

BILL, RECORD LECTURE!!!!

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Small Ramsey Numbers

Exposition by **William Gasarch**

June 2, 2026

The First Theorem in Ramsey Theory

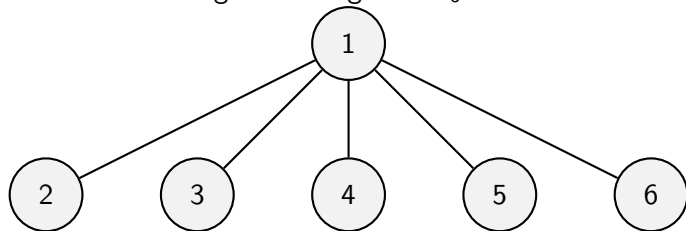
Thm For all COL: $\binom{[6]}{2} \rightarrow [2]$ there exists a homog set of size 3.

Focus on Vertex 1

Given a 2-coloring of the edges of K_6 we look at vertex 1.

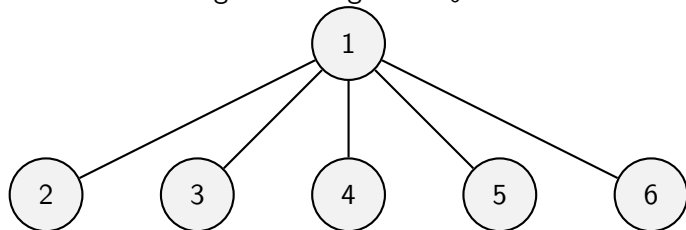
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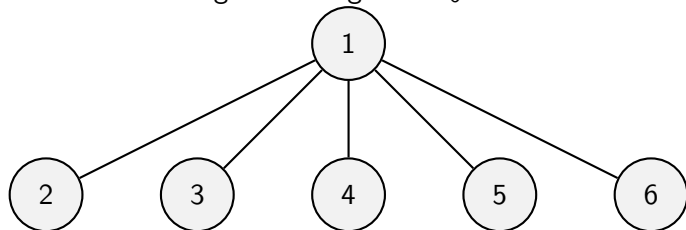
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There are 5 edges coming out of vertex 1.

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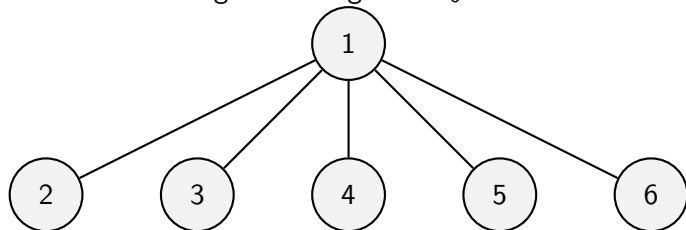


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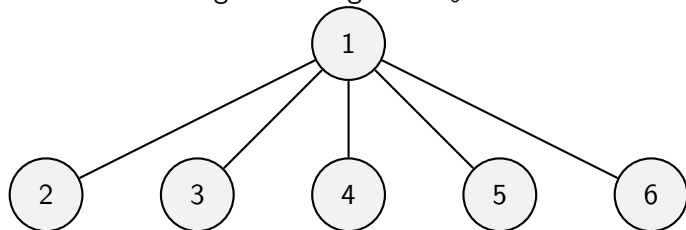
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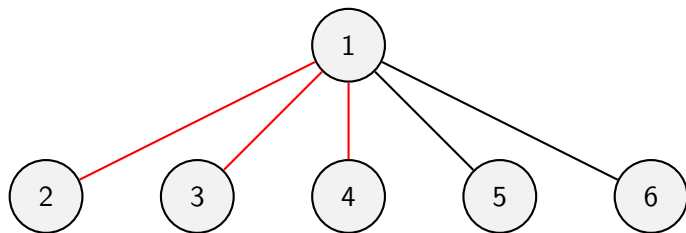
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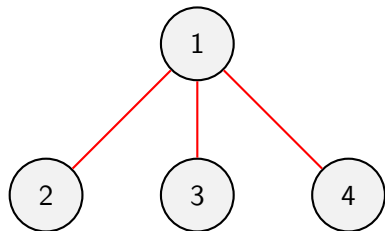
\exists 3 edges from vertex 1 that are the same color.

We can assume $(1, 2)$, $(1, 3)$, $(1, 4)$ are all **RED**.

