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All two-color Rado numbers for $a(x+y)=bz$

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Abstract

If it exists, the smallest number $N=R_k(\Sigma)$ is called the kth Rado number of a given system Σ of linear equations if it is guaranteed that any k-coloring of the numbers $1,2,\dots,N$ contains a monochromatic solution of Σ. For the family of equations $a(x+y)=bz$, all Rado numbers $R_2(a,b)$ are determined.

Keywords: Rado numbers; k-colouring; Schur number

1. Introduction

In 1917 Schur [8] proved the existence of a natural number N such that in any coloring of the numbers $1,2,\dots,N$ by k colors there are three numbers x, y, and z which are of the same color and satisfy $x+y=z$.

More generally, in 1936 Rado [6] considered systems of linear equations (see [2]). Given a system of linear equations Σ, and a natural number k, the smallest natural number N, such that for every k-coloring of the numbers $1,2,\dots,N$ there exists among the colored numbers a monochromatic solution of Σ, is denoted by $R_k(\Sigma)$ and called a Rado number.

Just a few Rado numbers are determined so far (see [1,3–5,7]), most of them recently. For $x+y=z$ the corresponding Rado numbers R_k are called Schur numbers and exact values are known only for $k\leqslant 4$. Here we will consider the generalized class of equations $a(x+y)=bz$ where a, b are positive integers and determine all of the corresponding Rado numbers $R(a,b)$ for $k=2$.

2. The two-color Rado numbers for $a(x+y)=bz$

In $a(x+y)=bz$, the coefficients a and b can be assumed to be coprime.

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Theorem 1. *For* $(a,b)=1$ *we have* $R(a,b)=r$ *where*

$$
\begin{array}{lll}
r=\alpha(a) & =(4a^2+1)a & \text{for } b=1,\\
r=\beta(a) & =a(a^2+1)/2 & \text{for } b=2,\\
r & =9 & \text{for } b=3,\ a=1,\\
r & =10 & \text{for } b=3,\ a=2,
\end{array}
$$

$$
r=\lambda(a)=\frac{a}{9}\left\{\begin{array}{ll}
(4a^2+2a+3) & \text{for } a\equiv 1\,(\mathrm{mod}\ 9),\\
(4a^2+a+9) & \text{for } a\equiv 2\,(\mathrm{mod}\ 9),\\
(4a^2+2a+9) & \text{for } a\equiv 4\,(\mathrm{mod}\ 9),\\
(4a^2+4a+6) & \text{for } a\equiv 5\,(\mathrm{mod}\ 9),\\
(4a^2+5a+3) & \text{for } a\equiv 7\,(\mathrm{mod}\ 9),\\
(4a^2+a+6) & \text{for } a\equiv 8\,(\mathrm{mod}\ 9),
\end{array}\right\}\quad \text{for } b=3, a\geqslant 4,
$$

$$
\begin{array}{lll}
r=\gamma(b) & =b(b+1)/2 & \text{for } b\geqslant 4,\ 1\leqslant a\leqslant \frac{b}{4},\\
r=\delta(b) & =\lceil b/2\rceil b & \text{for } b\geqslant 4,\ \frac{b}{4}<a<\frac{b}{2},\\
r & =ab & \text{for } b\geqslant 4,\ \frac{b}{2}<a<b,\\
r & =\varepsilon(a,b)=\lceil a^2/b\rceil a & \text{for } b\geqslant 4,\ b<a.
\end{array}
$$

Table 1 indicates those regions where the Rado numbers $R(a,b)$ have equal formulas when $(a,b)=1$. For $(a,b)>1$ the asterisks can be replaced by $R(a/(a,b), b/(a,b))$.

The Rado numbers $R(a,1)=(4a^2+1)a$ were proved by Burr and Loo (on Rado Numbers I, II, preprints 1992). The Rado numbers $R(a,2)=a(a^2+1)/2$ were proved by Harborth and Maasberg in [5]. It remains to determine the Rado numbers for $b\geqslant 3$.

Throughout this paper the abbreviation r will be as in Theorem 1. The two colors are called green and red.

For $R(a,b)\geqslant r$ a special 2-coloring of $1,2,\dots,r-1$ is described which does not contain a monochromatic solution (x,y,z) of $a(x+y)=bz$.

For $R(a,b)\leqslant r$ it is assumed that a 2-coloring of $1,2,\dots,r$ exists without a monochromatic solution (x,y,z) of $a(x+y)=bz$ and then a contradiction is deduced.

The Rado numbers $R(1,3)=9$ and $R(2,3)=10$ can be proved as follows.

The 2-coloring of $1,2,\dots,8$ in which $1,3,4,7$ are green and $2,5,6,8$ red does not contain a monochromatic solution (x,y,z) of $x+y=3z$ which proves $R(1,3)\geqslant 9$. In each 2-coloring of $1,2,\dots,9$ at least one of the solutions $(3,6,3)$, $(6,6,4)$, $(9,9,6)$, and $(3,9,4)$ is monochromatic, and thus $R(1,3)\leqslant 9$ holds.

For $R(2,3)\geqslant 10$ the numbers $1,4,6,7,9$ are colored green and $2,3,5,8$ red. For $R(2,3)\leqslant 10$ it is easy to verify that in each 2-coloring of $1,2,\dots,10$ one of the solutions $(3,6,6)$, $(3,3,4)$, $(6,6,8)$, $(2,4,4)$, $(9,3,8)$, $(10,2,8)$, and $(6,9,10)$ is monochromatic.

