

# Problems with a Point: Exploring Math and Computer Science

August 9, 2021

**Authors:**  
**William Gasarch**  
**Clyde Kruskal**

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# How This Book Came to Be

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# Book's Origin

- ▶ In 2003 Lance Fortnow started **Complexity Blog**
- ▶ In 2007 Bill Gasarch joined and it was a co-blog.
- ▶ In 2015 various book publishers asked us

**Can you make a book out of your blog?**

- ▶ Lance declined but Bill said **YES**.

# Book's Point

Bill took the posts that had the following format:

- ▶ make a point **about** mathematics
- ▶ do some math to underscore those points

and made those into chapters.

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and made those into chapters.

**Caveat:** Not every chapter is quite like that.

To quote Ralph Waldo Emerson

*A foolish consistency is the hobgoblin of small minds.*

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The publisher wisely decided to be less cute and more informative:

**Problems with a Point: Exploring Math and Computer Science**

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**Please Procure People to Polish Prose and Proofs of Problems with a Point**

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Now onto some samples of the book!



# Point: Students Can Give Strange Answers

August 9, 2021

# The Paint Can Problem

From the Year 2000 Maryland Math Competition:

*There are 2000 cans of paint. Show that at least one of the following two statements is true:*

- ▶ There are at least 45 cans of the same color.
- ▶ There are at least 45 cans that are different colors.

**Work on it in groups! Prove a General Theorem.**

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**Answer:**

If there are 45 different colors of paint then we are done. Assume there are  $\leq 44$  different colors. If all colors appear  $\leq 44$  times then there are  $44 \times 44 = 1936 < 2000$  cans of paint, a contradiction.

**Note:** this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.

# One of the Wrong Answers. Or is it?

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## **ANSWER:**

*Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.*

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### **ANSWER:**

*Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.*

What do you think:

- ▶ That's just stupid. 0 points.
- ▶ Question says *cans of the same color*. ... The full 30 pts.
- ▶ Not only does he get 30 points, but everyone else should get 0.

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### **ANSWER:**

*If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.*



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What do you think:

- ▶ That's just stupid. 0 points.
- ▶ Well... he's got a point. 30 points in fact.
- ▶ Not only does he get 30 points, but everyone else should get 0.

# A Triangle Problem

From the year 2007 Maryland Math Competition.

**QUESTION** *Let  $ABC$  be a fixed triangle. Let  $COL$  be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle  $DEF$  in the plane such that  $DEF$  is similar to  $ABC$  and the vertices of  $DEF$  all have the same color.*

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**Note** I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn't. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit

# Funny Answers One

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## **Funny Answer One:**

*All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like  $2 + 2 = 5$  if thats what my math teacher says. Math is pretty subjective anyway.*

## Was Student One Serious?

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**Theorem** The students is not serious.

**Proof** Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.

## Funny Answers Two

**QUESTION** *Let  $ABC$  be a fixed triangle. Let  $COL$  be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle  $DEF$  in the plane such that  $DEF$  is similar to  $ABC$  and the vertices of  $DEF$  all have the same color.*



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*I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.*

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Was Student Two Serious? Yes. About **Justice!**.

# The Real Answer to Points in the Plane Problem

*Each point in the plane is colored either red or green. Let  $ABC$  be a fixed triangle. Prove that there is a triangle  $DEF$  in the plane such that  $DEF$  is similar to  $ABC$  and the vertices of  $DEF$  all have the same color.*

Fix a 2-coloring of the plane.

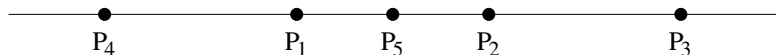
## There are 3 equally-spaced mono points on $x$ -axis

**Proof** Clearly there are two points on the  $x$ -axis of the same color:  $p_1, p_2$  are RED. If  $p_3$ , the midpoint of  $p_1, p_2$ , is RED then  $p_1, p_3, p_2$  are all RED. DONE. Hence we assume  $p_3$  is GREEN.

