1 Introduction

Notation 1.1 Let $n \in \mathsf{N}$, $n \geq 2$.

1. E^n is R^n with the Euclidean distance function $d(x, y)$.
2. Let G_n be the graph with vertex set E^n and edge set

$$\{\{x, y\} : d(x, y) = 1\}.$$

3. $\chi(\mathsf{E}^n)$ is the chromatic number of G_n.

In this document we prove the following:

- $\chi(\mathsf{E}^n) = \Omega(n^2)$.
- For every fixed d, $\chi(\mathsf{E}^n) = \Omega(n^d)$.
- For sufficiently large n, $\chi(\mathsf{E}^n) \geq (1.2)^n$.

This document draws heavily from a paper by Frankl and Wilson [1].
In this document we use the following conventions.

Convention 1.2

1. We will often have sets $w \subseteq [n]$. We will use w for both the set and its bit vector.
2. We use linear algebra with those vectors over F_2 or F_p or R.
3. In the expression $(\forall i, j)$ it is understood that $i \neq j$.
4. In the expression $(\exists i, j)$ it is understood that $i \neq j$.
5. In the expression $w_1, \ldots, w_m$, the w_i's are distinct (though not necessarily disjoint).

2 $\chi(\mathsf{E}^n) = \Omega(n^2)$

In this section we will prove $\chi(\mathsf{E}^n) = \Omega(n^2)$. This result is not impressive; however, the proof contains many of the ideas needed to obtain better lower bounds.

2.1 Oddtown

We present and prove a well-known theorem that will be helpful for obtaining lower bounds on $\chi(\mathsf{E}^n)$.

Imagine a town of n people where there are m clubs $w_1, \ldots, w_m$. Each club has an odd number of members. Every pair of distinct clubs has an even intersection. Does this impose any condition on n and m?

We give some examples:

Example 1 For all $1 \leq i \leq n$, $w_i = \{i\}$. Every club has 1 member (odd), and every intersection has 0 members (even). There can be at most n such clubs.

Example 2 $w_1 = \{1, 2, 3\}$, $w_2 = \{1, 2, 4\}$, ..., $w_m = \{1, 2, m+2\}$. Every club has 3 members (odd) and every intersection has 2 members (even). There can be at most $n - 2$ such clubs since each club has a different third member.

Exercise 1 Come up with more interesting examples. Note whether you ever get $m > n$. Spoiler alert: you won't.

Convention 2.1 Throughout this section, congruences are mod 2.

Theorem 2.2 *Let $w_1, \ldots, w_m$ be subsets of $[n]$ such that the following hold:*

- *For all i, $|w_i| \equiv 1$.*
- *For all $i < j$, $|w_i \cap w_j| \equiv 0$.*

Then $m \leq n$.

Proof:

View $w_1, \ldots, w_m$ as vectors in F_2^n.

Note that:

$w_i \cdot w_i \equiv |w_i| \equiv 1$.

$w_i \cdot w_j \equiv |w_i \cap w_j| \equiv 0$.

We show that $w_1, \ldots, w_m$ are linearly independent over F_2.

Assume $\alpha_1 w_1 + \cdots + \alpha_m w_m \equiv 0$. We show $(\forall i)[\alpha_i \equiv 0]$.

Dot both sides with w_i.

$\alpha_1(w_1 \cdot w_i) + \cdots + \alpha_m(w_m \cdot w_i) \equiv 0$.

For $i \neq j$, $w_i \cdot w_j \equiv 0$, so only the ith term remains.

$\alpha_i(w_i \cdot w_i) \equiv 0$.

Since $w_i \cdot w_i \equiv 1$:

$\alpha_i \equiv 0$.

So $(\forall i)[\alpha_i \equiv 0]$. Hence $w_1, \ldots, w_m$ are linearly independent.

Since $w_1, \ldots, w_m \in \mathsf{F}_2^n$ are linearly independent over F_2, $m \leq n$.

∎

We will need the following easy corollary to Oddtown.