Table 1
Rado numbers $R(a,b)$ with $(a,b)=1$

$b\backslash a$	1	2	3	4	5	6	7	8	9	...
1	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	$\alpha(a)$	...
2	$\beta(a)$	*	$\beta(a)$	*	$\beta(a)$	*	$\beta(a)$	*	$\beta(a)$	...
3	9	10	*	$\lambda(a)$	$\lambda(a)$	*	$\lambda(a)$	$\lambda(a)$	*	...
4	$\gamma(b)$	*	ab	*	$\varepsilon(a,b)$	*	$\varepsilon(a,b)$	*	$\varepsilon(a,b)$	...
5	$\gamma(b)$	$\delta(b)$	ab	ab	*	$\varepsilon(a,b)$	$\varepsilon(a,b)$	$\varepsilon(a,b)$	$\varepsilon(a,b)$	
6	$\gamma(b)$	*	*	*	ab	*	$\varepsilon(a,b)$	*	*	
7	$\gamma(b)$	$\delta(b)$	$\delta(b)$	ab	ab	ab	*	$\varepsilon(a,b)$	$\varepsilon(a,b)$	
8	$\gamma(b)$	*	$\delta(b)$	*	ab	*	ab	*	$\varepsilon(a,b)$	
9	$\gamma(b)$	$\gamma(b)$	*	$\delta(b)$	ab	*	ab	ab	*	
⋮	⋮									⋱

Table 2
Proofs of $R(a,b)=r$ (Theorem 1)

			Proof of $R(a,b)\geqslant r$	Proof of $R(a,b)\leqslant r$	
$b=3$	$a\geqslant 4$		Lemma 1	Lemma 5	Lemma 6
$b\geqslant 4$	$1\leqslant a<b/2$	b odd	Lemma 2		Lemma 7
		b even	Lemma 3		
	$b/2<a<b$	b odd	Lemma 2		Lemma 8
		b even	Lemma 3		
	$b<a$		Lemma 4		Lemma 9

The remaining part of the proof of Theorem 1 ($b\geqslant 4$ and $a\geqslant 4$ for $b=3$) is partitioned into Lemmata 1–9, as shown in Table 2. The first steps of the proofs for $R(a,b)\leqslant r$ are similar and therefore given in Lemma 5 which precedes Lemmata 6–9.

Lemma 1. *For $b=3$, $a\geqslant 4$, and $(a,3)=1$ we have $R(a,3)\geqslant r$ where $r=\lambda(a)$ from Theorem* 1.

Proof. First, r will be written in a different way. Let d and t be defined by $a\equiv d\,(\mathrm{mod}\,3)$ with $d\in\{1,2\}$ and by $t=(a-d)/3$, that is, $a=3t+d$. For $a\equiv 1\,(\mathrm{mod}\,9)$ we have $d=1$ and $t\equiv 0\,(\mathrm{mod}\,3)$ which implies $a+t+1\equiv 2\,(\mathrm{mod}\,3)$ and $r=((a+t+1)a+d)a/3$. Corresponding transformations reduce the six cases of Lemma 1 to the following three cases:

$$r=\frac{a}{3}\begin{cases}((a+t+1)a+3) & \text{for } a+t+1\equiv 0\,(\mathrm{mod}\,3),\\ ((a+t+2)a+d) & \text{for } a+t+1\equiv 1\,(\mathrm{mod}\,3),\\ ((a+t+1)a+d) & \text{for } a+t+1\equiv 2\,(\mathrm{mod}\,3).\end{cases}$$

Now for $R(a,3) \geqslant r$ the numbers $1,2,\ldots,r-1$ are colored as follows. The number a, all numbers $(a+t+1)a,(a+t+2)a,\ldots,r-a$, all numbers $3,6,\ldots,3\lfloor(a+t)/2\rfloor$, and all numbers $d,d+3,d+6,\ldots,d+3(a+t-1)$ are colored red and the remaining numbers are colored green. It may be noted that $2a,3a,\ldots,(a+t)a$ are green since $3\lfloor(a+t)/2\rfloor<2a$, and since $3(a+t-1)+d<4a$ and $2a,3a\not\equiv d \pmod 3$.

For every solution (x,y,z) of $a(x+y)=3z$ we have $3|(x+y)$ since $(a,3)=1$. Only solutions (x,y,z) with $1\leqslant x,y,z<r$ are considered, and the cases $3|x$ and $3\nmid x$ will be distinguished.

If $3|x$ then $3|y$ holds as well.

If x and y both are green then $x,y\geqslant 3(\lfloor(a+t)/2\rfloor+1)$. Then $z=a(x+y)/3\geqslant (a+t+1)a$ and therefore z is red since it is a multiple of a.

Let x and y both be red. If $3\leqslant x,y\leqslant 3\lfloor(a+t)/2\rfloor$ then $2a\leqslant z\leqslant 2\lfloor(a+t)/2\rfloor a\leqslant(a+t)a$ and z is green. It remains that at least one of the variables x and y, say x, can be a multiple of a, that is, $x\geqslant(a+t+1)a$. Then

$$x\geqslant\begin{cases}(a+t+1)a & \text{for } a+t+1\equiv 0 \pmod 3,\\ (a+t+3)a & \text{for } a+t+1\equiv 1 \pmod 3,\\ (a+t+2)a & \text{for } a+t+1\equiv 2 \pmod 3)\end{cases}$$

can be concluded since $3|x$ and $3\nmid a$. Together with $y\geqslant 3$ it follows that $z=a(x+y)/3\geqslant r$.

