1 Introduction

in Chapter X we proved some combinatorial lemmas and then used them to obtain lower bounds on $\chi(\mathsf{E}^n)$. Those combinatorial lemmas, together with a few more combinatorial lemmas, can be used to get constructive lower bounds on the Ramsey numbres.

In this chapter we:

1. Define the Ramsey number $R(k)$.
2. Prove the finite Ramsey theory on graphs. This prove also gives an upper bound on $R(k)$.
3. Prove a lower bound on $R(k)$. This proof will be non-constructive.
4. Discuss some easy by weak constructive lower bounds on $R(k)$.
5. Prove a some combinatorial lemmas.
6. Use the combinatorial lemmas in Chapter X, together with the new combinatorial lemmas, to obtain better constructive lower bounds on $R(k)$.

2 Ramsey's Theorem for the Finite Complete Graphs

Definition 2.1

1. Let $n \in \mathsf{N}$. Let $K_n = ([n], \binom{[n]}{2})$ be the complete graph on n vertices. Let COL: $\binom{[n]}{2} \to [2]$. A set $H \subseteq [n]$ is a *homogenous set with respect to* COL (henceforth *homog set*) if the COL: $\binom{H}{2} \to [2]$ is constant. Alternatively, the vertices of H form a homog $K_{|H|}$.
2. Let $k \in \mathsf{N}$. Let n be the least natural number such that for all

Theorem 2.2 *For all k there exists n such that for every* COL: $\binom{[n]}{[2]} \to [2]$ *there is a homog set of size k. KEY: We can take $n = 2^{2k-1}$.*

Proof: Let COL: $\binom{[n]}{[2]} \to [2]$. We define an finite sequence of vertices,

$$x_1, x_2, \ldots, x_{2k-1}$$

and an finite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots, V_{2k-1}$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has $n-1$ edges coming out of it. Some are RED, and some are BLUE. Hence there are either $\geq \frac{n-1}{2}$ RED edges coming out of x_1, or there are

$\geq \frac{n-1}{2}$ BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has $\frac{n-1}{2}$ edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $\text{COL}(v, x_1) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= [n] \\ x_1 &= 1 \end{aligned}$$

$$\begin{aligned} c_1 &= \text{RED} \ \text{ if } |\{v \in V_0 \mid \text{COL}(v, x_1) = \text{RED}\}| \geq \tfrac{|V_0|-1}{2} \\ &= \text{BLUE} \ \text{ otherwise} \\ V_1 &= \{v \in V_0 \mid \text{COL}(v, x_1) = c_1\} \text{ (note that } |V_1| \geq \tfrac{|V_0|-1}{2}) \end{aligned}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \text{ the least number in } V_{i-1}$$

$$\begin{aligned} c_i &= \text{RED} \ \text{ if } |\{v \in V_{i-1} \mid \text{COL}(v, x_i) = \text{RED}\}| \geq \tfrac{|V_{i-1}|-1}{2} \\ &= \text{BLUE} \ \text{ otherwise} \\ V_i &= \{v \in V_{i-1} \mid \text{COL}(v, x_i) = c_i\} \text{ (note that } |V_i| \geq \tfrac{V_{i-1}}{2} \text{)} \end{aligned}$$

(NOTE- look at the step where we define c_i. We are using the fact that if you 2-color X you are guaranteed some color appears $|X|/2$ times. we are using the 1-hypergraph Ramsey Theorem. Later when we prove Ramsey on 3-hypergraphs we will use Ramsey on 2-hypergraphs.)

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We can show by induction that, for every i, $|V_i| \geq \frac{n}{2^i}$ (this is not quite right because of the -1 but we ignore this detail). Hence the sequence

$$x_1, x_2, \ldots$$

will go until V_i is empty. Since $|V_0| = n$ and at every stage the set is cut in half, this will go on for $\log_2(n)$ iterations. Hence we want $2k - 1 \geq \log_2(n)$ so we need $n = 2^{2k-1}$.

Consider the infinite sequence

$$c_1, c_2, \ldots, c_{2k-1}.$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be a a subset of k of them that are the same color

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

Denote this color by c, and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$$

We leave it to the reader to show that H is homog. ∎