Corollary 2.3

1. *Let $w_1, \ldots, w_m \subseteq [n]$ such that the following hold:*

   - $(\forall i)[|w_i| = 3]$.
   - $(\forall i, j)[|w_i \cap w_j| \in \{0, 2\}]$.

   Then $m \leq n$.

2. *Let $w_1, \ldots, w_m \subseteq [n]$ such that the following hold:*

   - $(\forall i)[|w_i| = 3]$.
   - $m > n$

   Then $(\exists i, j)[|w_i \cap w_j| = 1]$.

2.2 $\chi(\mathsf{E}^n) = \Omega(n^2)$

Theorem 2.4 *Let $c, n \in \mathsf{N}$ such that $c < \binom{n}{3}/n$.*

1. *For all colorings $\text{COL}\colon \mathsf{E}^n \to [c]$ there exist two points of the same color that are distance one apart.*

2. *$\chi(\mathsf{E}^n) = \Omega(n^2)$. This follows from Part 1.*

Proof:
1) Let c, n be as in the premise. Let $\text{COL}\colon \mathsf{E}^n \to [c]$.
For $A \in \binom{[n]}{3}$, $A = \{i, j, k\}$ let w_A be the following point in E^n:

$$(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0).$$

The $\frac{1}{2}$'s are at the ith, jth, and kth coordinates.
Let $W = \{w_A\colon A \in \binom{[n]}{3}\}$.
By the pigeonhole principle, there is an $X \subseteq \binom{[n]}{3}$ such that the following hold:

- $|X| \geq \binom{n}{3}/c > n$.
- For all $A \in X$, $|A| = 3$. (This is trivial.)
- All the points in $\{w_A\colon A \in X\}$ are the same color.

Since $|X|$ n and every $A \in X$ has size 3, Corollary 2.3.2 applies. Hence there exists $A, B \in \binom{[n]}{3}$ with $|A \cap B| = 1$. Look at w_A and w_B.

- There is 1 coordinate where both have $\frac{1}{2}$.
- There are 2 coordinates where w_A is $\frac{1}{2}$ and w_B is 0.
- There are 2 coordinates where w_B is $\frac{1}{2}$ and w_A is 0.
- In all remaining coordinates both have 0.

Hence

$$d(w_A, w_B) = 4(\frac{1}{2})^2 = 1$$

∎

3 For Every Fixed d, $\chi(\mathsf{E}^n) = \Omega(n^d)$

We want to improve the lower bound $\chi(\mathsf{E}^n) = \Omega(n^2)$. We do this by improving Oddtown.

3.1 Generalized Oddtown

We present and prove a generalization of Oddtown that will be helpful for obtaining better lower bounds on $\chi(\mathsf{E}^n)$ than Oddtown enabled us to obtain.

Let p be a prime. Let $L \subseteq \{0, \ldots, p-1\}$. Imagine a town of n people where there are m clubs $w_1, \ldots, w_m$. The size of each club, reduced mod p, is not in L. The size of the intersection of any two distinct clubs, reduced mod p, is in L. (This is one funky town!) Does this impose any condition on n and m?

We give some examples:

Example 1 $p = 5$, $n = 4$, $L = \{0, 1\}$, $m = 6$. The clubs are $\binom{[4]}{2}$. If A is a club then $|A| = 2$, so $|A| \pmod 5 \notin L$. If A and B are distinct clubs then $|A \cap B| \in \{0, 1\}$ so $|A \cap B| \pmod 5 \in L$. (The congruence conditions here are already implied by the ordinary intersection sizes.)

Note that we have $n < m$. So the simple condition from Oddtown, $m \leq n$, will not hold here. However $m - n = 2$. Are bigger gaps possible?

Example 2 $p = 7$, $n = 6$, $L = \{2, 3\}$, $m = 15$. The clubs are $\binom{[6]}{4}$. If A is a club then $|A| = 4$, so $|A| \pmod 7 \notin L$. If A and B are distinct clubs then $|A \cap B| \in \{2, 3\}$ so $|A \cap B| \pmod 7 \in L$. (The congruence conditions here are already implied by the ordinary intersection sizes.)