If $3\nmid x$ then $3\nmid y$ and $x\equiv d \pmod 3$ and $y\equiv 3-d \pmod 3$ can be assumed.

If x and y both are green then $x\geqslant 4a$ since $d+3(a+t-1)=4a-3$. With $y\geqslant 3-d$ it follows that $z=a(x+y)/3\geqslant(a+t+1)a$, so z is red.

If x and y both are red then y is a multiple of a with $y\geqslant(a+t+1)a$. Then

$$y\geqslant\begin{cases}(a+t+3)a & \text{for } a+t+1\equiv 0 \pmod 3,\\ (a+t+2)a & \text{for } a+t+1\equiv 1 \pmod 3,\\ (a+t+1)a & \text{for } a+t+1\equiv 2 \pmod 3)\end{cases}$$

can be concluded since $y\equiv 3-d \pmod 3$ and $a\equiv d \pmod 3$. Together with $x\geqslant d$ it follows that $z=(y+x)a/3\geqslant r$. □

Lemma 2. *For $b\geqslant 4, 1\leqslant a<b, (a,b)=1$, and b odd we have $R(a,b)\geqslant r$ where*

$$r=\begin{cases} b(b+1)/2 & \textit{for } 1\leqslant a<b/2,\\ ab & \textit{for } b/2<a<b.\end{cases}$$

Proof. The numbers $1,2,\ldots,r-1$ are colored as follows. The multiples kb of b are colored green for $1\leqslant k<b/4$ and red for $b/4<k<r/b$. The remaining numbers $kb+m<r$ with $0\leqslant k<r/b$ and $1\leqslant m<b$ are colored red for $1\leqslant q<b/2$ and green for $b/2<q\leqslant b-1$ where q is determined by $qa\equiv m \pmod b$.

For every solution of $a(x+y)=bz$ we see that $b|(x+y)$ since $(a,b)=1$. The cases $b|x$ and $b\nmid x$ will be distinguished.

If $x = k_1 b$ then $y = k_2 b$ and $z = (k_1 + k_2)a$.

Let $x = k_1 b$ and $y = k_2 b$ both be green. Then $1 \leqslant k_1, k_2 < b/4$, that is, $2 \leqslant k_1 + k_2 < b/2$, and $z = (k_1 + k_2)a$ is colored red.

If $x = k_1 b$ and $y = k_2 b$ both are red then $b/4 < k_1, k_2 < r/b$, that is,

$$b/2 < k_1 + k_2 \leqslant 2((r/b) - 1) = \begin{cases} b - 1 & \text{for } 1 \leqslant a < b/2, \\ 2(a-1) & \text{for } b/2 < a < b. \end{cases}$$

Then $z = (k_1 + k_2)a$ is colored green whenever $z < r$.

If $b \nmid x$ then $b \nmid y$ and for $x \equiv q_1 a \pmod b$ and $y \equiv q_2 a \pmod b$ we have $1 \leqslant q_1, q_2 < b$. From $b|(x+y)$ it follows that $0 \equiv x + y \equiv (q_1 + q_2)a \pmod b$ and thus $q_1 + q_2 \equiv 0 \pmod b$ since $(a,b) = 1$. However, this contradicts $1 \leqslant q_1, q_2 < b/2$ or $b/2 < q_1, q_2 < b$, that is, if x and y both are of the same color. □

Lemma 3. *For $b \geqslant 4$, $1 \leqslant a < b$, $(a,b) = 1$ and b even we have $R(a,b) \geqslant r$ where*

$$r = \begin{cases} (b+1)b/2 & \text{for } 1 \leqslant a \leqslant b/4, \\ bb/2 & \text{for } b/4 < a < b/2, \\ ab & \text{for } b/2 < a < b. \end{cases}$$

Proof. Since $(a,b) = 1$, only $a = 1$ and $b = 4$ satisfy $a = b/4$. In this case $1, 4, 5, 6, 9$ red and $2, 3, 7, 8$ green describes a 2-coloring without a monochromatic solution (x, y, z) of $x + y = 4z$. This proves $R(1,4) \geqslant 10$, and $a \neq b/4$ can be assumed in the following.

The numbers $1, 2, \ldots, r - 1$ are colored as follows. The multiples $k(b/2)$ of $b/2$ are colored green for $1 \leqslant k < b/2$, red for $b/2 < k < 2r/b$, and for $k = b/2$ green if $a > b/2$ and red if $a < b/2$. The remaining numbers $kb + m < r$ with $0 \leqslant k < r/b$ and $1 \leqslant m < b$, $m \neq b/2$, are colored red for $1 \leqslant q < b/2$ and green for $b/2 < q \leqslant b - 1$ where q is determined by $qa \equiv m \pmod b$.

Again $b|(x+y)$ for every solution of $a(x+y) = bz$ since $(a,b) = 1$. The cases $(b/2)|x$ and $(b/2) \nmid x$ will be distinguished.

If $x = k_1(b/2)$ then $y = k_2(b/2)$ and $z = a(k_1 + k_2)/2$.

