The Problem

Alice is A, Bob is B, Carol is C.
The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length $n$ on their foreheads.
The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length $n$ on their foreheads.
2. Foreheads: A’s has $a$; B’s has $b$; C’s has $c$.
The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length $n$ on their foreheads.
2. Foreheads: A’s has $a$; B’s has $b$; C’s has $c$.
3. A knows $b, c$; B knows $a, c$; C knows $a, b$. 

4. They want to know if $a + b + c = 2^n + 1 - 1$.

5. Solution
   A says $b$, B then computes $a + b + c$ and then says YES if $a + b + c = 2^n + 1 - 1$, NO if not.

6. Solution uses $n + 1$ bits of comm. Can do better?
The Problem

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Solution

A says $b$, B then computes $a + b + c$ and then says YES if $a + b + c = 2^{n+1} - 1$, NO if not.

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The Problem

Alice is A, Bob is B, Carol is C.

1. A, B, and C have a string of length \( n \) on their foreheads.
2. Foreheads: A’s has \( a \); B’s has \( b \); C’s has \( c \).
3. A knows \( b, c \); B knows \( a, c \); C knows \( a, b \).
4. They want to know if \( a + b + c = 2^{n+1} - 1 \).
5. **Solution** A says \( b \), B then computes \( a + b + c \) and then says YES if \( a + b + c = 2^{n+1} - 1 \), NO if not.
The Problem

Alice is A, Bob is B, Carol is C.

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3. A knows $b, c$; B knows $a, c$; C knows $a, b$.
4. They want to know if $a + b + c = 2^{n+1} - 1$.
5. **Solution** A says $b$, B then computes $a + b + c$ and then says YES if $a + b + c = 2^{n+1} - 1$, NO if not.
6. **Solution** uses $n + 1$ bits of comm. Can do better?
1. Any protocol requires \( n + 1 \) bits, hence the one that takes \( n + 1 \) is the best you can do.

2. There is a protocol that takes \( \alpha n \) bits for some \( \alpha < 1 \) but any protocol requires \( \Omega(n) \) bits.

3. There is a protocol that takes \( \ll n \) bits.
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**STUDENTS WORK IN GROUPS TO BEAT $n + 1$ OR SHOW YOU CAN’T**
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. $A: a_0 \cdots a_{n-1}$, $B: b_0 \cdots b_{n-1}$, $C: c_0 \cdots c_{n-1}$. 
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.
2. A says: $c_0 \oplus b_{n/2}, \cdots, c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.
2. A says: $c_0 \oplus b_{n/2}, \cdots, c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.
3. Bob knows $c_i$'s so he now knows $b_{n/2}, \cdots, b_{n-1}$. 
4. Carol knows $b_i$'s so she now knows $c_0, \cdots, c_{n/2-1}$.

Carol knows the carry bit $z$ so she can compute $a_0 \cdots a_{n/2} + b_0 \cdots b_{n/2} + c_0 \cdots c_{n/2} + z = t$.
$t = 1$: Carol says YES.
$t \neq 1$: Carol says NO.
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.

2. A says: $c_0 \oplus b_{n/2}$, $c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.

3. Bob knows $c_i$'s so he now knows $b_{n/2}, \ldots, b_{n-1}$.
   Bob knows $a_i$'s and $c_i$'s so he can compute
   
   $$a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$$
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

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   Bob knows $a_i$'s and $c_i$'s so he can compute
   $a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$
   $s = 1^{n/2}$: Bob says (MAYBE,$z$). $s \neq 1^{n/2}$: Bob says NO.
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.

2. A says: $c_0 \oplus b_{n/2}, \cdots, c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.

3. Bob knows $c_i$’s so he now knows $b_{n/2}, \ldots, b_{n-1}$.
   Bob knows $a_i$’s and $c_i$’s so he can compute
   $a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$
   $s = 1^{n/2}$: Bob says (MAYBE, $z$). $s \neq 1^{n/2}$: Bob says NO.

4. Carol knows $b_i$’s so she now knows $c_0, \ldots, c_{n/2-1}$. 


Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.

2. A says: $c_0 \oplus b_{n/2}$, $\cdots$, $c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.

3. Bob knows $c_i$’s so he now knows $b_{n/2}, \ldots, b_{n-1}$.
   Bob knows $a_i$’s and $c_i$’s so he can compute
   
   $a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$

   $s = 1^{n/2}$: Bob says (MAYBE, z). $s \neq 1^{n/2}$: Bob says NO.