Note that we have $n < m$ with $m - n = 9$. Can the additive gap be arbitrary large?

Example 3 $p = 11$, $n = 9$, $L = \{0, 1, 2\}$, $m = \binom{5}{3} + 1 = 11$. The clubs are all 3-element sets of [5] and also the four-element set $\{6, 7, 8, 9\}$. Formally $\binom{[5]}{3} \cup \{\{6, 7, 8, 9\}\}$. We include this example since not all clubs are the same size. If A is a club then $|A| \in \{3, 4\}$, so $|A|$

(mod 11) $\notin L$. If A and B are distinct clubs then $|A \cap B| \in \{0,1,2\}$ so $|A \cap B| \pmod{11} \in L$. (The congruence conditions here are already implied by the ordinary intersection sizes.)

$n < m$ with $m - n = 2$.

Example 4 Let $p = 5$, $n = 1000$, $L = \{0,1\}$, $m = \binom{1000}{2}$. The clubs are $\binom{[1000]}{2}$. If $A \in \binom{[1000]}{2}$ then $|A| = 2$ hence $|A| \pmod 5 \notin L$. If A and B are distinct clubs then $|A \cap B| \in \{0,1\}$ hence $|A \cap B| \pmod 5 \in L$. (The congruence conditions here are already implied by the ordinary intersection sizes.)

$n < m$ by a large margin. In fact m is roughly $\frac{n^2}{2}$. What is the function f such that $m \leq f(n)$?

Example 5 Let $p = 5$, $n = 19$, $L = \{1,2\}$, and $m = \binom{10}{8} = 45$. The clubs are all 8-element subsets of [10], regarded as subsets of [19]. Note that the numbers $11, 12, \ldots, 19$ do not belong to any club. We include this example since, unlike the other examples, the mod p matters.

If $A \in \binom{[10]}{8}$ then $|A| = 8$ so $|A| \pmod 5 = 3 \notin L$. If A and B are distinct clubs then $|A \cap B| \in \{6,7\}$ hence $|A \cap B| \pmod 5 \in \{1,2\} = L$.

Exercise 2 For each of these either find the sequence or show that none exists.

1. Find an infinite sequence $(n_i, p_i, L_i, m_i, S_i)$ (S_i is a set of m_i subsets of $[n_i]$) such that (1) $n_1 < n_2 < \cdots$, (2) $|S_1| < |S_2| < |S_3| < \cdots$, (3) the condition is satisfied, and (4) there exists c such that $m_i \geq cn_i^2$.

2. Do the problem above but with $m_i \geq cn_i^3$ or higher.

3. Do the problem above but with $m_i \geq 2^{cn_i}$. (Warning- this problem might be open.)

4. Try to formulate and prove a theorem that gives an upper bound on m as a function of n. (This is next so if you are going to attempt it, do so before looking at the next theorem.)

Theorem 3.1 *Let $n \in \mathsf{N}$ and p be a prime. Let $L \subseteq \{0, \ldots, p-1\}$. Let $w_1, \ldots, w_m \subseteq [n]$ such that the following hold:*

- $(\forall i)[|w_i| \pmod p \notin L]$.
- $(\forall i, j)[|w_i \cap w_j| \pmod p \in L]$.

Then $m \leq \sum_{j=0}^{|L|} \binom{n}{j}$ (This does not match the Oddtown Theorem when $p = 2$. We discuss this after the proof.)

Proof: Associate to w_i the function $f_i\colon \{0,1\}^n \to \mathsf{F_p}$ defined by

$$f_i(x) = \prod_{\ell \in L}((x \cdot w_i) - \ell) \pmod{p}.$$

Example $p = 5$, $L = \{1, 3, 4\}$, $n = 8$, $m = 4$. All congruences are mod 5. We only present w_1 and w_2.