Let  $p_4$  be such that  $|p_1 - p_4| = |p_2 - p_1|$ . If  $p_4$  is RED then  $p_4, p_1, p_2$  are all RED. DONE. Hence we assume  $p_4$  is GREEN.

Let  $p_5$  be such that  $|p_5 - p_2| = |p_2 - p_1|$ . If  $p_5$  is RED then  $p_1, p_2, p_5$  are all RED. DONE. Hence we assume  $p_5$  is GREEN.

Only case left  $p_3, p_4, p_5$  are all GREEN. DONE.



## Finish Proof By Picture

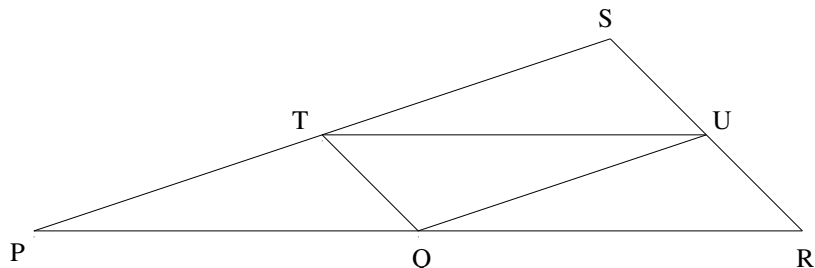


Figure: Triangle Similar to  $ABC$  with Monochromatic Vertices

$P, Q, R$  are RED.

If  $T$  or  $U$  or  $S$  are RED then get RED Triangle similar to  $ABC$ .

If not then ALL of  $T, U, S$  are GREEN, so get GREEN triangle similar to  $ABC$ .

# Point: What is a Pattern?

August 9, 2021



# Simple Functions

Bill assigned the following in Discrete Math: For each of the following sequences find a **simple function**  $A(n)$  such that the sequence is  $A(1), A(2), A(3), \dots$

1. 10, -17, 24, -31, 38, -45, 52,  $\dots$
2. -1, 1, 5, 13, 29, 61, 125,  $\dots$
3. 6, 9, 14, 21, 30, 41, 54,  $\dots$

**Caveat:** These are NOT trick questions.

**Work on it in groups.**

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1. 10, -17, 24, -31, 38, -45, 52,  $\dots$   $A(n) = (-1)^{n+1}(7n + 3)$ .
2. -1, 1, 5, 13, 29, 61, 125,  $\dots$   $A(n) = 2^n - 3$ .
3. 6, 9, 14, 21, 30, 41, 54,  $\dots$   $A(n) = n^2 + 5$ .

# A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets.** Is that what you want us to use?*

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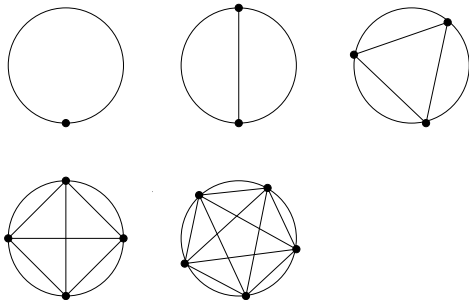
The student got the first one right, but left the other two blank.

# When Do Patterns Hold?

The last question brings up the question of when patterns do and don't hold. We looked for cases where a pattern *did not* hold.

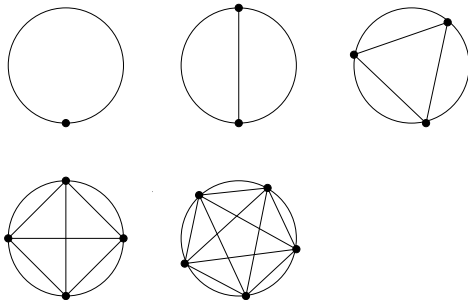
## First Non-Pattern: $n$ Points on a circle

What is the max number of regions formed by connecting every pair of  $n$  points on a circle. For  $n = 1, 2, 3, 4, 5$ :