Let $x = k_1(b/2)$ and $y = k_2(b/2)$ both be green, that is, $1 \leqslant k_1, k_2 \leqslant b/2$, and thus $1 \leqslant (k_1 + k_2)/2 \leqslant b/2$. Then $z = qa$ with $q = (k_1 + k_2)/2$ is red. Note that $q = b/2$ implies $k_1 = k_2 = b/2$ and $a > b/2$ so that $z = a(b/2)$ is red.

Let $x = k_1(b/2)$ and $y = k_2(b/2)$ both be red, so that $b/2 \leqslant k_1, k_2 < 2r/b$, that is,

$$\frac{b}{2} \leqslant \frac{k_1 + k_2}{2} \leqslant \frac{r}{b/2} - 1 = \begin{cases} b & \text{for } 1 \leqslant a < b/4, \\ b - 1 & \text{for } b/4 < a < b/2, \\ 2a - 1 & \text{for } b/2 < a < b. \end{cases}$$

Then $z = qa$ with $q = (k_1 + k_2)/2$ is green for $b/2 \leqslant (k_1 + k_2)/2 < b$. Since $(k_1 + k_2)/2 = b/2$ implies $k_1 = k_2 = b/2$ and $a < b/2$ so that $z = ab/2$ is green. If $(k_1 + k_2)/2 \geqslant b$ then either $a < b/4$ and $z = ab = 2a(b/2)$ is green or $a > b/2$ and $z \geqslant ab$ is not contained in $1, 2, \ldots, r - 1$.

If $(b/2) \nmid x$ then $(b/2) \nmid y$ and then $x \equiv q_1 a \pmod b$ and $y \equiv q_2 a \pmod b$ with $1 \leqslant q_1, q_2 < b$. From $b|(x + y)$ it follows that $0 \equiv x + y \equiv (q_1 + q_2)a \pmod b$ and thus $q_1 + q_2 \equiv 0 \pmod b$ since $(a,b)=1$. However, this contradicts $1 \leqslant q_1, q_2 < b/2$ or $b/2 < q_1, q_2 < b$, that is, if x and y both are of the same color. □

Lemma 4. *For $4 \leqslant b < a$ and $(a,b)=1$ we have $R(a,b) \geqslant r$ where $r = \lceil a^2/b \rceil a$.*

Proof. Here $1,2,\ldots,r-1$ are colored as follows. The multiples ka of a are colored green for $1 \leqslant k < a$ and red for $a \leqslant k < \lceil a^2/b \rceil$. All multiples kb of b with $1 \leqslant k \leqslant \lfloor (a-1)/2 \rfloor$ are colored red and with $\lfloor (a+1)/2 \rfloor \leqslant k < a$ are colored green. Since $(a,b)=1$ every remaining number x can be written as $kb+ma$ with $1 \leqslant k < a$ and $m \neq 0$. These numbers are colored red for $m<0$ and green for $m>0$.

For solutions (x,y,z) of $a(x+y)=bz$ we see that $z=k_1 a$ since $(a,b)=1$.

For $x=k_2 a$ it follows that $y=k_1 b - k_2 a$.

If $x=k_2 a$ and $z=k_1 a$ both are green then $1 \leqslant k_1, k_2 < a$ and thus $y=k_1 b - k_2 a$ with $m=-k_2$ is red or negative, that is, not contained in $1,2,\ldots,r-1$.

If $x=k_2 a$ and $z=k_1 a$ are both red, then $a \leqslant k_1, k_2 < \lceil a^2/b \rceil$ and thus $y \leqslant (\lceil a^2/b \rceil - 1)b - a^2 < 0$ is not contained in $1,2,\ldots,r-1$.

For $x=k_2 b$ with $k_2 < a$ it follows that $y=(k_1-k_2)b$.

If x and z both are green, then $\lfloor (a+1)/2 \rfloor \leqslant k_2 < a$ and $1 \leqslant k_1 < a$ so that $y=k_3 b$ with $k_3 = k_1 - k_2 \leqslant a - 1 - \lfloor (a+1)/2 \rfloor \leqslant \lfloor (a-1)/2 \rfloor$ is red or negative.

If x and z both are red, then $1 \leqslant k_2 \leqslant \lfloor (a-1)/2 \rfloor$ and $a \leqslant k_1 < \lceil a^2/b \rceil$ so that $y=k_3 b$ with $k_3 = k_1 - k_2 \geqslant a - \lfloor (a-1)/2 \rfloor \geqslant \lfloor (a+1)/2 \rfloor$ is green for $a \nmid k_3$ and for $a \mid k_3$ as well since $y = k_1 b - x < k_1 b \leqslant (\lceil a^2/b \rceil - 1)b < a^2$.

For the remaining numbers x in $1,2,\ldots,r-1$, that are $x = k_2 b \pm ma$ with $1 \leqslant k_2 < a$ and $m \geqslant 1$, we have $y = k_3 b \mp ma$ with $k_3 = k_1 - k_2$.

If x and z both are green then $y = k_3 b - ma$ with $k_3 < a$ is red or negative.