$w_1 = 11100011$. Note $|w_1| \equiv 0 \notin \{1, 3, 4\}$.
$w_2 = 00110000$. Note $|w_2| \equiv 2 \notin \{1, 3, 4\}$.
$|w_1 \cap w_2| \equiv 1 \in \{1, 3, 4\}$.

1. $$f_1(w_1) = ((w_1 \cdot w_1) - 1)((w_1 \cdot w_1) - 3)((w_1 \cdot w_1) - 4) \pmod{5}$$

   $f_1(w_1)$ is nonzero because $w_1 \cdot w_1 = |w_1| \pmod{p}$ is not in L. More generally $f_i(w_i)$ is nonzero because $w_i \cdot w_i = |w_i| \pmod{p}$ is not in L.

   We have just shown

   Claim 1 For all $1 \leq i \leq m$, $f_i(w_i) \not\equiv 0 \pmod{p}$.

2. $$f_1(w_2) = ((w_1 \cdot w_2) - 1)((w_1 \cdot w_2) - 3)((w_1 \cdot w_2) - 4) \pmod{5}$$

   We know this (and $f_i(w_j)$ for $i \neq j$ in general) is zero since $w_i \cdot w_j \equiv |w_i \cap w_j| \in L$ so one of the factors is 0.

   We have just shown

   Claim 2 For all $1 \leq i \neq j \leq m$, $f_i(w_j) \equiv 0 \pmod{p}$.

3. We expand out $f_1(x)$. We let $x = (x_1, \ldots, x_8)$. Since $x_i \in \{0, 1\}$ if we ever encounter x_i^t with $t \geq 2$ we can replace it with x_i. Since this is all in mod 5 we replace every number with its value mod 5.

   $x \cdot w_1 = x_1 + x_2 + x_3 + x_7 + x_8$

   $$f_1(x) = (x_1+x_2+x_3+x_7+x_8-1)(x_1+x_2+x_3+x_7+x_8-3)(x_1+x_2+x_3+x_7+x_8-4)$$

   $$\equiv (x_1 + x_2 + x_3 + x_7 + x_8 + 4)(x_1 + x_2 + x_3 + x_7 + x_8 + 2)(x_1 + x_2 + x_3 + x_7 + x_8 + 1)$$

   Expanding this and replacing each power x_i^t, $t \geq 2$, by x_i gives a multilinear polynomial. The degree is at most 3 since, for example, a term such as $x_1x_7x_8$ can appear (in fact, it does), but $x_1x_2x_3x_8$ cannot, because there are only three factors.

   We have just shown

   Claim 3 f_i is a multilinear polynomial of degree at most $|L|$.

Key The set of multilinear polys of degree at most $|L|$ is a vector space of dim $\sum_{j=0}^{|L|} \binom{n}{j}$. Note that $f_1, \ldots, f_m$ are all in that vector space. So we need to show that the f_i's are linearly independent.

Assume $\sum_{i=1}^{m} \alpha_i f_i = 0$.

Evaluate at w_j and apply Claim 1 and Claim 2.

$0 = \sum_{i=1}^{m} \alpha_i f_i(w_j) = \alpha_j f_j(w_j)$.

Since $f_j(w_j) \neq 0$, $\alpha_j = 0$.

Since j was arbitrary, all coefficients are zero; hence the f_i's are linearly independent.

Therefore $m \leq \sum_{j=0}^{|L|} \binom{n}{j}$ ∎

Note 3.2 For $p = 2$ and $L = \{0\}$, the generic bound is $m \leq n+1$. However, the polynomials have constant term $\prod_{\ell \in L}(-\ell) = 0$, so the sharper dimension count gives $m \leq n$, recovering Oddtown. Note that, for all p, if $0 \in L$ then $m \leq (\sum_{j=0}^{|L|} \binom{n}{j}) - 1$.

We will need the following easy corollary to generalized Oddtown.

Corollary 3.3 *Let $n \in \mathsf{N}$ and p be a prime. Let $L = \{0, 1, 2, 3, \ldots, p-2\}$.*

1. *Let $w_1, \ldots, w_m \subseteq [n]$ such that the following hold:*

   - $(\forall i)[|w_i| = 2p - 1]$.
   - $(\forall i, j)[|w_i \cap w_j| \not\equiv p - 1 \pmod{p}]$.

