Funky Dice: An Exposition

William Gasarch - University of MD

If You Roll Two Standard 6-Sided Dice Then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.

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6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob $\frac{1}{6}$.

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9. 10: (4,6), (5,5), (6,4) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.

Questions about Dice

1. Can we load two 6-sided dice so that every number from 2 to 12 has the **same** probability. Called **fair sums**.

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2. Can you label the dice something other than $\{1, \ldots, 6\}$ and $\{1, \ldots, 6\}$ and get the same probabilities you get with standard dice?

Loaded Dice

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Fair Dice Yield Unfair Sums

Fair Die:

$$\Pr(1)=\Pr(2)=\Pr(3)=\Pr(4)=\Pr(5)=\Pr(6) = 1/6 \sim 0.167$$

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How Unfair?: $1/6 - 1/36 \sim 0.139$ unfair.

What Are Loaded Dice?

Def: A **Die** is a 6-tuple $(p_1, p_2, p_3, p_4, p_5, p_6)$ such that $0 \leq p_i \leq 1$ and $\sum_{i=1}^{6} p_i = 1$.

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The coefficient of x^i is $\mathrm{Prob}(\mathrm{sum} = \mathrm{i})$

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Continued on Next Slide.

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1. r root of $x^{10} + \cdots + x + 1 \implies r$ root of $x^{11} - 1$ & $r \neq 1$.
2. r root of $x^{11} - 1$ & $r \neq 1 \implies r$ root of $x^{10} + \cdots + x + 1$.

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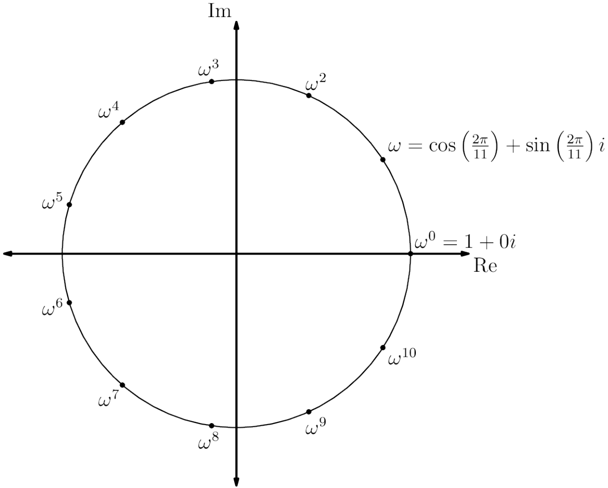
The roots of $x^{11} - 1$ are on the complex unit circle. See Next Slide.

The 11th Roots of Unity: Only Real one is 1



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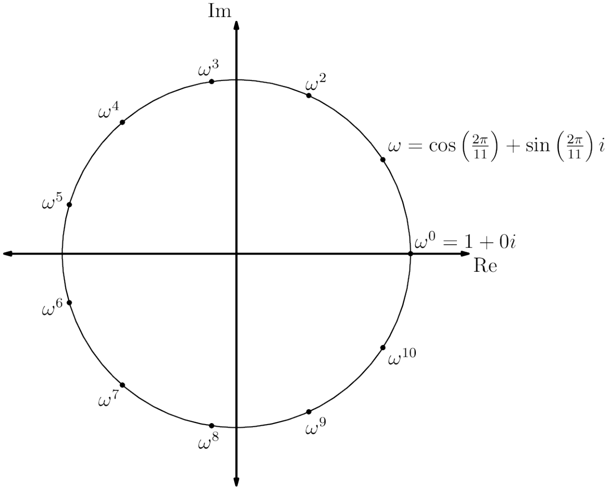

1 is only real 11th root of unity. $x^{10} + \cdots + 1 = 0$: **no** real roots.

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Recap

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Contradiction

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Can You Ever Load Dice to Get Fair Sums?

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2. Gasarch & Kruskal `https://www.cs.umd.edu/~gasarch/papers/dice.pdf` looked at allowing 0. They proved the following:

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Fame! One paper refers to **The Gasarch-Kruskal Thm**.