If x and z both are red then $y = k_3 b + ma$ and $a \leqslant k_1 < \lceil a^2/b \rceil$ so that y is green for $a \nmid k_3$ and also for $a \mid k_3$ since y is a multiple of a and $y = k_1 b - x < a^2$. □

Lemma 5. *Assume the existence of a 2-coloring of $1,2,\ldots,r$ without a monochromatic solution (x,y,z) of $a(x+y)=bz$. Let b be green (without loss of generality). If m_0 is the greatest integer such that $b, 2b, \ldots, m_0 b$ are green, then $(m_0+1)b, (m_0+2)b, \ldots, hb$ and $2a, 3a, \ldots, (\alpha-1)a$ are red and $\alpha a, (\alpha+1)a, \ldots, ga$ are green, where*

$$h = \begin{cases} \dfrac{r}{a} - 1 & \text{for } 3 \leqslant b < a, \\ \dfrac{r}{b+1} & \text{for } b \geqslant 4,\ 1 \leqslant a \leqslant \dfrac{b}{4},\ b \text{ even}, \\ \dfrac{r}{b} & \text{for } b \geqslant 4,\ 1 \leqslant a \leqslant \dfrac{b}{4},\ b \text{ odd} \\ & \text{and } b \geqslant 4,\ \dfrac{b}{4} < a < b, \end{cases}$$

$$\alpha = \begin{cases} 2m_0 + 1 & \text{if } (2m_0+1)a \text{ green}, \\ 2m_0 + 2 & \text{if } (2m_0+1)a \text{ red}, \end{cases}$$

$$g = \min\{\lfloor r/a \rfloor, 2h\}.$$

Proof. Every component of the solutions $(wb,(b-w)b,ab)$ with

$$w=\begin{cases}1 & \text{for } 3\leqslant b<a,\\ h & \text{otherwise}\end{cases}$$

is contained in $1,2,\dots,r$. This can be verified straightforward using the following steps. First, w and if necessary h (Lemma 5) and r (Theorem 1) are substituted so that the components of the solution depend on a and b only. Then the maximum component is determined as

$$\max\{wb,(b-w)b,ab\}=\begin{cases}ab & \text{for } 3\leqslant b<a,\\ hb & \text{otherwise.}\end{cases}$$

At last the maximum component can be verified to be at most r by Theorem 1.

Since no solution $(wb,(b-w)b,ab)$ is monochromatic and $b,2b,\dots,m_0b$ are green it follows $m_0<\max\{w,b-w,a\}$, that is, $(m_0+1)b\leqslant\max\{wb,(b-w)b,ab\}\leqslant r$. Thus $(m_0+1)b$ belongs to $1,2,\dots,r$ and then it is red by definition of m_0.

The solutions $(ub,vb,(u+v)a)$ with $1\leqslant u,\ v\leqslant m_0$ force $2a,3a,\dots,2m_0a$ to be red as long as these values are in $1,2,\dots,r$. In the following $(2m_0+2)a\leqslant r$ can be shown. From the arguments above

$$m_0+1\leqslant\begin{cases}a & \text{for } 3\leqslant b<a,\\ h & \text{otherwise}\end{cases}$$

will be used.

For $b=3$ it follows that $2(m_0+1)a\leqslant 2a^2$ and $2a^2\leqslant r$ holds (with r from Theorem 1).

For $b\geqslant 4$ and $a<b/2$ it follows that $2(m_0+1)a\leqslant 2ha<hb\leqslant r$.

For $b\geqslant 4$ and $a>b/2$ the components of the solution $(2a,(b-2)a,a^2)$ are contained in $1,2,\dots,r$ since $r=ab$ or $r=\lceil a^2/b\rceil a$. This solution is not monochromatic and thus $2m_0a<\max\{2a,(b-2)a,a^2\}<r$. Moreover, $(2m_0+2)a\leqslant a+a\max\{2,b-2,a\}\leqslant r$ with $r=ab$ if $a<b$ and $r=\lceil a^2/b\rceil a$ for $a>b$.

The solution $((m_0+1)b,(m_0+1)b,(2m_0+2)a)$ forces $(2m_0+2)a$ to be green.

Let t_0 be the greatest integer such that $(m_0+1)b,(m_0+2)b,\dots,t_0b$ are red.

If $t_0\leqslant h-1$ is assumed then the components of the solution $(b,(t_0+1)b,(t_0+2)a)$ are contained in $1,2,\dots,r$, since $(t_0+1)b\leqslant hb$ and $hb\leqslant r$ by definition of h (Lemma 1), and since $(t_0+2)a\leqslant(h+1)a$ and $(h+1)a\leqslant r$ can be verified straightforward by substituting h (Lemma 1) and if necessary r (Theorem 1). Then $(t_0+1)b$ is green and $(t_0+2)a$ is forced to be red. Then the solution $((m_0+1)b,(t_0+1-m_0)b,(t_0+2)a)$ determines $(t_0+1-m_0)b$ to be green. Since $t_0+1-m_0\leqslant t_0$ it follows that $t_0-1-m_0\leqslant m_0$ which is equivalent to $2m_0-t_0+1\geqslant 2$. On the other hand, $m_0+1\leqslant t_0$ implies $2m_0-t_0+1\leqslant m_0$ so that $(2m_0-t_0+1)b$ is green. Now the solution $((t_0+1)b,(2m_0-t_0+1)b,(2m_0+2)a)$ is green, a contradiction.

It remains $t_0\geqslant h$. Then $(m_0+1)b,(m_0+2)b,\dots,hb$ are red by definition of t_0. The solutions $(ub,vb,(u+v)a)$ with $m_0+1\leqslant u,\ v\leqslant h$ force $(2m_0+2)a,(2m_0+3)a,\dots,2ha$ to be green as long as these values are in $1,2,\dots,r$, that is, up to ga.