   Then $m \leq \sum_{j=0}^{|L|} \binom{n}{j}$

2. *Let $w_1, \ldots, w_m \subseteq [n]$ such that the following hold:*

   - $(\forall i)[|w_i| = 2p - 1]$.
   - $m > \sum_{j=0}^{|L|} \binom{n}{j}$

   Then there exists $1 \leq i \neq j \leq m$ such that $|w_i \cap w_j| = p - 1$.

Proof:

Part 1: This is a subcase of Generalized Oddtown (Theorem 3.1).

Part 2: By Part 1 the two conditions imply that $(\exists i, j)[|w_i \cap w_j| \equiv p-1 \pmod{p}]$. (Recall that by convention $i \neq j$.) Hence $|w_i \cap w_j| \in \{p-1, 2p-1, 3p-1, \ldots\}$. Since $|w_i| = 2p-1$, $|w_i \cap w_j| \in \{p-1, 2p-1\}$. Since $w_i \neq w_j$, $|w_i \cap w_j| = p-1$. ∎

3.2 A General Lower Bound

Theorem 3.4 *Let $c, n, p \in \mathsf{N}$ such that p is prime and $c < \binom{n}{2p-1} / \sum_{j=0}^{p-1} \binom{n}{j}$.*

1. *For all colorings* COL: $\mathsf{E}^n \to [c]$ *there exist two points of the same color that are distance one apart.*
2. $\chi(\mathsf{E}^n) \geq \binom{n}{2p-1} / \sum_{j=0}^{p-1} \binom{n}{j}$. *(This follows from Part 1.)*

Proof:

Let c, n be as in the premise. Let COL: $\mathsf{E}^n \to [c]$.

For $A \in \binom{[n]}{2p-1}$, $A = \{i_1, \ldots, i_{2p-1}\}$, let w_A be the following point in E^n:

$$(0, \ldots, 0, \frac{1}{\sqrt{2p}}, 0, \ldots, 0, \ldots, 0, \frac{1}{\sqrt{2p}}, 0, \ldots, 0, \ldots, 0, \frac{1}{\sqrt{2p}}, 0, \ldots, 0).$$

The $\frac{1}{\sqrt{2p}}$'s are at the i_1th, i_2th, ..., i_{2p-1}th coordinates.

Let $W = \{w_A \colon A \in \binom{[n]}{2p-1}\}$.

By the pigeonhole principle, there is an $X \subseteq \binom{[n]}{2p-1}$ such that the following hold:

- $|X| \geq \binom{n}{2p-1}/c > \sum_{j=0}^{p-1} \binom{n}{j}$.
- For all $A \in X$, $|A| = 2p - 1$. (This is trivial.)
- All the points in $\{w_A \colon A \in X\}$ are the same color.

Since $|X| > \sum_{j=0}^{p-1} \binom{n}{j}$ and every $A \in X$ has size $2p-1$, Corollary 3.3.2 applies. Hence there exists $A, B \in \binom{[n]}{2p-1}$ with $|A \cap B| = p - 1$. Look at w_A and w_B.

- There are $p-1$ coordinates where both have $\frac{1}{\sqrt{2p}}$.
- There are p coordinates where w_A is $\frac{1}{\sqrt{2p}}$ and w_B is 0.
- There are p coordinates where w_B is $\frac{1}{\sqrt{2p}}$ and w_A is 0.
- In all remaining coordinates both have 0.

Hence

$$d(w_A, w_B) = 2p\left(\frac{1}{\sqrt{2p}}\right)^2 = 1$$

∎

3.3 For every fixed d, $\chi(\mathsf{E}^n) = \Omega(n^d)$

We apply Theorem 3.4 to obtain the result stated in the title section.

