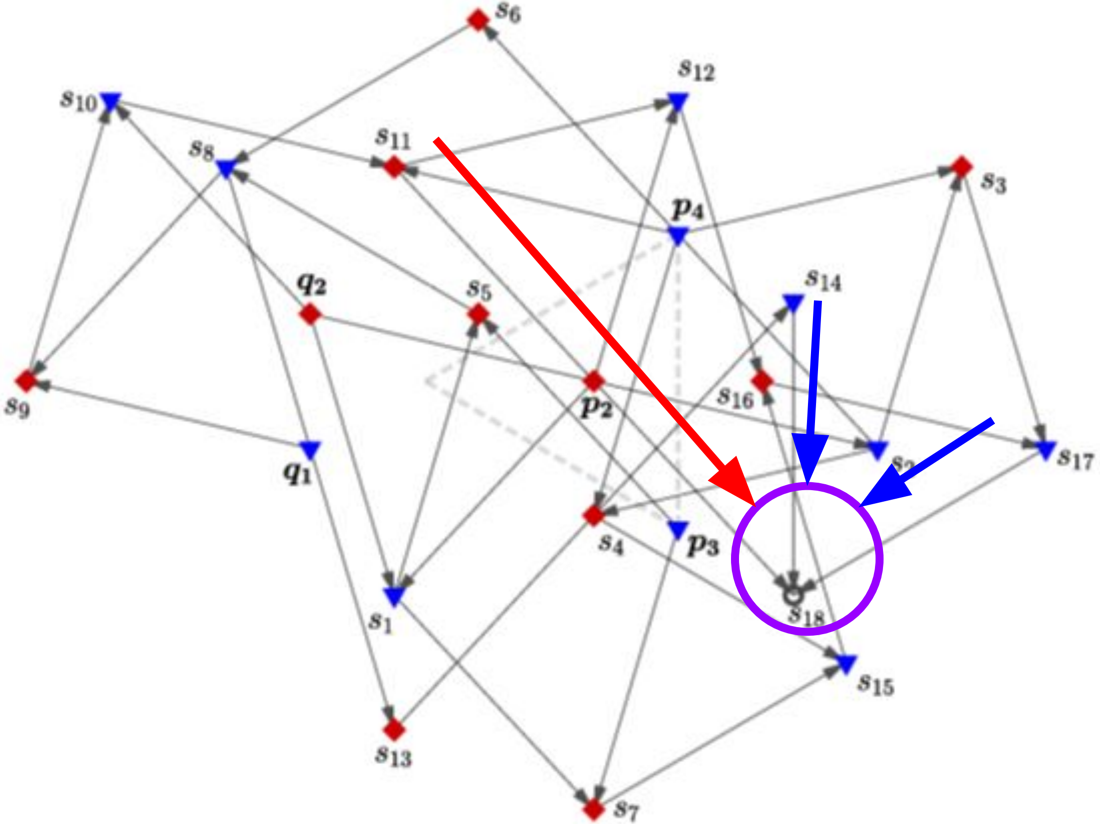


FIGURE 3. The color implication steps



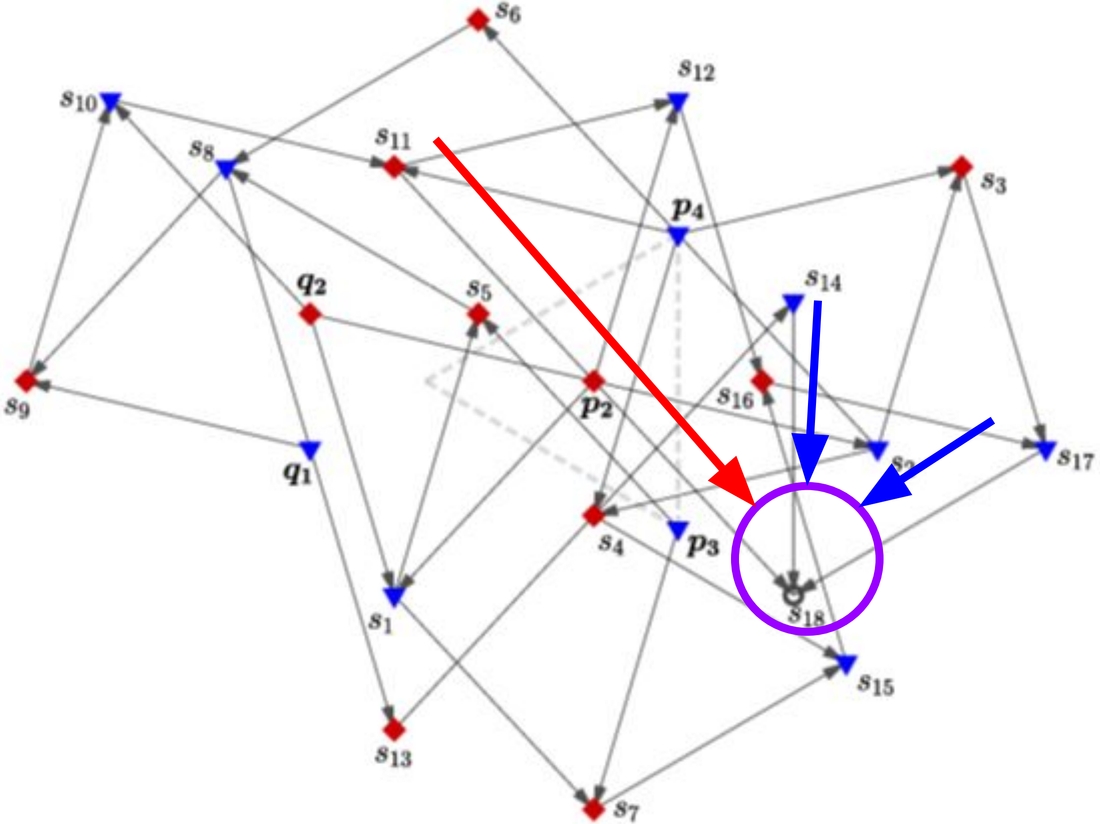
Casework on q1 and q2

q1 R and q2 B results in:



Casework on q1 and q2

q1 R and q2 B results in:



End of proof for Lemma 2.1.

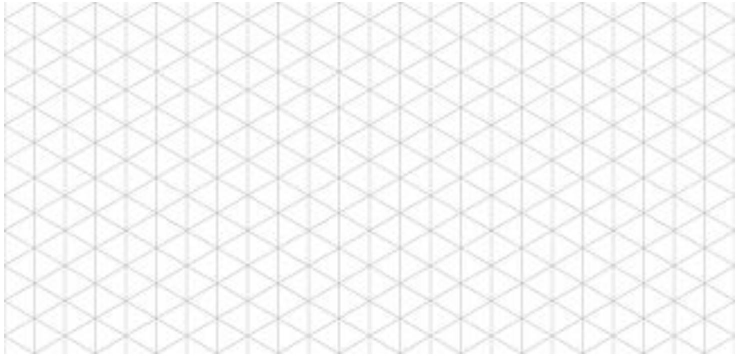
Lemmas we will use

Theorem 1.2. (Erdős et. al.) Will Not Prove

Lemma 2.1. If E^2 is 2-colored without a monochromatic L_3, any unit eq. triangle colored R-R-B has B centroid Just Proved

Lemma 2.2. If E^2 is 2-colored without a monochromatic L_3, a $\frac{1}{\sqrt{3}}$ scaled

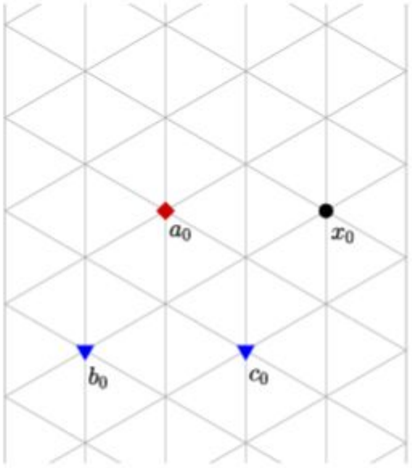
Hexagonal grid has one valid coloring



Proof for Lemma 2.2:
If E2 is 2-colored without a monochromatic L3, a $\frac{1}{\sqrt{3}}$ scaled
Hexagonal grid has one valid coloring

W.L.O.G. Suppose a unit triangle on points $\{a_0, b_0, c_0\}$ with R,B,B coloring, respectively

Now define x_0 as:

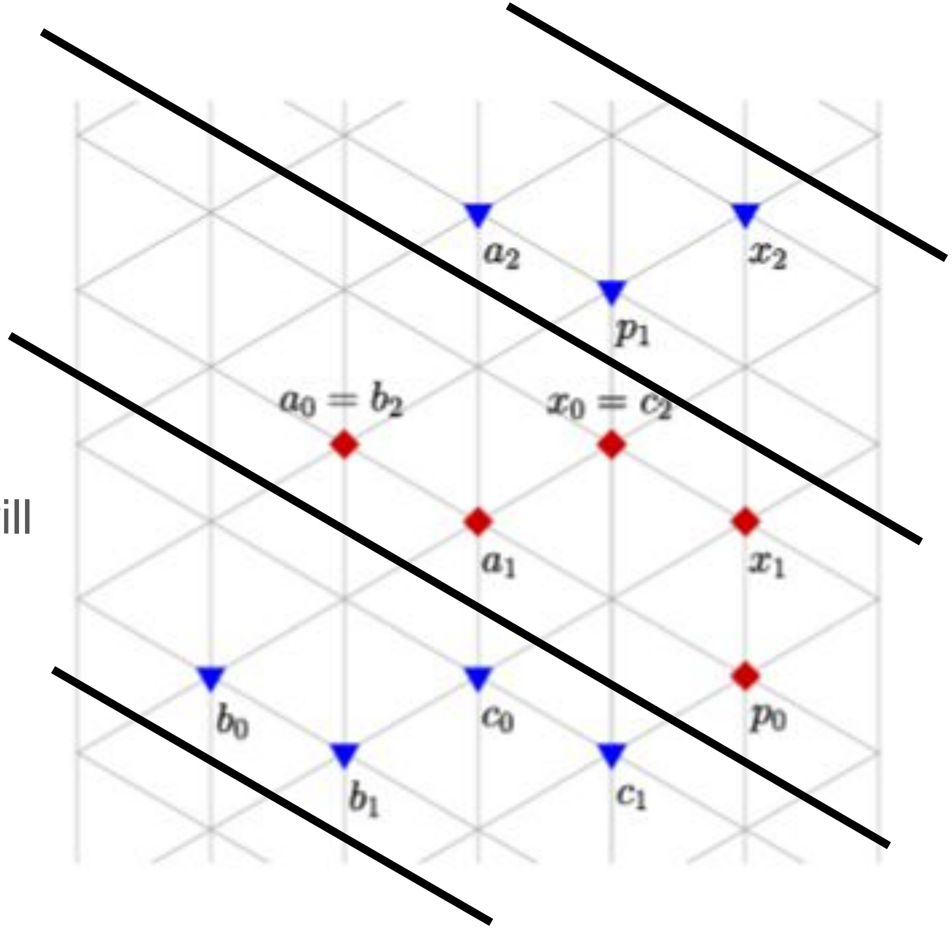


Casework again on x_0

If x_0 is R:

By continuing the steps, the grid will

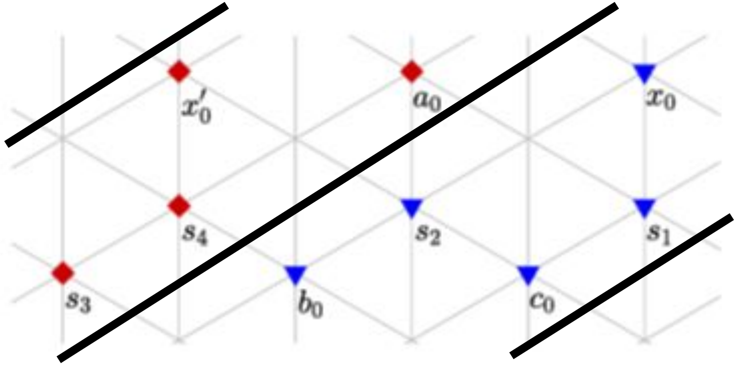
result in repetition of:
2 rows of R and 2 rows of B



Casework again on x_0

If x_0 is B:

By repetition, it will result the same as when x_0 is R



End of proof for Lemma 2.2.

Using Lemma 2.2 repeatedly results…

A unique grid!

