Regular Graph Properties
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1 Introduction

Consider the following question:

Is the set of Hamiltonian graphs regular?

To pose this question properly we need to specify (a) which strings represent graphs, (b) what do to if a string that does not represent a graph is input.

Def 1.1 All strings are over the alphabet $\{0, 1, \$\}$. Let x be a string of the form $\$x_1\$x_2\$ \cdots \$x_n\$$ where the following happen:

1. $(\forall i)[x_i \in \{0, 1\}^n]$. We will let

$$x_i = x_{i1} \cdots x_{in}.$$

2. View the string x as the following matrix:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

3. For all $1 \le i < j \le n$, $x_{ij} = x_{ji}$.

4. For all $1 \le i \le n$, $x_{ii} = 0$.

Any string of the form above is interpreted as the adjacency matrix of a graph.

We identify a graph G with the string that is its adjacency matrix, as above. Hence we will say things like *Run DFA M on G.*

Example 1.2 The graph K_4 is the string

$$\$0111\$1011\$1101\$1110\$$$

We will never express a graph that way. We will instead write down the matrix. In this case

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Def 1.3 Let $\mathcal{G}$ be a set of graphs.

1. Let $\mathcal{G}$ is a *graph property* if $\mathcal{G}$ satisfies the following: for all pairs of graphs, (G_1, G_2), if G_1 and G_2 are isomorphic then either $G_1, G_2 \in \mathcal{G}$ or $G_1, G_2 \notin \mathcal{G}$.

2. A graph property $\mathcal{G}$ is *regular* if there exists a DFA M such that the following hold:

   (a) If $G \in \mathcal{G}$ then $M(G)$ accepts.

   (b) If $G \notin \mathcal{G}$ then $M(G)$ rejects.

   (c) If w is a string that does not represent an adjacency matrix then we have *no condition* on what $M(w)$ is.

2 Graph Properties that are Regular

Def 2.1

1. If $w \in \{0,1\}^*$ then $\#_1(w)$ is the number of 1's in w.

2. Let $A \subseteq \mathsf{N}$. A is *regular* if the following set is regular:

$$\{w \in \{0,1\}^* : \#_1(w) \in A\}$$

   is regular.

Theorem 2.2 *Let $A \subseteq \mathsf{N}$ be regular.*

1. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : (\forall v \in V)[|\deg(v)| \in A]\}.$$

2. *The following graph property is regular:*

$$\mathcal{G} = \{G = (V, E) : |E| \in A\}.$$

Proof:
1) Let $W = \{w \in \{0,1\}^* : \#_1(w) \in A\}$. W is regular by the definition of $A \subseteq \mathsf{N}$ being regular. Let α be the regular expression such that $L(W) = \alpha$.

A graph G is in $\mathcal{G}$ iff every row of its adjacency matrix is in L. Consider the regular expression

$$\beta = \$(\alpha\$)^*$$

It is easy to see that
$G \in \mathcal{G}$ implies $G \in L(\beta)$.
$G \notin \mathcal{G}$ implies $G \notin L(\beta)$.

2) We leave this to the reader.
∎

Corollary 2.3 *Let $d, m \in 2$. The following graph properties are regular.*

1. *The set of graphs where $(\forall v)[\deg(v) \geq d]$.*
2. *The set of graphs where $(\forall v)[\deg(v) = d]$.*
3. *The set of graphs where $(\forall v)[\deg(v) \leq d]$.*
4. *Let $A \subseteq \{0, 1, \ldots, d-1\}$. The set of graphs where*

$$\{\deg(v) \pmod{m} : v \in V\} \subseteq A\}.$$

5. *The set of Eulerian graphs. (This is the $m = 2$, $A = \{0\}$ case.)*
6. *The set of graphs $G = (V, E)$ such that $|E| \equiv d \pmod{m}$.*

Open Problem 2.4 Aside from Eulerian graphs and (arguably) the set of graphs of constant degree, are there any other *interesting* graph properties that are regular.

3 Graph Properties that are not Regular

3.1 Graphs that are the Union of Two Isomorphic Graphs

Theorem 3.1 *Let $\mathcal{G}$ be the set of graphs that are the union of two isomorphic graphs. Then $\mathcal{G}$ is not regular.*

Proof:

Assume, by way of contradiction, that $\mathcal{G}$ is regular via DFA M which has s states. Let n be a large even number to be picked later.

Here is an example of what we plan to do. Consider the partial graphs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(The partial graph is vertex 1 and one neighbor which is $\{2\}$.)

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The parital graph is vertex 1 and two neighbors which is $\{2, 3\}$.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The partial graph is vertex 1 and three neighbors which is $\{2, 3, 4\}$.

Say we fed these three graphs into the DFA and two of them ended in the same state q, say the first and third. Now look at the following inputs

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The graph is vertex 1 and one neighbor which is $\{2\}$, and vertex 5 and one neighbor which is $\{6\}$. Hence this graph is in $\mathcal{G}$.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The graph is vertex 1 and three neighbors which is $\{2, 3, 4\}$, and vertex 5 and one neighbor which is $\{6\}$. Hence this graph is not in $\mathcal{G}$.

Both of these graphs go to the same state since the first 4 rows of both of them end up in state q and the next four rows are the same for each graph.

Now of course there is no guarantee that two of these partial graphs will end up in the same state. But if we take a large enough number of vertices then it will be guranteed that two of the partial graphs go to the same state.

Now onto the formal proof.

Let n be a large even number to be picked later. Feed the following partial graphs which are the first $n/2$ rows into M.

1. The only neighbor of 1 is 2.

2. The neighbors of 1 are 2,3. $\vdots$

3. The neighbors of 1 are $2, 3, \ldots, n/2$.

Take n such that $n/2 \geq s+1$. Hence there are (at least) two partial graphs H_1 and H_2 that map to the same state q. Let d_1 and d_2 be such that H_1 is vertex 1 with neighbors $\{1, \ldots, d_1\}$ and H_2 is vertex 1 with neighbors $\{1, \ldots, d_2\}$.

Let H_3 be the partial graph where vertex $n/2$ has neighbors $\{n/2 + 1, \ldots, n/2 + d_1\}$.

Input to M the graph which is H_1 followed by H_3. Note that this graph is in $\mathcal{G}$. Input to M the graph which is H_2 followed by H_3. Note that this graph is not in $\mathcal{G}$. However, they will end up in the same state. This is a contradiction. ∎