Lemma 3.5 *Let $n, p \in \mathsf{N}$ such that $n \geq 4p$. Then $\sum_{j=0}^{p-1} \binom{n}{j} < 2\binom{n}{p-1}$.*

Proof:

Let $1 \leq j \leq p-1$. Then $\frac{\binom{n}{j-1}}{\binom{n}{j}} = \frac{j}{n-j+1} \leq \frac{p-1}{n-p+2}$.

Since $n \geq 4p$ we have:

$$\frac{p-1}{n-p+2} \leq \frac{p-1}{4p-p+2} = \frac{p-1}{3p+2} \leq \frac{1}{3}.$$

Hence $\binom{n}{j-1} \leq \frac{1}{3}\binom{n}{j}$, so $\binom{n}{j} \leq \frac{1}{3^{p-j-1}}\binom{n}{p-1}$

Therefore

$$\sum_{j=0}^{p-1} \binom{n}{j} < \binom{n}{p-1}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right) < 2\binom{n}{p-1}.$$

∎

Theorem 3.6 *Let $c, n, p \in \mathsf{N}$ such that p is prime and $c < \binom{n}{2p-1} / \sum_{a=j}^{p-1} \binom{n}{j}$.*

1. *If $n \geq 4p$ then $\chi(\mathsf{E}^n) \geq \binom{n}{2p-1}/2\binom{n}{p-1}$.*
2. *For every fixed d, $\chi(\mathsf{E}^n) = \Omega(n^d)$.*

Proof:

1) From Theorem 3.4

$$\chi(\mathsf{E}^n) \geq \binom{n}{2p-1} / \sum_{j=0}^{p-1} \binom{n}{j}.$$

By Lemma 3.5, for $n \geq 4p$ we have

$$\chi(\mathsf{E}^n) \geq \binom{n}{2p-1}/2\binom{n}{p-1}.$$

2) Algebraic manipulation yields

$$\binom{n}{2p-1}/2\binom{n}{p-1} \geq \frac{(p-1)!}{2((2p-1)!} \prod_{j=p}^{2p-1} (n-j+1).$$

Each factor in the product is

$$\geq n - (2p-1) + 1 = n - 2p + 2 \geq \frac{n}{2}$$

The last inequality follows from $n \geq 4p$.

The product is $\geq (\frac{n}{2})^p$. Hence

$$\binom{n}{2p-1}/2\binom{n}{p-1} \geq \frac{(p-1)!}{2^{p+1}(2p-1)!}n^p.$$

Hence

$$\chi(\mathsf{E}^n) \geq \frac{(p-1)!}{2^{p+1}(2p-1)!}n^p.$$

We show that, for every fixed d, $\chi(\mathsf{E}^n) = \Omega(n^d)$. Given d, choose a prime $p \geq d$ and hold p fixed as n grows. For $n \geq 4p$ the displayed bound gives $\chi(\mathsf{E}^n) \geq C_p n^p \geq C_p n^d$.

∎

3.4 For Sufficiently Large n, $\chi(\mathsf{E}^n) \geq (1.2)^n$

In this section we obtain better lower bounds on $\chi(\mathsf{E}^n)$ by applying the lower bound in Theorem 3.4 more carefully than we did in Theorem 3.6.

Convention 3.7 In this section log is natural log.

Definition 3.8 We first define H on $(0,1)$ and then define $H(0)$ and $H(1)$ so that the function H on $[0,1]$ is continuous.

1. On the interval $(0,1)$

$$H(t) = -t\log t - (1-t)\log(1-t).$$

This is the entropy function; however, we will not be using that fact.

2. Since $\lim_{x\to 0^+} x\ln x = 0$, H has a continuous extension to $[0,1]$ given by $H(0) = H(1) = 0$.

The following lemma is easy to verify and left to the reader.

Lemma 3.9

1. $H'(t) = \ln((1-t)/t)$.

2. $H(1/2) = \ln 2 \approx 0.6931$.