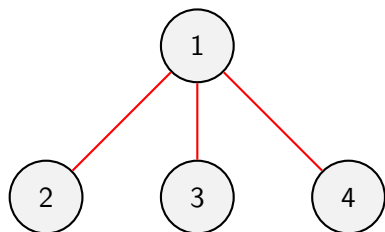
(1,2), (1,3), (1,4) are **RED**



We Look Just at Vertices 1,2,3,4



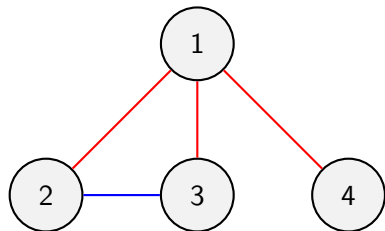
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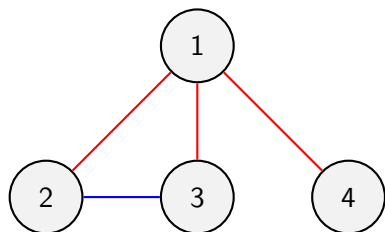
If (2,3) is **RED** then get **RED** Triangle. So assume (2,3) is **BLUE**.

(2,3) is BLUE

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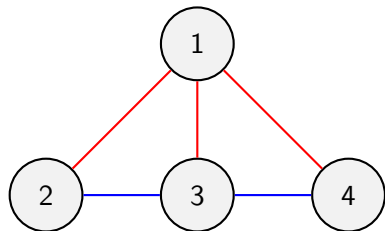
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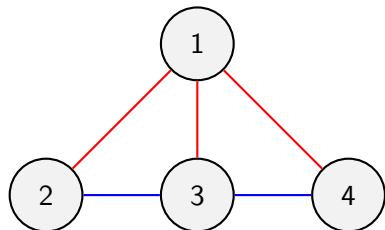
If (3,4) is **RED** then get **RED** triangle. So assume (3,4) is **BLUE**.

(2,3) and (3,4) are **BLUE**

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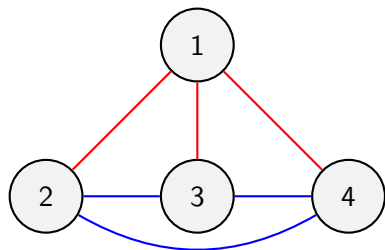
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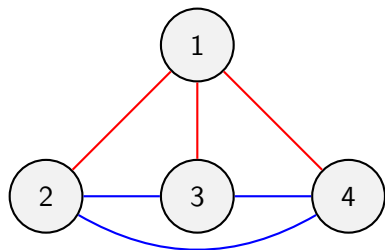
If (2,4) is **RED** then get **RED** triangle. So assume (2,4) is **BLUE**.

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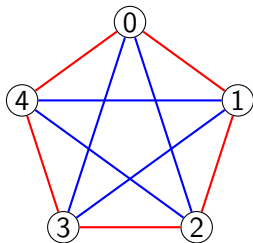
(2,4) is **BLUE**



Note that there is a **BLUE** triangle with verts 2, 3, 4. Done!

What If We Color Edges Of K_5 ?

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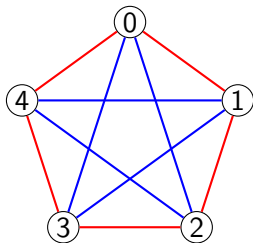


This graph is not arbitrary.

$$SQ_5 = \{x^2 \pmod{5} : 0 \leq x \leq 4\} = \{0, 1, 4\}.$$

- ▶ If $i - j \in SQ_5$ then **RED**.
- ▶ If $i - j \notin SQ_5$ then **BLUE**.

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Two ways to show no mono \triangle s on next slide.

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Need to show there are no mono \triangle .

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Method 1

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The **R** edges are all symmetric so if there is a \triangle edge can assume one of the edges is $(0, 1)$. No x with $\text{COL}(0, x) = \text{COL}(0, 1) = \mathbf{R}$.

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UPSHOT $R(3, 3) = 6$.

Asymmetric Ramsey Numbers

Definition Let $a, b \geq 2$. $R(a, b)$ is least n such that for all 2-colorings of K_n there is **either** a red K_a or a blue K_b .