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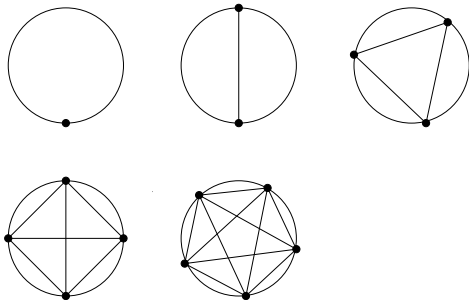
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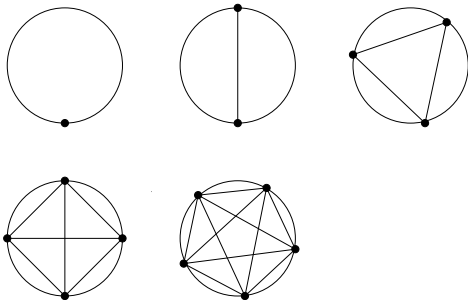
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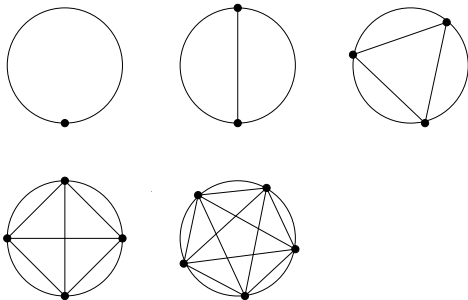
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But for  $n = 6$ , the number of regions is only 31.

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But for  $n = 6$ , the number of regions is only 31.

The actual number of regions for  $n$  points is  $\binom{n}{4} + \binom{n}{2} + 1$ .

## Second Non-Pattern: Borwein Integrals

$$\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2}$$

⋮

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$$\frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \sim 0.9999999999852937186 \times \frac{\pi}{2}$$

# Why the breakdown at 15?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1.$$

For more Google

Borwein Integral

# Computers to FIND proofs vs Computers to DO Proofs

August 9, 2021

# Colorings and Square Differences

The following are all true:

1. There exists a number  $W_2$  such that, for all 2-colorings of  $\{1, \dots, W_2\}$  there exists 2 nums, square-apart, same color.

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4. For all  $c$  there exists a number  $W_c \dots$

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The proofs in the literature of these theorems give EEEEEEEEEENORMOUS bounds on  $W_2, W_3, W_4, W_c$ . We look at easier proofs with two **points** in mind:

- ▶ Would they be good questions on a HS math competition?
- ▶ What is the role of Computers in these proofs?



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Let COL be a 2-coloring of  $\{1, 2, 3, \dots\}$  with colorings  $R$  and  $B$ .  
We can assume  $\text{COL}(1) = R$ .

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**Upshot** Could be easy HS Math Comp Prob. No computer used.

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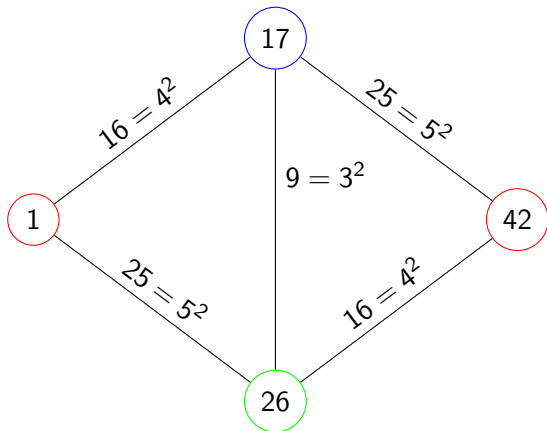


Figure:  $\text{COL}(x) = \text{COL}(x + 41)$

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Can we get better bound on  $W_3$ ?