3. *Let $0 < a < 1/2$. Let ϵ_n be a sequence such that $\lim_{n\to\infty} \epsilon_n = 0$ and $(\forall n)[2a + \epsilon_n \in (0, 1)]$. Then $H(2a + \epsilon_n) = H(2a) + o(1)$. This follows from the Taylor series of H around $2a$ and using the derivative in Part 1.*

Notation 3.10 Let $r\colon \mathsf{N} \to \mathsf{R}$.

1. r is $o(1)$ (we write $r(n) \in o(1)$) if $\lim_{n\to\infty} r(n) = 0$. Note that r may take both positive and negative values.
2. r is $o(n)$ (we write $r(n) \in o(n)$) if $\lim_{n\to\infty} \frac{r(n)}{n} = 0$. Note that r may take both positive and negative values.
3. We will write things like

$$\log(\chi(\mathsf{E}^n)) \geq n \log(\beta) + o(n)$$

to mean that there exists some function $r(n) \in o(n)$ such that

$$\log(\chi(\mathsf{E}^n)) \geq n \log(\beta) + r(n)$$

4. We will use Lemma 3.9.3 by saying $H(2a + o(1)) = H(2a) + o(1)$.

The following lemma is standard and can be derived from Stirling's Formula.

Lemma 3.11 *Let $k, n \in \mathsf{N}$ with $k < n$.*

1.

$$\frac{1}{n+1} \times e^{nH(k/n)} \leq \binom{n}{k} \leq e^{nH(k/n)}.$$

2.

$$\log\left(\frac{1}{n+1}\right) + nH(k/n) \leq \log\left(\binom{n}{k}\right) \leq nH(k/n).$$

Theorem 3.12 *Let $\beta = \frac{1+\sqrt{2}}{2} > 1.2071$.*

1. $\chi(\mathsf{E}^n) \geq e^{n \log \beta + o(n)}$.
2. *For sufficiently large n, $\chi(\mathsf{E}^n) \geq (1.2)^n$.*

Proof: From Theorem 3.4 we have

$$\chi(\mathsf{E}^n) \geq \frac{\binom{n}{2p-1}}{\sum_{j=0}^{p-1}\binom{n}{j}}.$$

We want a lower bound on the right hand side. We will later choose $p < \frac{n}{2}$ so we have $\sum_{j=0}^{p-1}\binom{n}{j} \leq p\binom{n}{p-1} \leq n\binom{n}{p-1}$. Hence we have

$$\chi(\mathsf{E}^n) \geq \frac{\binom{n}{2p-1}}{n\binom{n}{p-1}}$$

By Lemma 3.11 we have

$$\chi(\mathsf{E}^n) \geq \frac{\frac{1}{n+1}\exp\left(nH\left(\frac{2p-1}{n}\right)\right)}{n\exp\left(nH\left(\frac{p-1}{n}\right)\right)} = \frac{1}{n(n+1)}\exp\left(nH\left(\frac{2p-1}{n}\right) - nH\left(\frac{p-1}{n}\right)\right)$$

We take the log of both sides.

$$\log(\chi(\mathsf{E}^n)) \geq \log\left(\frac{1}{n(n+1)}\right) + \left(nH\left(\frac{2p-1}{n}\right) - nH\left(\frac{p-1}{n}\right)\right)$$

$$\log(\chi(\mathsf{E}^n)) \geq \log\left(\frac{1}{n(n+1)}\right) + n\left(H\left(\frac{2p-1}{n}\right) - H\left(\frac{p-1}{n}\right)\right)$$

We now choose p. Let $0 < a < 1/2$ be a number to be chosen later. Let p be the largest prime such that $p \leq an$. By the prime number theorem $p = an + o(n)$ (recall that $o(n)$ can be negative). Hence we have, using Lemma 3.9.3:
$H((2p-1)/n) = H(2a + o(1)) = H(2a) + o(1)$.
$H((p-1)/n) = H(a + o(1)) = H(a) + o(1)$.
Hence

$$\log(\chi(\mathsf{E}^n)) \geq n(H(2a) - H(a) + o(1)) = n(H(2a) - H(a)) + o(n).$$

Let

$$f(a) = H(2a) - H(a)$$

We want to find the value of a that maximizes $f(a)$ on the interval $[0, 1/2]$. We hope the value is not 0 or $1/2$ since we need $0 < a < 1/2$. Spoiler alert: the maximum value will not be achieved at either endpoint.