1. $R(a, b) = R(b, a)$.
2. $R(2, b) = b$
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Proof left to the reader, but its easy.

$$R(a, b) \leq R(a - 1, b) + R(a, b - 1)$$

Theorem $R(a, b) \leq R(a - 1, b) + R(a, b - 1)$

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The proof has three cases on the next three slides.

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1. There is a vertex with large **Red** Deg.

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1. There is a vertex with large **Red** Deg.
2. There is a vertex with large **Blue** Deg.
3. All verts have small **Red** degree and small **Blue** degree.

Some Vertex v Has Large Red Deg

Case 1 $(\exists v)[\deg_R(v) \geq R(a-1, b)]$.

Some Vertex v Has Large Red Deg

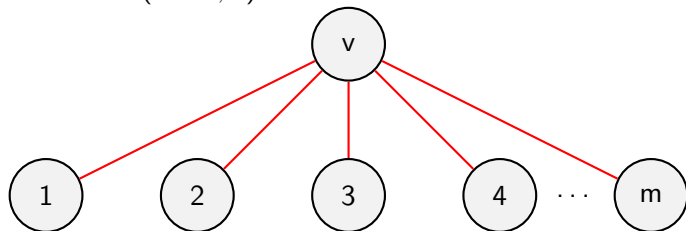
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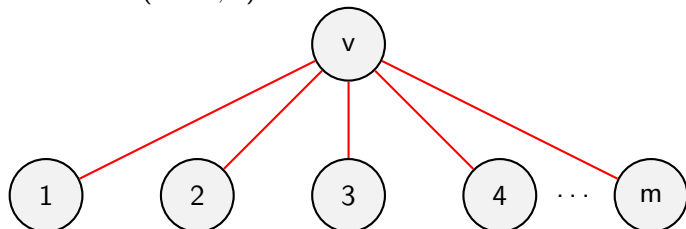
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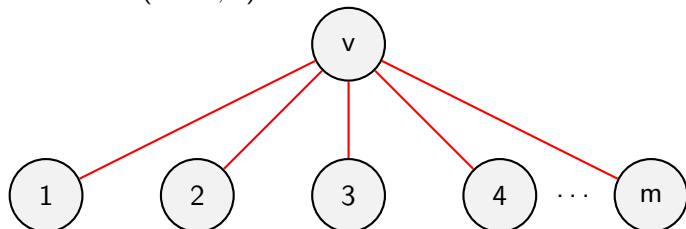


Case 1.1 There is a **Red** K_{a-1} in $\{1, \dots, m\}$. This set together with vertex v is a **Red** K_a .

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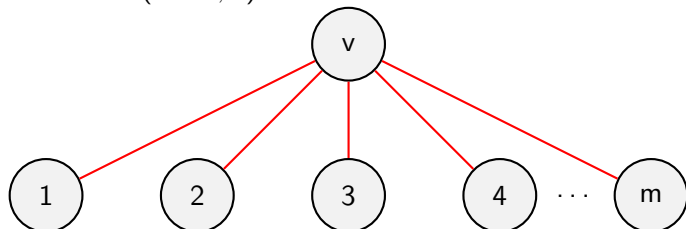
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Case 1.3 Neither. **Impossible** since $m = R(a-1, b)$.

Some Vertex v Has Large **Blue** Deg

Case 2 $(\exists v)[\deg_B(v) \geq R(a, b - 1)]$.

Some Vertex v Has Large Blue Deg

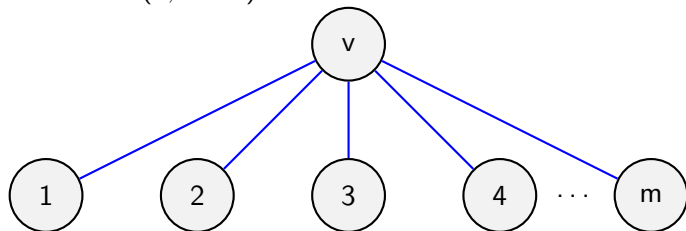
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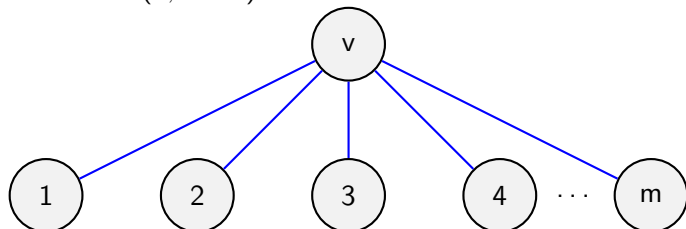
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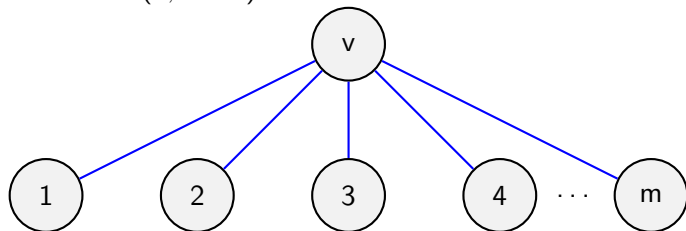


