Hat Problem: People Standing in a Line

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The Set Up

100 people working together as a team, must stand in a line. Each person can see the heads of everyone in front of her, but not her own head, or the heads of those in back of her. BEFORE hats are placed (the next step) they can discuss strategy; however, the adversary listens in on that conversation.

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The Contestants Move: After the hats have been placed, each contestant, in turn starting from the back of the line and proceeding one by one to the front of the line, will call out one of the two colors, **red** or **blue**. Their goal is to get as many people as possible to correctly call out their own hat color.

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Work on the Following in Groups

n people. 2 hat colors:

1. Is there a strategy that is guaranteed to get MORE THAN $n/2$ hats correct?
2. What is the best they can do?
3. If finish early work on 3 colors, 4 colors, etc.

n people, 2 Hat Colors, Several Answers

p_i is person i.

1. For all $1 \le i \le n/3$
   p_{3i} says **R** if p_{3i+1}, p_{3i+2} are same, **B** otherwise. p_{3i+1} can deduce his color, then p_{3i+2} can deduce her color.

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2. $p_1, p_2, \ldots, p_{\lg_2 n}$ spell out in binary the **number of red hats** among $p_{\lg_2 n+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances.

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4. BILL- TELL the Story!

More Hat Colors!

What if n people, 3 hats colors? 4 ? c?

(If you finish early than look at an infinite number of people and 2 hat colors.)

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Let s_j be what p_j says. p_i can deduce that

$$h_i \equiv s_1 - \sum_{j=2}^{i-1} s_j \pmod 3$$

Infinite Number of People!

Infinite number of people and 2 colors of hats.

Want a protocol such that all but a finite number get it right.

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They all end up collectively saying REP, which is only a finite number of hats away from the real answer.

Can They Do Better?

Vote

1. There is a protocol and a constant C so that the protocol always results in $\le C$ hats wrong, and this is known.
2. For all protocols and all constant C there is a way for the adversary to put hats on peoples heads so that the protocol gets $\ge C$ wrong, and this is known.
3. The question

    Is there a protocol and a C such that BLAH BLAH

    is independent of ZFC.
4. Which of 1,2, or 3 happens is **Unknown to Science**.

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Work on it in small groups.

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   (From what p_1 said)

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The only one who might get it wrong is p_1.