So far the color of $(2m_0+1)a$ remains unknown and by α the green and red multiples of a are separated. □

Lemma 6. *For $b=3$, $a \geqslant 4$, and $(a,3)=1$ we have $R(a,3) \leqslant r$ where $r=\lambda(a)$ from Theorem* 1.

Proof. As in the proof of Lemma 1 we can take for r,

$$r=\frac{a}{3}\begin{cases} ((a+t+1)a+3) & \text{for } a+t+1 \equiv 0 \pmod 3, \\ ((a+t+2)a+d) & \text{for } a+t+1 \equiv 1 \pmod 3, \\ ((a+t+1)a+d) & \text{for } a+t+1 \equiv 2 \pmod 3 \end{cases}$$

with $a \equiv d \pmod 3$, $d \in \{1,2\}$, and $t=(a-d)/3$ so that $a=3t+d$.

By Lemma 5 it follows that $2a, 3a, \ldots, (\alpha-1)a$ are red and αa, $(\alpha+1)a, \ldots, r$ are green. The cases $a<\alpha$ and $a \geqslant \alpha$ will be distinguished.

If $a \leqslant \alpha-1$ then the solution $(a, 2a, a^2)$ forces a to be green. The number $(2t+d)a$ is red since $2<2t+d<a \leqslant \alpha-1$, and $(d, 2a, (2t+d)a)$ forces d to be green and then $(d, 3-d, a)$ forces $3-d$ to be red. The number $4a$ is red since $4 \leqslant a \leqslant \alpha-1$ and thus $(4a, 3-d, (a+t+1)a)$ forces $(a+t+1)a$ and therefore also $(a+t+2)a$ to be green. Now in all three cases $a+t+1 \equiv 0$, 1, and $2 \pmod 3$ completely green solutions $((a+t+1)a, 3, r)$, $((a+t+2)a, d, r)$, and $((a+t+1)a, d, r)$, respectively, are determined in contradiction to the first assumption of Lemma 5.

If $a \geqslant \alpha$ then the solutions $(\alpha a, 3, a(\alpha a+3)/3)$, $((\alpha+2)a, 3, a((\alpha+2)a+3)/3)$, and $((\alpha+1)a, 3, a((\alpha+1)a+3)/3)$ are considered for $\alpha=0$, 1, and $2 \pmod 3$, respectively. Every component is contained in $1, 2, \ldots, r$ since already the largest component $a((\alpha+2)a+3)/3$ is at most r which follows for $t \geqslant 2$ from $(\alpha+2)a+3 \leqslant (a+t)a+3$ and for $t=1$, that is, $a=4$ or $a=5$, from $(\alpha+2)a+3 \leqslant (a+t+1)a+3$. Up to $b=3$ each component is a multiples of a and of size at least αa. Thus all three solutions are green, which is a contradiction. □

Lemma 7. *For $b \geqslant 4$, $1 \leqslant a<b/2$, and $(a,b)=1$ we have $R(a,b) \leqslant r$ where*

$$r=\begin{cases} (b+1)b/2 & \text{for } 1 \leqslant a \leqslant (b-1)/2 \text{ and } b \text{ odd (Case A)}, \\ (b+1)b/2 & \text{for } 1 \leqslant a \leqslant b/4 \text{ and } b \text{ even (Case B)}, \\ bb/2 & \text{for } b/4<a<b/2 \text{ and } b \text{ even (Case C)}. \end{cases}$$

Proof. Starting with Lemma 5 it follows that $b, 2b, \ldots, m_0 b$ are green, $(m_0+1)b, (m_0+2)b, \ldots, hb$ and $2a, 3a, \ldots, (\alpha-1)a$ are red and $\alpha a, (\alpha+1)a, \ldots, 2ha$ are green with

$$h=\begin{cases} (b+1)/2 & \text{in Case A}, \\ b/2 & \text{in Cases B and C}, \end{cases}$$

$$\alpha=\begin{cases} 2m_0+1 & \text{if } (2m_0+1)a \text{ green}, \\ 2m_0+2 & \text{if } (2m_0+1)a \text{ red}. \end{cases}$$

As in the proof of Lemma 5 the solution $(hb, (b-h)b, ab)$ is considered. Since $ab \leqslant (b-h)b \leqslant hb$ can be verified and hb is red this solution forces ab to be green and thus $a \leqslant m_0$. Furthermore, $m_0 < 2m_0 + 1 \leqslant \alpha$ so that $a < \alpha$ and then a^2 is red with one possible exception if $a = 1$.

At first, Case C will be discussed. Here $a > b/4 \geqslant 1$ and a^2 is red so that the solution (ha, ha, a^2) forces ha to be green. If the solution $(\lfloor h/2 \rfloor b, \lceil h/2 \rceil b, ha)$ is not green then $m_0 + 1 \leqslant \lceil h/2 \rceil$, that is, $m_0 < h/2$. However, this contradicts $h/2 = b/4 < a \leqslant m_0$ and Case C is proved.

