Egg Problems

William Gasarch-U of MD

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- You only have a limited number of eggs.
- If an egg **does not break** then you **can** re-use it.
- If an egg **breaks** then you can **cannot** re-use it.

One Egg

Get an answer AND prove its optimal.

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1. How many drops needed if 100 floors, 2 eggs.
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   Optimal If g, $2g$, etc and then 1-egg within a gap the worst case is roughly $\frac{n}{g} + g$. This is minimized when $g = n^{1/2}$.

YOU"VE BEEN PUNKED

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- If last drop SPLAT then takes 19 drops.

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A Clue That This is Not Optimal

- If first drop SPLAT then takes 11 drops.
- If last drop SPLAT then takes 19 drops.
- There should be a way to make the worst case better and the best case worse.

A Trick! A Technique!

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15th floor. If SPLAT, 14 left, **1+13=14 total**

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(15+14)th Floor. If SPLAT, 13 left, **2+12=14 total**

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(15+14+13)th Floor. If SPLAT, 12 left, **3+11=14 total**

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$(15 + \cdots + 4)$th Floor. If SPLAT, 3 left, **12+2=14 total**
Sum is 101, so actually works for 101 floors.

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14 drops

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Find least k with $1 + 2 + \cdots + k \geq n$. Note: $k \sim 2^{1/2} \times n^{1/2}$.

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Old Method: $2 \times n^{1/2}$ drops

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New Method: $2^{1/2} \times n^{1/2}$ drops

Is $\sim 2^{1/2} \times n^{1/2}$ optimal?

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Find least k with $1+2+\cdots+k \geq n$. Note: $k \sim 2^{1/2} \times n^{1/2}$.

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kth floor. If SPLAT, $k-1$ left, $1+(k-1)=k+1$ **total**

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$(k+\cdots+1)$th Floor.

Contrast

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New Method: $2^{1/2} \times n^{1/2}$ drops

Is $\sim 2^{1/2} \times n^{1/2}$ optimal? Vote YES or NO

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Contrast

Old Method: $2 \times n^{1/2}$ drops

New Method: $2^{1/2} \times n^{1/2}$ drops

Is $\sim 2^{1/2} \times n^{1/2}$ optimal? Vote YES or NO Answer is YES.

Optimal

Two eggs.
Given n, the optimal number of drops is least k with $1 + 2 + \cdots + k \geq n$.

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least k with $1 + 2 + \cdots + k \geq n$.
Proof Sketch Any algorithm that deviates from the one we give has to do worse. Formally you would look at the first step where the algorithm differs from ours.

Three Eggs

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1. How many drops needed if 100 floors, 3 eggs.
2. ow many drops needed if n floors, 3 eggs.
   (Can ignore $+O(1)$ terms.)

Work on in groups.

Three Eggs Answer for 100

First need to know some two-egg answers

n	Sum	number of drops
1		0
$2, 3$	$1 + 2 \geq 3$	2
$4, 5, 6$	$1 + 2 + 3 \geq 6$	3
$7, 8, 9, 10$	$1 + 2 + 3 + 4 \geq 10$	4
$11, \ldots, 15$	$1 + 2 + 3 + 4 + 5 \geq 15$	5
$16, \ldots, 21$	$1 + 2 + 3 + 4 + 5 + 6 \geq 21$	6
$21, \ldots, 28$	$1 + 2 + 3 + 4 + 5 + 6 + 7 \geq 28$	7
$28, \ldots, 36$	$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \geq 36$	8

How many drops needed if 100 floors, 3 eggs.

36th floor. If SPLAT use 2-egg sol. **1+8=9**

(36+28)th floor. If SPLAT use 2-egg sol. **2+7=9**

(36+28+21)th floor. If SPLAT use 2-egg sol. **3+6=9**

(36+28+21+15+10))th floor. If SPLAT use 2-egg sol. **3+6=9**

Three Eggs Answer for n

Recall 2-egg for n least k such that

$$1 + 2 + \cdots + k \geq n \;\; k \sim 2^{1/2} n^{1/2}$$

Rephrase:

Recall 2-egg for n least k such that

$$\binom{1}{1} + \binom{2}{1} + \cdots + \binom{k}{1} \geq n \;\; k \sim 2^{1/2} n^{1/2}$$

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Approximate:

$$\binom{2}{2} + \cdots + \binom{k}{2} \sim \sum_{i=2}^{k} \frac{i^2}{2} \sim \frac{k^3}{6}$$

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$$\frac{k^3}{6} \geq n$$

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$$\binom{e-1}{e-1} + \cdots + \binom{k}{e-1} = \sum_{i=e-1}^{k} \frac{i^{e-1}}{(e-1)!} \sim \frac{k^e}{e!}$$

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$$\frac{k^e}{e!} \geq n$$

$$k \geq (e!)^e n^{1/e}$$