Hat Problem: People Standing in a Line

William Gasarch-U of MD
The Set Up

100 people working together as a team, must stand in a line. Each person can see the heads of everyone in front of her, but not her own head, or the heads of those in back of her. BEFORE hats are placed (the next step) they can discuss strategy; however, the adversary listens in on that conversation.
The Set Up

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The Adversary’s Move: The Adversary places either a red hat or a blue hat on top of each contestant’s head. The contestants cannot communicate at all except as specified in the next step.
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The Adversary’s Move: The Adversary places either a red hat or a blue hat on top of each contestant’s head. The contestants cannot communicate at all except as specified in the next step.

The Contestants Move: After the hats have been placed, each contestant, in turn starting from the back of the line and proceeding one by one to the front of the line, will call out one of the two colors, red or blue. Their goal is to get as many people as possible to correctly call out their own hat color.
Work on the Following in Groups

$n$ people. 2 hat colors:

1. Is there a strategy that is guaranteed to get at least $n/2$ hats correct? (YES)

2. Is there a strategy that is guaranteed to get MORE THAN $n/2$ hats correct?

3. What is the best they can do? (If finish early work on 3 colors, 4 colors, etc.)
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$. 

Optimal!
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. $p_i$ says the majority color, they all say that color.
$n$ people, 2 Hat Colors, Several Answers

$p_i$ is person $i$.

1. $p_i$ says the majority color, they all say that color. $n/2$. 

2. 

3. 

4. $p_1, p_2, \ldots, p_{\log_2(n)}$ spell out in binary the number of red hats among $p_{\log_2(n)+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \log_2(n)$. 

5. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances.

6. BILL - TELL the Story!
\[ n \text{ people, 2 Hat Colors, Several Answers} \]

\( p_i \) is person \( i \).

1. \( p_i \) says the majority color, they all say that color. \( n/2 \).
2. \( p_{2x+1} \) says \( p_{2x+2} \)'s color. \( p_{2x+2} \) says her color.
$n$ people, 2 Hat Colors, Several Answers

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3. $p_{3x}$ says $R$ if $p_{3x+1}, p_{3x+2}$ are same, $B$ otherwise. $p_{3x+1}$ can deduce his color, then $p_{3x+2}$ can deduce her color.

Optimal! BILL TELL the Story!
\( n \) people, 2 Hat Colors, Several Answers

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BILL - TELL the Story!
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3. $p_{3x}$ says $R$ if $p_{3x+1}, p_{3x+2}$ are same, $B$ otherwise. $p_{3x+1}$ can deduce his color, then $p_{3x+2}$ can deduce her color. $2n/3$.
4. $p_1, p_2, \ldots, p_{\lfloor \log_2 n \rfloor}$ spell out in binary the number of red hats among $p_{\lfloor \log_2 n \rfloor+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances.
$n$ people, 2 Hat Colors, Several Answers

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4. $p_1, p_2, \ldots, p_{\lg n}$ spell out in binary the number of red hats among $p_{\lg n+1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \lg(n)$.

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2. $p_{2x+1}$ says $p_{2x+2}$'s color. $p_{2x+2}$ says her color. $n/2$.
3. $p_{3x}$ says R if $p_{3x+1}$, $p_{3x+2}$ are same, B otherwise. $p_{3x+1}$ can deduce his color, then $p_{3x+2}$ can deduce her color. $2n/3$.
4. $p_1$, $p_2$, \ldots, $p_{\lg n}$ spell out in binary the number of red hats among $p_{\lg n+1}$, \ldots, $p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \lg(n)$.
5. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances.
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4. $p_1, p_2, \ldots, p_{\lfloor \log_2 n \rfloor}$ spell out in binary the number of red hats among $p_{\lfloor \log_2 n \rfloor + 1}, \ldots, p_n$. Each person can deduce their color based on the number and the prior utterances. $n - \log(n)$.
5. $p_1$ says red if the number of red hats she sees is even, blue otherwise. Each person can deduce their color based on the number and the prior utterances. $n - 1$.

Optimal!
$n$ people, 2 Hat Colors, Several Answers

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6. BILL- TELL the Story!
More Hat Colors!

What if $n$ people, 3 hats colors? 4 ? $c$?

(If you finish early than look at an infinite number of people and 2 hat colors.)
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$. 

Rephrase.

For 3 colors:

$p_1$ says

$$n \sum_{i} h_i \equiv 0 \pmod{3}$$ 

Let $s_j$ be what $p_j$ says. $p_i$ can deduce that $h_i \equiv s_1 - i - 1 \sum_{j} s_j \pmod{3}$.