If a^2 is red the solution $((\alpha - 1)a, (b - \alpha + 1)a, \alpha a)$ forces $(b - \alpha + 1)a$ to be green since ba green implies $b \geqslant \alpha$ which is equivalent to $b - \alpha + 1 \geqslant 1$. From $(b - \alpha + 1)a$ green it follows $b - \alpha + 1 = 1$, that is, a is green, or $b - \alpha + 1 \geqslant \alpha$.

In the following, three alternatives are distinguished, (1) a red, (2) a green for $a \geqslant 2$, and (3) a green for $a = 1$.

(1) If a is red then $b - \alpha + 1 \geqslant \alpha$ which is equivalent to $\alpha \leqslant h$ and thus $(\alpha - a)b - a < hb \leqslant r$ so that the solution $(a, (\alpha - a)b - a, (\alpha - a)a)$ forces $(\alpha - a)b - a$ to be green. Since $(b+1)a < (b+1)b/2 = r$ the solution $((b+1)a, (\alpha - a)b - a, \alpha a)$ determines $(b + 1)a$ to be red in contradiction to $(b + 1)a = 2ha$ green in Case A.

In Case B the solution $((b + 1)h, (b + 1)h, (b + 1)a)$ implies $(b + 1)h$ green. Since a red implied $\alpha < h + 1$, it follows that $(h + 1)a$ is green and thus $((b + 1)h, h, (h + 1)a)$ forces to be red. Now the solution (h, h, a) is red, a contradiction.

(2) If a is green for $a \geqslant 2$ then a^2 was determined to be red and either $b = \alpha$ or $b - \alpha + 1 \geqslant \alpha$.

For $b = \alpha$ the solution $(2a, (b - 2)a, a^2)$ is red, a contradiction.

For $b - \alpha + 1 \geqslant \alpha$, which is equivalent to $\alpha \leqslant h$, if follows from $\alpha < 2\alpha - 1 \leqslant b \leqslant 2h$ that $(b - 1)a$ is green. Furthermore, $(\alpha - a)b + a < \alpha b \leqslant hb \leqslant r$ so that the solution $((b - 1)a, (\alpha - a)b + a, \alpha a)$ forces $(\alpha - a)b + a$ to be red. Since $2 \leqslant a \leqslant m_0 < \alpha$ the third component of $(b - a, (\alpha - a)b + a, (\alpha + 1 - a)a)$ is red which forces $b - a$ to be green. Then $(b - a, a, a)$ is green, a contradiction.

(3) If a is green for $a = 1$ then $(1, b - 1, 1)$ forces $b - 1$ to be red and thus also $b - 2$ is red. Since $ba = b$ was assumed to be green it follows that $b = \alpha$ and $2 \leqslant h \leqslant (\alpha + 1)/2 < \alpha$ determines $ha = h$ to be red. Then $(2, hb - 2, h)$ implies that $hb - 2$ is green.

In Case A the solution $(b - 2, (h - 1)b + 2, h)$ forces $(h - 1)b + 2$ to be green and then $(hb - 2, (h - 1)b + 2, 2h - 1)$ forces $2h - 1$ to be red in contradiction to $b = 2h - 1$ being green.

In Case B from $h = b/2 = \alpha/2 < \alpha - 1$ it follows that $h + 1$ is red. Then $(b - 2, hb + 2, h + 1)$ implies that $bh + 2$ is green since $bh + 2 \leqslant bh + h = r$. Now the green solution $(hb - 2, hb + 2, b)$ is a contradiction. □

Lemma 8. *For $b \geqslant 4$, $b/2 < a < b$, and $(a, b) = 1$ we have $R(a, b) \leqslant r$ where $r = ab$.*

Proof. Using Lemma 5 it follows that hb is red and ga is green. However, $hb = r = ab$ and $ga = a \min\{b, 2a\} = ab$ determine $hb = ga$, a contradiction. □

Lemma 9. *For $4 \leqslant b < a$ and $(a, b) = 1$ we have $R(a, b) \leqslant r$ where $r = \lceil a^2/b \rceil a$.*

Proof. Using Lemma 5 it follows that $b, 2b, \ldots, m_0 b$ are green, $(m_0+1)b, (m_0+2)b, \ldots, hb$ and $2a, 3a, \ldots, (\alpha-1)a$ are red and $\alpha a, (\alpha+1)a, \ldots, r$ are green with

$$h = (r/a) - 1 = \lceil a^2/b \rceil - 1,$$

$$\alpha = \begin{cases} 2m_0+1 & \text{if } (2m_0+1)a \text{ green,} \\ 2m_0+2 & \text{if } (2m_0+1)a \text{ red.} \end{cases}$$

Since $(2a, (b-2)a, a^2)$ is a solution and $2a \leqslant (b-2)a < a^2 < r$ it follows that $\alpha \leqslant a$.

Some abbreviations are needed. The number δ is determined by $r/a = \lceil a^2/b \rceil = (a^2+\delta)/b$ with $1 \leqslant \delta < b$. The numbers γ and ε are determined by $a = \gamma b + \varepsilon$ with $1 \leqslant \gamma$ and $1 \leqslant \varepsilon < b$. Furthermore, there are numbers σ and τ so that $\alpha a + \tau = \sigma b$ with $1 \leqslant \tau < b$ and $\alpha < \sigma$. Also $\sigma \leqslant r/a$ follows from $\sigma = (r/a)(\alpha a + \tau)/(a^2+\delta)$ for $\alpha < a$, and for $\alpha = a$ if $\delta = \tau$ is noted. Then σa is green and $(\alpha a, \tau, \sigma a)$ forces τ to be red.