Case 2.1 There is a **Red** K_a in $\{1, \dots, m\}$. DONE

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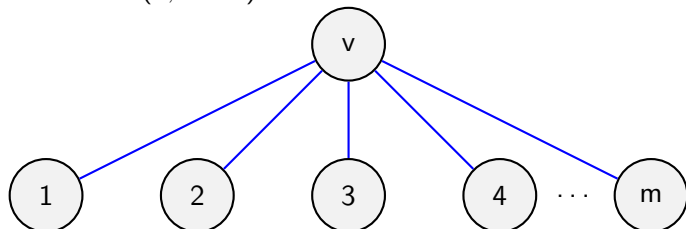
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Case 2.2 There is a **Blue** K_{b-1} in $\{1, \dots, m\}$. This set together with vertex v is a **Blue** K_b .

Case 2.3 Neither. **Impossible** since $m = R(a, b - 1)$.

All Verts: Small Red Deg and Small Blue Deg

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Hence

$$(\forall v)[\deg(v) \leq R(a-1, b) + R(a, b-1) - 2 = n - 2]$$

Not possible since every vertex of K_n has degree $n - 1$.

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Table of Bounds

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We need a theorem.

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Can we make some improvements to this? YES!

We need a theorem. We first do an example.

A Graph on 9 Vertices with all verts Deg 3?

Thm There is NO graph on 9 verts, with every vertex of deg 3.

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We generalize this on the next slide.

Handshake Lemma

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Handshake Lemma If all pairs of people in a room shake hands, even number of shakes. (Pre-COVID when people shook hands.)

Corollary of Handshake Lemma

Impossible to have a graph on an odd number of vertices where every vertex is of odd degree.

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And NOW to our improvements on small Ramsey numbers.

$R(3, 4) \leq 9$ Case 1

Assume we have a 2-coloring of the edges of K_9 .

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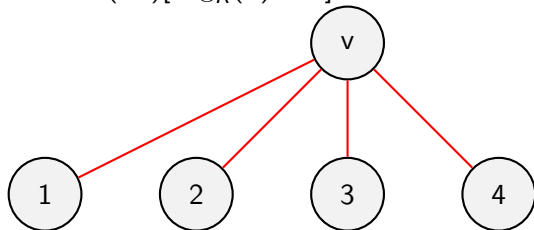
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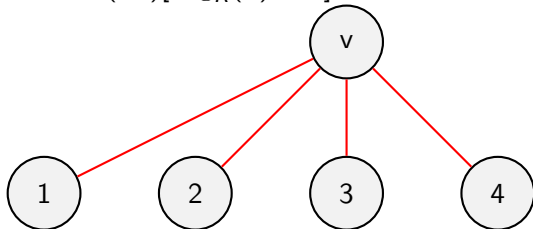
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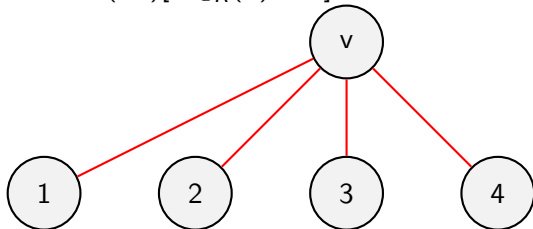


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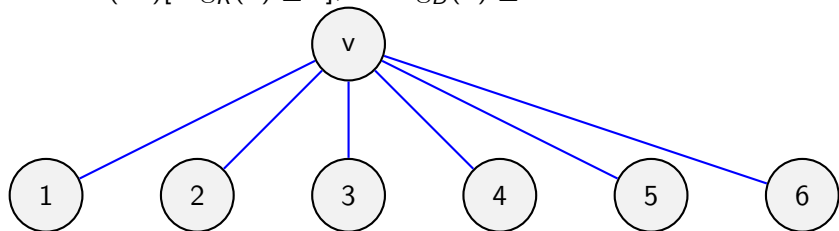


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2) If **all** of $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$ are **BLUE**, have **BLUE** K_4 .

