We present and prove a generalization of Oddtown that will be helpful for obtaining better lower bounds on $\chi(\mathsf{E}^n)$ than Oddtown enabled us to obtain.

Let p be a prime. Let $L \subseteq \{0, \ldots, p-1\}$. Imagine a town of n people where there are m clubs $w_1, \ldots, w_n$. Let w_i, w_j be two of the clubs. The number of people in w_i mod p is not in L. The number of people in both w_i and w_j mod p is in L. (This is one weird down!) Does this impose any condition on m, n?

We give some examples

Example 1 $p = 5$, $n = 4$, $L = \{0, 1\}$, $m = 6$. The clubs are $\binom{[6]}{2}$. If A is a club then $|A| = 2$, so $|A| \pmod 5 \notin L$. If A, B are clubs then $|A \cap B| \in \{0, 1\}$ so $|A \cap B| \pmod 5 \in L$. (The mod had no effect here.)

Note that we have $n < m$. So the simple condition from Oddtown, $m \leq n$, will not hold here. However $m - n = 2$. Are bigger gaps possible?

Example 2 $p = 7$, $n = 6$, $L = \{2, 3\}$, $m = 15$. The clubs are $\binom{[6]}{4}$. If A is a club then $|A| = 4$, so $|A| \pmod 7 \notin L$. If A, B are clubs then $|A \cap B| \in \{2, 3\}$ so $|A \cap B| \pmod 7 \in L$. (The mod had no effect here.)

Note that we have $n < m$ with $m - n = 9$. Can the additive gap be arbitrary large?

Example 3 $p = 11$, $n = 9$, $L = \{0, 1, 2\}$, $m = \binom{5}{3} + 1 = 11$. The clubs are $\binom{[5]}{3} \cup \{6, 7, 8, 9\}$. We include this example since not all clubs are the same size. If A is a club then $|A| \in \{3, 4\}$, so $|A| \pmod{11} \notin L$. If A, B are clubs then $|A \cap B| \in \{0, 1, 2\}$ so $|A \cap B| \pmod{11} \in L$. (The mod had no effect here.)

$n < m$ with $m - n = 2$.

Example 4 Let $p = 5$, $n = 1000$, $L = \{0, 1\}$, $m = \binom{1000}{2}$. The clubs are $\binom{[1000]}{2}$. If $A \in \binom{[1000]}{2}$ then $|A| = 2$ hence $|A| \pmod 5 \notin L$. If $A, B \in \binom{[1000]}{2}$ then $|A \cap B| \in \{0, 1\}$ hence $|A \cap B| \pmod 5 \in L$.

$n < m$ by a large margin. In fact m is roughly $\frac{n^2}{2}$. What is the function f such that $m \leq f(n)$?

Example 5 Let $p = 5$, $n = 19$, $L = \{1, 2\}$, and $m = \binom{10}{8} = 45$. The clubs are $\binom{[10]}{8}$. We include this example since, unlike the other examples, the mod p matters.

If $A \in \binom{[10]}{8}$ then $|A| = 8$ so $|A| \pmod 5 = 3 \notin L$. If $A, B \in \binom{[10]}{8}$ then $|A \cap B| \in \{6, 7\}$ hence $|A \cap B| \pmod 5 \in \{1, 2\} = L$.

Exercise For each of these either find the sequence or show that none exists.

1. Find an infinite sequence $(n_i, p_i, L_i, m_i, S_i)$ (S_i is a set of m_i subsets of $[n]$) such that (1) $n_1 < n_2 < \cdots$, (2) the condition is satisfied, and (3) there exists c such that $m_i \geq cn_i^2$.
2. Do the problem above but with $n_i \geq cn_i^3$ or higher.
3. Do the problem above but with $n_i \geq 2^{cn_i}$. (Warning- this problem might be open.)
4. Try to formulate and prove a theorem that gives a bound on m. (This is next so if you are going to attempt it, do so before looking at the next theorem.)