4. Carol knows $b_i$’s so she now knows $c_0, \ldots, c_{n/2-1}$.
   Carol knows the carry bit $z$ so she can compute
   
   $a_0 \cdots a_{n/2} + b_0 \cdots b_{n/2} + c_0 \cdots c_{n/2} + z = t$
Protocol in $\frac{n}{2} + O(1)$ bits Due to Dean Foster

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$.

2. A says: $c_0 \oplus b_{n/2}$, $\cdots$, $c_{n/2-1} \oplus b_{n-1}$. $n/2$ bits.

3. Bob knows $c_i$'s so he now knows $b_{n/2}, \ldots, b_{n-1}$.
Bob knows $a_i$'s and $c_i$'s so he can compute
$a_{n/2} \cdots a_{n-1} + b_{n/2} \cdots b_{n-1} + c_{n/2} \cdots c_{n-1} = s + \text{carry } z$
$s = 1^{n/2}$: Bob says (MAYBE, z). $s \neq 1^{n/2}$: Bob says NO.

4. Carol knows $b_i$'s so she now knows $c_0, \ldots, c_{n/2-1}$.
Carol knows the carry bit $z$ so she can compute
$a_0 \cdots a_{n/2} + b_0 \cdots b_{n/2} + c_0 \cdots c_{n/2} + z = t$
t $t = 1^{n/2}$: Carol says YES. $t \neq 1^{n/2}$: Carol says NO.
Alice is A, Bob is B, Carol is C, Donna is D.
Four People

Alice is A, Bob is B, Carol is C, Donna is D.

1. A, B, C, D have a string of length $n$ on their foreheads.
Alice is A, Bob is B, Carol is C, Donna is D.

1. A, B, C, D have a string of length $n$ on their foreheads.
2. A’s forehead has $a$, B’s forehead has $b$, . . .
Four People

Alice is A, Bob is B, Carol is C, Donna is D.

1. A, B, C, D have a string of length $n$ on their foreheads.
2. A's forehead has $a$, B's forehead has $b$, ... .
3. They want to know if $a + b + c + d = 2^{n+1} - 1$. 

Obvious Solution uses $n+1$ bits of comm. Can do better?

STUDENTS WORK IN GROUPS TO EITHER DO BETTER THAN $n+1$ OR SHOW YOU CAN'T
Four People

Alice is A, Bob is B, Carol is C, Donna is D.

1. A, B, C, D have a string of length $n$ on their foreheads.
2. A’s forehead has $a$, B’s forehead has $b$, . . . .
3. They want to know if $a + b + c + d = 2^{n+1} - 1$.
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Four People

Alice is A, Bob is B, Carol is C, Donna is D.

1. A, B, C, D have a string of length $n$ on their foreheads.
2. A’s forehead has $a$, B’s forehead has $b$, . . .
3. They want to know if $a + b + c + d = 2^{n+1} - 1$.
4. **Obvious Solution** uses $n + 1$ bits of comm. Can do better?

**STUDENTS WORK IN GROUPS TO EITHER DO BETTER THAN $n + 1$ OR SHOW YOU CAN’T**
Protocol for 4 in $\frac{n}{3} + O(1)$ bits
Protocol for 4 in $\frac{n}{3} + O(1)$ bits

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$, D: $d_0 \cdots d_{n-1}$. 

2. A says: $c_0 \oplus b_{n/3-1} \oplus d_{2n/3-1} \cdots c_{n/3-1} \oplus b_{2n/3-1} \oplus d_{n-1}$.

3. Carol can add first 1/3 of the bits, sees if it's $1_{n/3}$, if it's not say NO, if it is say MAYBE and the carry bit.

4. Bob can add second 1/3 of the bits along with the carry bit, sees if it's $1_{n/3}$, if it's not say NO, if it is say MAYBE and the carry bit.

5. Donna can add third 1/3 of the bits along with the carry bit, sees if it's $1_{n/3}$, if it's not say NO, if it is say YES.
Protocol for $4\text{ in } \frac{n}{3} + O(1) \text{ bits}$

1. $A:a_0 \cdots a_{n-1}$, $B:b_0 \cdots b_{n-1}$, $C:c_0 \cdots c_{n-1}$, $D:d_0 \cdots d_{n-1}$.

2. $A$ says: $c_0 \oplus b_{n/3-1} \oplus d_{2n/3-1}$, $\cdots$, $c_{n/3-1} \oplus b_{2n/3-1} \oplus d_{n-1}$.
Protocol for 4 in $\frac{n}{3} + O(1)$ bits

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$, D: $d_0 \cdots d_{n-1}$.