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How far are normal dice from uniform?

$$2(1/11-1/36)^2+2(1/11-1/18)^2+2(1/11-1/12)^2+2(1/9-1/11)^2+$$

$$2(5/36-1/11)^2)+(1/6-1/11)^2 \sim 0.0217$$

How Close To Uniform Can You Get? (cont)

Thm The optimal pair of 6-sided dice is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$ and $(\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8})$.

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Distance Optimal dice from uniform is $\sim$ **0.0028**.

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What About n-sided Dice?

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The optimal pair of n-sided dice is
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The distance from uniform is $\frac{1}{2(2n-1)(3n-2)} \sim \frac{1}{12n^2}$.

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Given that I am giving this talk on short notice I didn't work out how close n normal dice are to uniform. I would like one of you to do that soon and email me your calculations and the answer. Would be happy with an approximation like $\frac{1}{an^b}$.

Different Labels on Dice

William Gasarch - University of MD

Can You Label Dice To Get Same Probs?

A **labeling** of a 6-sided die has any positive natural numbers as labels. We allow using a number twice. We allow using numbers higher than 6. So $(1, 2, 2, 3, 5, 8)$ would be allowed.

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Answer on next slide.

Yes We Kam!

YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

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YES. There a non-standard labeling of a pair of 6-sided dice so that the dice yield the SAME probabilities as the standard dice.

We prove this on the next slide.

Let Polynomials Do The Work For You!

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Coefficient of x^n is number of ways to get n.

Example of Non-Standard Labelings

What if we label the dice $(1, 2, 2, 2, 5, 5)$ and $(1, 3, 3, 3, 3, 7)$?

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$$(2x^5+3x^2+x)(x^7+4x^3+x) = 2x^{12}+3x^9+9x^8+2x^6+12x^5+4x^4+3x^3+x^2$$

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1. 12: TWO ways. Prob $\frac{1}{18}$.
2. 9: THREE ways. Prob $\frac{1}{12}$.
3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.

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3. 8: NINE ways. Prob $\frac{1}{4}$.
4. 6: TWO ways. Prob $\frac{1}{18}$.
5. 5: TWELVE ways. Prob $\frac{1}{3}$.
6. 4: FOUR ways. Prob $\frac{1}{9}$.
7. 3: THREE ways. Prob $\frac{1}{12}$.
8. 2: ONE ways. Prob $\frac{1}{36}$.

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Question Is there a nonstandard labeling of two 6-sided dice that gives the same probabilities as the standard dice?

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$$(x^6+x^5+x^4+x^3+x^2+x)^2.$$

Is there a Non-Standard Labeling That... Cont.

$$(x^{a_1}+x^{a_2}+x^{a_3}+x^{a_4}+x^{a_5}+x^{a_6})(x^{b_1}+x^{b_2}+x^{b_3}+x^{b_4}+x^{b_5}+x^{b_6}) =$$

$$(x^6+x^5+x^4+x^3+x^2+x)^2 = x^2(x^5+x^4+x^3+x^2+x+1)^2 =$$

$$x^2(x+1)^2(x^2-x+1)^2(x^2+x+1)^2.$$

Need to Factor. . .

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$x(x+1)(x^2-x+1)^2(x^2+x+1) = x^8+x^6+x^5+x^4+x^3+x$.
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So desired dice are $(1,2,2,3,3,4)$ and $(1,3,4,5,6,8)$.

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What About Two d-Sided Dice?

For which $d \geq 2$ are there two non-standard d-sided dice that have the same prob as standard dice? **VOTE:**

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Answer on Next Slide

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

What About Two d-Sided Dice?

Answer There are two non-standard d-sided dice iff d is non-prime.

The proof is similar to what we did, though requires some thought.

What About d_1, d_2-Sided Dice?

For which $d_1, d_2 \geq 2$ are there non-standard d_1-sided die and d_2-sided die that have the same prob as standard dice?
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4. UNKNOWN TO SCIENCE!

Answer on Next Slide.

Unknown to . . .

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   Or maybe just **Unknown to Bill**.

Parting Thoughts

William Gasarch - University of MD

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3. It is remarkable that a problem about dice lead to looking at complex roots of polynomials!