By Lemma 3.9.1 and some calculation:

$$f'(a) = \log\left(\frac{(1-2a)^2}{4a(1-a)}\right).$$

$f'(a) = 0$ iff $\frac{(1-2a)^2}{4a(1-a)} = 1$ iff $a = \frac{2\pm\sqrt{2}}{4}$.

Since we are only interested in $a \in [0, 1/2]$ the value of interest is $a_* = \frac{2-\sqrt{2}}{4}$. Hence the maximum is achieved at either $a = 0$, $a = a_*$, or $a = \frac{1}{2}$. We will compute $f(a_*)$ and then compare $f(0)$, $f(1/2)$, and $f(a_*)$ to see which is the maximum.

$a = 0$: $f(0) = H(0) - H(0) = 0$.

$a = 1/2$: $f(1/2) = H(1) - H(1/2) = 0 - \ln 2 < 0$.

$a = a_*$: We do this in a clever way.

First get $f(a)$ in a different form.

$f(a) = H(2a) - H(a)$

$= -2a\log(2a) - (1-2a)\log(1-2a) + a\log(a) + (1-a)\log(1-a)$

$= -2a\log(2a) + a\log(a) - (1-2a)\log(1-2a) + (1-a)\log(1-a)$.

$= -a(2\log(2a) - \log(a)) - (1-2a)\log(1-2a) + (1-a)\log(1-a)$.

Note that $-a(2\log(2a) - \log(a)) = -a\log(4a)$.

Hence we have:

Equation 1: $f(a) = -a\log(4a) - (1-2a)\log(1-2a) + (1-a)\log(1-a)$.

Second we get an expression for $-a_*\log(4a_*)$.

$$\frac{(1-2a_*)^2}{4a_*(1-a_*)} = 1$$

$$4a_* = \frac{(1-2a_*)^2}{1-a_*}.$$

$$\log(4a_*) = 2\log(1-2a_*) - \log(1-a_*).$$

Equation 2: $-a_*\log(4a_*) = -2a_*\log(1-2a_*) + a_*\log(1-a_*)$.

Third we combine Equations 1 and 2.

$f(a_*) = -a_*\log(4a_*) - (1-2a_*)\log(1-2a_*) + (1-a_*)\log(1-a_*)$.

$= -2a_*\log(1-2a_*) + a_*\log(1-a_*) - (1-2a_*)\log(1-2a_*) + (1-a_*)\log(1-a_*)$.

$= \log(1-a_*) - \log(1-2a_*) = \log\left(\frac{(1-a_*}{1-2a_*}\right)$

$= \log\frac{1+\sqrt{2}}{2} = \log\beta > 0$.

Now back to our narrative.

We present $f(0)$, $f(1/2)$, and $f(a_*)$.

- $f(0) = H(0) - H(0) = 0$.
- $f(1/2) = H(1) - H(1/2) = 0 - \ln 2 < 0$.
- $f(a_*) = \log\beta$. Since $\beta > 1$, $f(a_*) > 0$.

Hence f achieves its maximum at a_*.

Back to $\chi(\mathsf{E}^n)$

$$\log(\chi(\mathsf{E}^n)) \geq n(H(2a) - H(a)) + o(n).$$

By setting a to a_* we get

$$\log(\chi(\mathsf{E}^n)) \geq n \log \beta + o(n).$$

Since $\beta > 1.2$, for large n,

$$n \log \beta + o(n) \geq n \log(1.2).$$

(This step is needed since $o(n)$ could be negative.)
So, for large n,

$$\log(\chi(\mathsf{E}^n)) \geq n \log(1.2)$$

$$\chi(\mathsf{E}^n) \geq e^{n \log(1.2)} = (1.2)^n.$$

∎

References

[1] P. Frankl and R. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1:357–368, 1981.