2. A says: $c_0 \oplus b_{n/3-1} \oplus d_{2n/3-1}, \cdots, c_{n/3-1} \oplus b_{2n/3-1} \oplus d_{n-1}$.

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Protocol for 4 in $\frac{n}{3} + O(1)$ bits

1. A: $a_0 \cdots a_{n-1}$, B: $b_0 \cdots b_{n-1}$, C: $c_0 \cdots c_{n-1}$, D: $d_0 \cdots d_{n-1}$.

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3. Carol can add first 1/3 of the bits, sees if its $1^{n/3}$, if its not say NO, if it is say MAYBE and the carry bit.

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5. Donna can add third 1/3 of the bits along with the carry bit, sees if its $1^{n/3}$, if its not say NO, if it is say YES.
People are $A_1, \ldots, A_k$. 

1. $A_i$ has a string of length $n$ on their foreheads.
2. $A_i$'s forehead has a $a_i$.
3. They want to know if $a_1 + \cdots + a_k = 2^n + 1 - 1$.
4. Can do in $n^k - 1 + O(1)$ bits, similar to the $k=3,4$ cases.
5. Caveat: The $O(1)$ term is really $O(k)$ which matters if $k$ is a function of $n$. 

$k$ People
People are $A_1, \ldots, A_k$.

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$k$ People

People are $A_1, \ldots, A_k$.

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2. $A_i$'s forehead has $a_i$
3. They want to know if $a_1 + \cdots + a_k = 2^{n+1} - 1$.
4. Can do in $\frac{n}{k-1} + O(1)$ bits, similar to the $k = 3, 4$ cases.
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Final Vote and the Answer

Let's go back to 3 people. We know we can do $\frac{n}{2} + O(1)$. 
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1. $\frac{n}{2} + O(1)$ is roughly optimal.
Final Vote and the Answer

Lets go back to 3 people. We know we can do $\frac{n}{2} + O(1)$.

1. $\frac{n}{2} + O(1)$ is roughly optimal.
2. There is an $O\left(\frac{n}{\log n}\right)$ protocol and it is roughly optimal.
Final Vote and the Answer

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1. $\frac{n}{2} + O(1)$ is roughly optimal.
2. There is an $O\left(\frac{n}{\log n}\right)$ protocol and it is roughly optimal.
3. There is an $O\left(\frac{n}{\log n}\right)$ protocol, optimal UNKNOWN.
Final Vote and the Answer

Lets go back to 3 people. We know we can do $\frac{n}{2} + O(1)$.

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2. There is an $O\left(\frac{n}{\log n}\right)$ protocol and it is roughly optimal.
3. There is an $O\left(\frac{n}{\log n}\right)$ protocol, optimal UNKNOWN.
4. There exists an $O(n^{1/2})$ protocol and it is roughly optimal.
Final Vote and the Answer

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VOTE!
$k$ people Answer and Some History

3 people:

Chandra-Furst-Lipton (CFL) (1983): $(\frac{1}{2})$ protocol. [Source](https://www.cs.umd.edu/~gasarch/TOPICS/ramsey/mpp.pdf), [Source](https://www.cs.umd.edu/~gasarch/TOPICS/ramsey/expositionofCFG.pdf). They needed this lemma to get lower bounds in computer science. Better lower bounds were later proven using easier techniques. However, by then The Forehead Problem had taken on a life of its own.

CFL showed lower bound $\Omega(1)$.

Gasarch (2006): Lower Bound $\Omega(\log \log n)$.
3 people:

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k people Answer and Some History

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- CFL: constructive and did not have the constants. Linial and Shraibman: explicitly protocol that uses $n^{1/2} + o(n^{1/2})$ bits. See [https://arxiv.org/pdf/2102.00421.pdf]
$k$ people Answer and Some History

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- CFL showed lower bound $\Omega(1)$.

- Gasarch (2006): Lower Bound $\Omega(\log \log n)$. 
$k$ people Answer and Some History

Gasarch 2006: there is an $O\left(\frac{1}{\log^2 (k - 1)}\right)$ protocol. (A more careful analysis of CFL protocol.)

CFL lower bound $\Omega(1)$.

Nothing else is known.
$k$ people Answer and Some History

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CFL lower bound $\Omega(1)$.

Nothing else is known.
Open Problem

For 3 people we have:

1. Elementary proof: Protocol $n^2 + O(1)$.
2. Hard proof: Protocol $O(n^{1/2})$. 

Open. Find an elementary proof for a protocol, $n^2 + O(1)$.

Open. Similar questions for 4 people, 5 people, etc.
Open Problem

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Open Problem

For 3 people we have:

1. Elementary proof: Protocol $\frac{n}{2} + O(1)$.
2. Hard proof: Protocol $O(n^{1/2})$.

Open Find an elementary proof for a protocol, $< \frac{n}{2} + O(1)$.
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