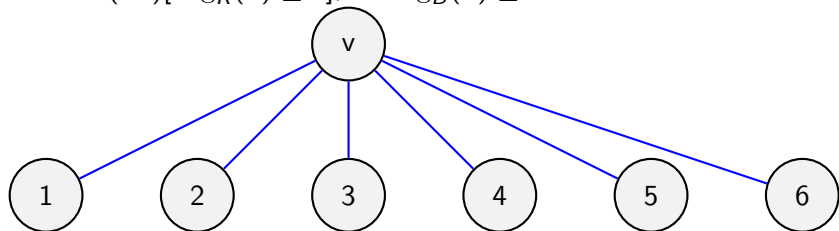
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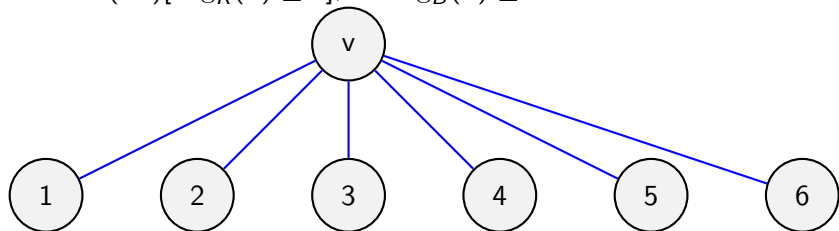
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- (1) There is a **RED** K_3 in $\{1, 2, 3, 4, 5, 6\}$. Have **RED** K_3 .
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$R(3, 4) \leq 9$ Case 3

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This is impossible!

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Theorem $R(a, b) \leq$

1. $R(a, b - 1) + R(a - 1, b)$ always.
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Proof left to the Reader.

Some Better Upper Bounds

- ▶ $R(3, 3) \leq R(2, 3) + R(3, 2) \leq 3 + 3 = 6.$
- ▶ $R(3, 4) \leq R(2, 4) + R(3, 3) \leq 4 + 6 - 1 = 9.$
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- ▶ $R(3, 7) \leq R(2, 7) + R(3, 6) \leq 7 + 19 = 26$
- ▶ $R(4, 4) \leq R(3, 4) + R(4, 3) \leq 9 + 9 = 18.$
- ▶ $R(4, 5) \leq R(3, 5) + R(4, 4) \leq 14 + 18 - 1 = 31.$
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Are these tight?

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Are these tight? Some yes, some no.

$$R(3, 3) \geq 6$$

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$R(3, 3) = 6$ as shown in prior slide.

$$R(4, 4) = 18$$

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Use

$COL(a, b) = \text{RED}$ if $a - b \in SQ_{17}$, BLUE OW.

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Same idea as above for K_5 , but more cases for algebra.

UPSHOT $R(4, 4) = 18$ and the coloring used math of interest!

$$R(3, 5) = 14$$

$R(3, 5) \geq 14$: Need coloring of K_{13} w/o **RED** K_3 or **BLUE** K_5 .

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THATS IT.

No other $R(a, b)$ are known using NICE methods.

Summary of Bounds

$R(a, b)$	Old Bound	New Bound	Opt	Int?
$R(3, 3)$	6	6	6	Y
$R(3, 4)$	10	9	9	Y
$R(3, 5)$	15	14	14	Y
$R(3, 6)$	21	19	18	Lower-Y
$R(3, 7)$	28	27	23	Lower-Y
$R(4, 4)$	20	18	18	Y
$R(4, 5)$	35	31	25	N
$R(5, 5)$	70	62	≤ 46	N

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$R(5, 5)$: See the assigned paper for more on this.

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(Joel Spencer) The Law of Small Numbers: Patterns that persist for small numbers will vanish when the calculations get to hard.
2. Seemed like a nice **Math** problem that would involve interesting and perhaps deep mathematics. No. The work on it is interesting and clever, but (1) the math is not deep, and (2) progress is slow.