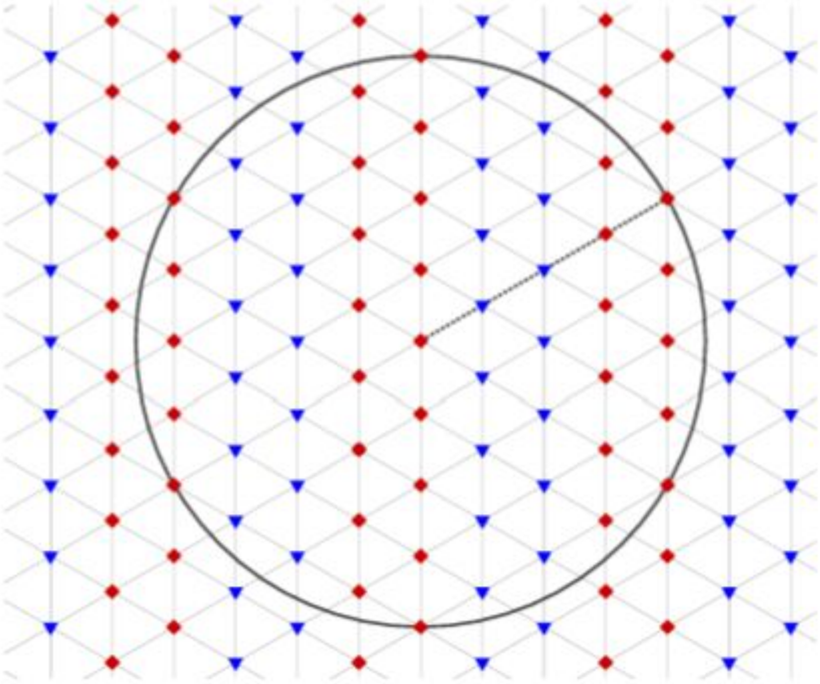


FIGURE 7. A circle with radius $\frac{4}{\sqrt{3}}$ in the colored grid

Proof of Theorem 1.1.

From the unique grid,

Now choose a R point:

All 6 points $\frac{4}{\sqrt{3}}$ units away from

the point are also red

Rotate the whole grid to make a

red circle of radius $\frac{4}{\sqrt{3}}$

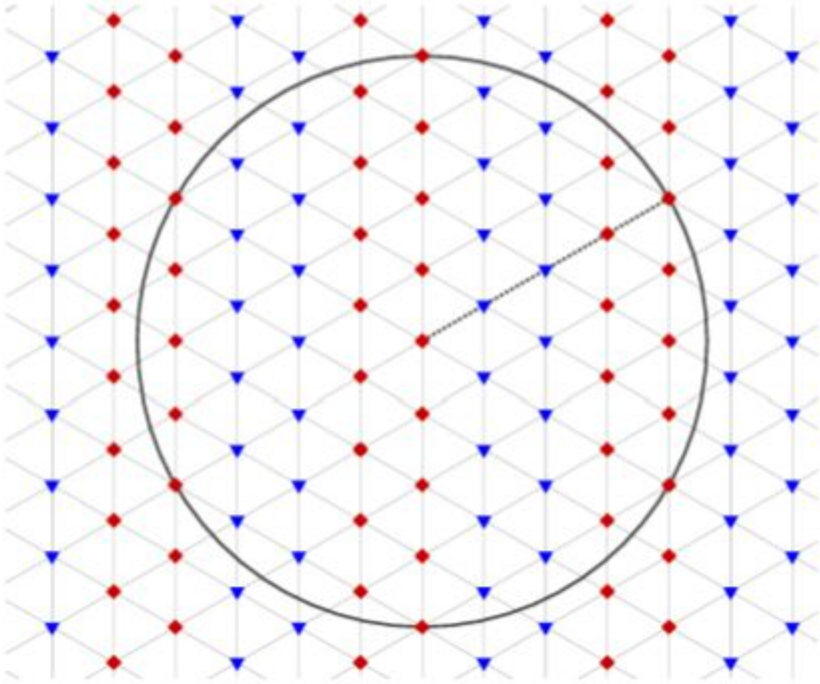


FIGURE 7. A circle with radius $\frac{4}{\sqrt{3}}$ in the colored grid

Now take a B point close enough

and make a congruent circle with B color, which would intersect

with R circle

The intersection are both B and R

Contradiction!!

End of the proof for the Theorem 1.1