## Better Bound on $W_3$

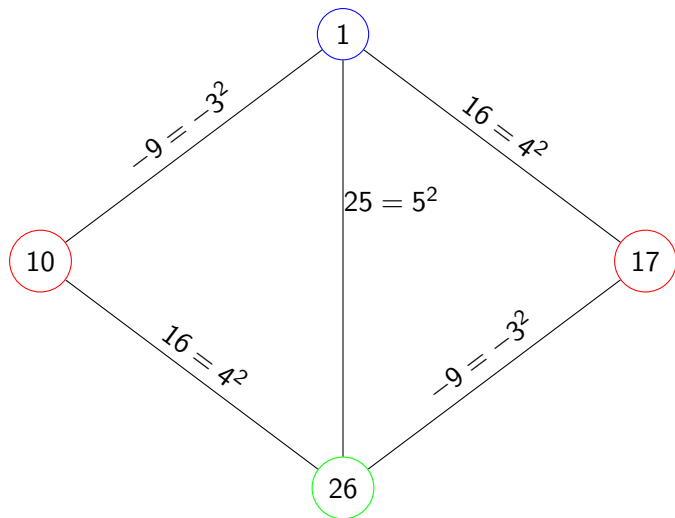


Figure: If  $x \geq 10$  then  $\text{COL}(x) = \text{COL}(x + 7)$ , so  $W_3 \leq 59$

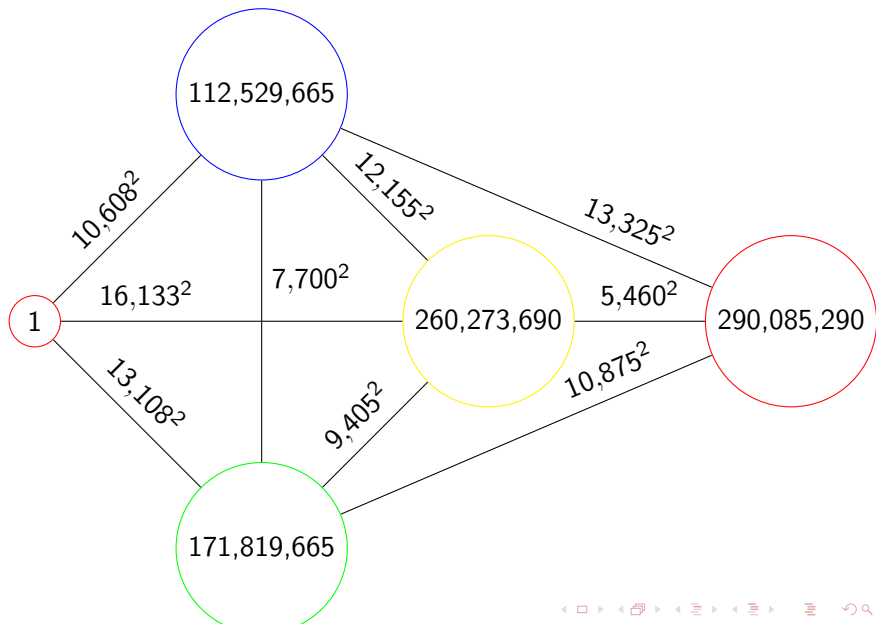
## Reflection on $W_3, W_4$

1. Problem 5 (so hard) on UMCP HS Math Comp, 2006:  
*Show that for all 3-colorings of  $\{1, \dots, 2006\}$  there exists 2 numbers that are a square apart that are the same color*
2. 240 took exam, 40 tried this problem, 10 got it right.
3. Bill Gasarch and Matt Jordan proved, by hand,  $W_3 = 29$ .
4. **Is there a HS-proof that  $W_4$  exists?** Bill wanted to put this problem on the next HS exam to find out. He was (wisely) told **NO**.
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$W_4$  Exists:  $\text{COL}(x) = \text{COL}(x + 290,085,290)$



## Reflection on $W_4$

1. Zach's proof shows  $W_4 \leq 1 + 299,085,290^2$ .  
**PRO** Proof is easy to verify  
**CON** Number is large, proof does not generalize to  $W_5$ .
2. The classical proof.  
**PRO** Gives bounds for  $W_c$ .  
**CON** Bounds are GINORMOUS, even for  $W_2$ .
3. A Computer Search showed that  $W_4 = 58$ .  
**PRO** Get exact value.  
**CON** not human-verifiable. Does not generalize to  $W_5$ .

Which do you prefer?

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