

Constructive Ramsey Lower Bounds AND Chromatic Number of $\mathbb{R}^n$!

Exposition by
William Gasarch-U of MD
and
Neal Shandilya-Howard HS

Credit Where Credit is Due

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- 3) A great set of slides by Kelin Zhu.
(An undergraduate working with William Gasarch.)

Easy But Weak Lower Bounds on $R(k)$

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Lemma Needed For $R(k) = \Omega(k^{\log_4 17})$

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Exercise Use the last exercise to obtain a better lower bound than $R(k) = \Omega(k^{\log_4 17})$.

Two Combinatorial Lemmas

**We Will Prove Two
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If I have $w_1, \dots, w_m$ then these are all distinct.

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Next slide is examples.

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Rest of proof on next slide. (So write down theorem.)

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Case 4 $1 \leq \lambda < v$. Next slide.

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Proof continued on next slide.

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End of Proof

Constructive Lower Bounds on $R(k)$

We Will Show
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Case 2 on next slide.

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Exercise Show that our proof that $R(k) > \binom{k-1}{3}$ is constructive.

3) There is a nonconstructive proof that shows $R(k) = \Omega(k2^{k/2})$.

Chromatic Number of $\mathbb{R}^n$

We Will Show
 $\chi(\mathbb{R}^n) = \Omega(n^2)$

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We take a shortcut and write $5 \leq \chi(\mathbb{R}^2) \leq 7$.

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What is $\chi(\mathbb{R}^n)$? The next slide gives some information.

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4) The best known results asymptotically are

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$

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Notation We call that last theorem **Oddtown'**.

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We will use $c < \frac{(n-1)(n-2)}{6}$.

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Proof continued on next slide.

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Note For $p = 2$ and $L = \{0\}$, the generic bound is $m \leq n + 1$.
However, the polynomials have constant term $\prod_{\ell \in L} (-\ell) = 0$, so the sharper dimension count gives $m \leq n$, recovering Oddtown.
Note that, for all p , if $0 \in L$ then $m \leq (\sum_{a=0}^{|L|} \binom{n}{a}) - 1$.

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Key The set of multilinear polys of degree $\leq |L|$ is a vector space of dim $\sum_{a=0}^{|L|} \binom{n}{a}$. So need to show that the f_i 's are linearly independent.

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Since $x \in \{0, 1\}^n$ we can replace every power of x_i^t with x_i for $t \geq 2$. This gives a multilinear polynomial representing the same function on $\{0, 1\}^n$, of degree $\leq |L|$.

Key The set of multilinear polys of degree $\leq |L|$ is a vector space of dim $\sum_{a=0}^{|L|} \binom{n}{a}$. So need to show that the f_i 's are linearly independent.

Proof continued on next slide.

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Better Lower Bound on $R(k)$

We use Gen of ODD and FISH to get better Lower Bound on $R(k)$.

A Bound We Will Need

Lemma Let $n, p \in \mathbb{N}$ such that $n \geq 4p$. Then $\sum_{a=0}^{p-1} \binom{n}{a} < 2\binom{n}{p-1}$.

Proof Use $1 \leq a \leq p-1$ on the first inequality, and $n \geq 4p$ on the second inequality to get:

$$\frac{\binom{n}{a-1}}{\binom{n}{a}} = \frac{a}{n-a+1} \leq \frac{p-1}{n-p+2} \leq \frac{p-1}{4p-p+2} = \frac{p-1}{3p+2} \leq \frac{1}{3}.$$

Hence $\binom{n}{a-1} \leq \frac{1}{3}\binom{n}{a}$, so $\binom{n}{a} \leq \frac{1}{3^{p-a-1}}\binom{n}{p-1}$

Therefore

$$\sum_{a=0}^{p-1} \binom{n}{a} < \binom{n}{p-1} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \right) < 2\binom{n}{p-1}.$$

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$(\forall i)[|w_i| = p^2 - 1], \quad (\forall i, j)[|w_i \cap w_j| \in L]$.

By Generalized Fisher $m \leq \sum_{a=0}^{|L|} \binom{n}{a} = \sum_{a=0}^{p-1} \binom{n}{a}$.

By Lemma, and $n > 2p^2 \geq 4p$, we have $\sum_{a=0}^{p-1} \binom{n}{a} \leq 2\binom{n}{p-1} = k$.

Case 2

$$\text{COL}(x, y) = \begin{cases} \mathbf{R} & \text{if } |x \cap y| \not\equiv p-1 \pmod{p} \\ \mathbf{B} & \text{if } |x \cap y| \equiv p-1 \pmod{p} \end{cases} \quad (4)$$

Case 2 H is **B**. We show that $w_1, \dots, w_m$ satisfy the premise of Generalized Fisher.

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End of Proof

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Exercise For every fixed $d \in \mathbb{N}$, prove that $R(k)/k^d \rightarrow \infty$. Hint: choose a fixed prime $p > d$ and use an argument similar to ours.

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Better Lower Bound on $\chi(\mathbb{R}^n)$

**We use Gen of ODD to
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Notation We call the above Theorem Gen-Oddtown'.

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Proof continues on next slide.

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Recap

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Note Only used Gen. Oddtown.

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Better is known. See next slide.

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Question From Bill Frankl-Wilson uses prime powers, not primes.
Does this improve the results?