The number γab is red since $\gamma a \geqslant a \geqslant \alpha \geqslant 2m_0+1$ and $\gamma a = (a-\varepsilon)a/b < h$. This implies $\gamma ba \leqslant (\alpha-1)a$ which is equivalent to $\alpha - \gamma b \geqslant 1$. The number $(\sigma - \gamma a)a$ is red since $\sigma - \gamma a \geqslant 2$ follows from $(\sigma - \gamma a)b = (\alpha - \gamma b)a + \tau \geqslant b + \tau$ and $\sigma - \gamma a < \alpha$ follows from $(\sigma - \gamma a)b - \alpha b = \alpha(a-b) - \gamma ba + \tau \leqslant (a - b - \gamma b)a + \tau = (\varepsilon - b)a + \tau < 0$. Now the solution $((\alpha - \gamma b)a, \tau, (\sigma - \gamma a)a)$ forces $(\alpha - \gamma b)a$ to be green and therfore it remains only the possibility that $\alpha - \gamma b = 1$ and a is green.

The cases $\varepsilon = 1$ and $\varepsilon \geqslant 2$ will be distinguished.

If $\varepsilon = 1$ then $a = \gamma b + 1 = \alpha$.

If b is even then $a = \gamma b + 1$ is odd and $(a+1)/2$ is an integer. Next, $(b/2)a$ and $((a+1)/2)a$ are red since $2 \leqslant b/2 < b \leqslant \gamma b = \alpha - 1$ and $2 \leqslant b/2 < (a+1)/2 \leqslant a - 1 = \alpha - 1$. Then $((b/2)a, b/2, ((a+1)/2)a)$ determines $b/2$ to be green. However, now $(b/2, b/2, a)$ is green, a contradiction.

If b is odd then the solutions $((b-1)/2, ((b+1)/2)a, (((b+1)/2)\gamma + 1)a)$ and $((b+1)/2, ((b-1)/2)a, (((b-1)/2)\gamma + 1)a)$ force $(b-1)/2$ and $(b+1)/2$ to be green since $2 \leqslant (b-1)/2 < (b+1)/2 \leqslant ((b-1)/2)\gamma + 1 < ((b+1)/2)\gamma + 1 \leqslant \gamma b = \alpha - 1$. Then the solution $((b-1)/2, (b+1)/2, a)$ is green, a contradiction.

If $\varepsilon \geqslant 2$ then εa is red since $\varepsilon < b \leqslant \gamma b = \alpha - 1$. The solution (a^2, δ, ha) forces δ to be red and then $(\varepsilon a, \delta, r - \gamma a^2)$ implies that $r - \gamma a^2$ is green. Together with $(r/a) - \gamma a = (\delta + \varepsilon a)/b > \delta/b + \varepsilon > 2$ it follows that $\alpha \leqslant (r/a) - \gamma a$.

The case $\varepsilon \leqslant b/2$ cannot occur since $\gamma b + 1 = \alpha \leqslant (r/a) - \gamma a = (\varepsilon a + \delta)/b = (\varepsilon(\gamma b + \varepsilon) + \delta)/b \leqslant ((b/2)(\gamma b + b/2) + \delta)/b$ leads to $(2\gamma - 1)b^2 \leqslant 4(\delta - b)$ which is impossible for $b > \delta$ and $\gamma \geqslant 1$.

If $\varepsilon > b/2$ then $(\gamma b + b - \varepsilon)a$ is green since $\gamma b + b - \varepsilon \geqslant \gamma b + 1 = \alpha$ and $\gamma b + b - \varepsilon < \gamma b + b/2 < \gamma b + \varepsilon = a < r/a$. The number $(2 + (\gamma+1)a - (r/a - \gamma a))a$ is green since $(\gamma+1)a - (r/a - \gamma a) \leqslant r/a - 2$ follows from $(r/a - 1)b + \delta = (\gamma b + b - \varepsilon)a + (2\varepsilon - b - 1)a + a - b + 2\delta > (\gamma b + b - \varepsilon)a = ((\gamma+1)a - (r/a - \gamma a))b + \delta$ and since $(\gamma+1)a - (r/a - \gamma a) = (\gamma+1)a - (\varepsilon a + \delta)/b > (\gamma+1)a - (ba+b)/b = \gamma a - 1 > \alpha - 2$. Then $((\gamma b + b - \varepsilon)a, 2b - \delta, (2 + (\gamma+1)a - (r/a - \gamma a))a)$ implies that $2b - \delta$ is red. This gives the contradiction that $(\delta, 2b - \delta, 2a)$ is red. □

3. Remarks

For $k \geqslant 4$ numbers $R_k(a,b)$ for $a \neq b$ and $b=2a$ do not exist (see [6]). If $2b \leqslant a$ or $4a \leqslant b$ then also for $k=3$ Rado numbers $R_3(a,b)$ do not exist (see [4, Theorem 2]). If $a<2b<8a$ in the case of $k=3$ then a general proof for the existence of $R_3(a,b)$ is unknown. Besides the Schur number $R_3(1,1)=14$ and the trivial Rado number $R_3(1,2)=1$ only the four numbers $R_3(3,1)=54$, $R_3(3,2)=54$, $R_3(4,3)=108$, and $R_3(5,2)=105$ which were determined with the help of a computer are known so far.

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