For $c$ color replace 3 with $c$. 
*n* people, 3 Hat Colors Answer

\( p_i \) is person \( i \).

\( p_1 : \text{red if the numb of reds is even, blue otherwise} \)
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: red if the numb of reds is even, blue otherwise

Rephrase. red is 0, blue is 1, $h_i$ is hat on $p_i$.

$$p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{2}$$
\(n\) people, 3 Hat Colors Answer

\(p_i\) is person \(i\).

\(p_1:\) red if the numb of reds is even, blue otherwise

Rephrase. red is 0, blue is 1, \(h_i\) is hat on \(p_i\).

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For 3 colors:

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p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{3}
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$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: **red** if the number of reds is even, **blue** otherwise

Rephrase. **red** is 0, **blue** is 1, $h_i$ is hat on $p_i$.

For $3$ colors:

$$p_1 \text{ says } \sum_{i=2}^{n} h_i \pmod{3}$$

Let $s_j$ be what $p_j$ says. $p_i$ can deduce that

$$h_i \equiv s_1 - \sum_{j=2}^{i-1} s_j \pmod{3}$$
$n$ people, 3 Hat Colors Answer

$p_i$ is person $i$.

$p_1$: **red** if the numb of **reds** is even, **blue** otherwise

Rephrase. **red** is 0, **blue** is 1, $h_i$ is hat on $p_i$.

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\[ h_i \equiv s_1 - \sum_{j=2}^{i-1} s_j \pmod{3} \]

For $c$ color replace 3 with $c$. 
Infinite Number of People!

Infinite number of people and 2 colors of hats.

Want a protocol such that all but a finite number get it right.
Infinite Number of People! 2 Hat Colors

People are $p_1, p_2, \ldots$
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They meet ahead of time. Let $H = \{R, B\}^\omega$. 

1. (Preprocess) $p_i$'s pick a representative from each part.
2. Each $p_i$ sees all but a finite number of hats. So they know which part they are in. Call representative of the part, REP.
3. Each $p_i$ says the color in the $i$th position in REP.

They all end up collectively saying REP, which is only a finite number of hats away from the real answer.
People are $p_1, p_2, \ldots$.
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They define
\[ x \equiv y \text{ if } x \text{ and } y \text{ differ only finitely often} \]
Infinite Number of People! 2 Hat Colors

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$\equiv$ is an equiv rel, so a partition. Every $x \in H$ is in one part.
Infinite Number of People! 2 Hat Colors

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1. (Preprocess) \( p_i \)'s pick a REPRESENTATIVE from each part.
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Infinite Number of People! 2 Hat Colors

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They all end up collectively saying REP, which is only a finite number of hats away from the real answer.
Can They Do Better?

Vote

1. There is a protocol and a constant $C$ so that the protocol always results in $\leq C$ hats wrong, and this is known.
2. For all protocols and all constant $C$ there is a way for the adversary to put hats on peoples heads so that the protocol gets $\geq c$ wrong, and this is known.
3. The question
   Is there a protocol and a $C$ such that BLAH BLAH is independent of ZFC.
4. Which of 1, 2, or 3 happens is Unknown to Science.
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Work on it in small groups.
Protocol that gets \( \leq 1 \) Wrong

1. \( p_1 \) determines REP. He says:
Protocol that gets $\leq 1$ Wrong

1. $p_1$ determines REP. He says:

   - $R$ if REP and $h_2, \ldots$ Differ In An Odd Number of Places

   - $B$ if REP and $h_2, \ldots$ Differ In An Even Number of Places

2. $p_2$ knows parity of how much $h_2, \ldots$ differs from REP (From what $p_1$ said)

   - $p_2$ knows parity of how much $h_3, \ldots$ differs from REP (She sees)

   - Hence she can deduce $h_2$.

3. Similar for all $p_i$ with $i \geq 2$.

Infinite people, chat colors.

Leave it to you.
Protocol that gets $\leq 1$ Wrong

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   (From what $p_1$ said)

   $p_2$ knows parity of how much $h_3, \ldots$, differs from REP
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   hence she can deduce $h_2$. 
Protocol that gets \( \leq 1 \) Wrong

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3. Similar for all \( p_i \) with \( i \geq 2 \).
Protocol that gets $\leq 1$ Wrong

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Infinite people, $c$ hat colors.
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Infinite people, $c$ hat colors. Leave it to you.