Discrete Mathematics 346 (2023) 113597

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Two-color Rado number of $x + y + c = kz$ for large c

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ARTICLE INFO

Article history:
Received 30 December 2022
Received in revised form 8 June 2023
Accepted 2 July 2023
Available online 18 July 2023

Keywords:
Rado number
Schur number
Ramsey theory

ABSTRACT

For positive integers c and k, we consider the equation $L: x + y + c = kz$. Two-color Rado number $R = R(c,k)$ of L is the least integer, provided that it exists, such that every two-coloring of $1, 2, \cdots, R$ admits a monochromatic solution to L. The Rado number $R(c,k)$ exists if and only if k is odd or c is even.

In this paper, we show that if k is odd or c is even with $k \geq 5$ and $c \geq 2k^3 + 2k^2 - k$, then the two-color Rado number of L is equal to $N = \left\lceil \frac{2\lceil \frac{c+2}{k} \rceil + c}{k} \right\rceil$, as predicted by Jones and Schaal.

1. Introduction

The Rado number theory is a Ramsey theory for an equation or system of equations. This originated in the work [17] of Schur who showed that there exists a natural number N such that every r-coloring of $[1, N] = \{1, 2, \cdots, N\}$ admits a monochromatic solution $x, y, z \in [1, N]$ to equation $x + y = z$. The smallest such number N is called the r-color Schur number, $S(r)$.

As is well known, $S(2) = 5$, $S(3) = 14$, $S(4) = 45$ [18], and $S(5) = 161$ [8], whereas $S(r)$ is unknown for $r \geq 6$. Rado [3,15] extended this theory to all linear equations and all systems of homogeneous linear equations. He determined the necessary and sufficient condition that for a given system of homogeneous linear equations, there exists a natural number N such that every r-coloring of $[1, N]$ admits a monochromatic solution. The smallest number N satisfying this property is called the *r-color Rado number* of the system. However, determining the Rado number for an equation (or a system of equations) is generally not an easy problem.

A natural generalization of the Schur equation $x + y = z$ is $x_1 + x_2 + \cdots + x_{m-1} = x_m$. Beutelspacher and Brestovansky [1] determined the two-color Rado number for it. Further, the two-color Rado number for $a_1x_1 + a_2x_2 + \ldots + a_mx_m = x_0$ was studied by Hopkins and Schaal [9], and computed by Guo and Sun [7]. Schaal [16] computed the two-color Rado number for $\sum_{i=1}^{m-1} x_i + c = x_m$ when $c > 0$. Kosek and Schaal [11] computed the two-color Rado number for the same equation when $c < 0$. The two-color Rado numbers for equations $x_1 + ax_2 - x_3 = c$ [4] and $\sum_{i=1}^{m-2} x_i + ax_{m-1} - x_m = c$ [5] were studied by Dwivedi and Tripathi. The two-color Rado number of $x + y + c = kz$ has been studied by several researchers [6,10,12–14]; its Rado number is denoted $R(c,k)$. In this paper, we also study the two-color Rado number $R(c,k)$ for this equation. We show that $R(c,k) = \left\lceil \frac{2\lceil \frac{c+2}{k} \rceil + c}{k} \right\rceil$ for $k \geq 5$ and $c \geq 2k^3 + 2k^2 - k$.

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https://doi.org/10.1016/j.disc.2023.113597
0012-365X/

2. Two-color Rado number of $x+y+c=kz$

One of the most studied equations in the theory of Rado numbers is $L: x+y+c=kz$ for integers c and $k\geq 1$. Let $R(c,k)$ be the two-color Rado number for L. If k is even and c is odd, then the two-coloring $\chi(x)=x \pmod 2$ admits no monochromatic solutions to L. Thus, throughout this paper, we consider $R(c,k)$ only when k is odd or c is even.

Burr et al. [2] showed that $R(0,1)=5$, $R(0,2)=1$, $R(0,3)=9$, $R(0,k)=\frac{k(k+1)}{2}$ for $k\geq 4$ and $R(c,1)=4c+5$ for all $c\geq 1$. Jones and Schaal [10] proved that for $c\geq 0$ and $k\geq 1$, $R(c,k)$ is finite if and only if k is odd or c is even. Martinelli and Schaal [14] determined that

$$LB(c,k)=\left\lceil \frac{2\lceil\frac{c+2}{k}\rceil+c}{k}\right\rceil$$

is a lower bound of $R(c,k)$ for all $c\geq 0$ and $k\geq 1$. Jones and Schaal [10] predicted that for each k, $R(c,k)=LB(c,k)$ for all sufficiently large c, based on empirical evidence. Martinelli and Schaal [14], meanwhile, proved that $R(c,2)=LB(c,2)$ for all even c, and studied $R(c,3)$. Kézdy et al. [12] proved that $R(c,3)=LB(c,3)$ for all $c\geq 13$. Additionally, they noted that $R(c,k)=LB(c,k)$ when $c=\frac{k^3-k}{2}$, and $R(c,k)>LB(c,k)$ when $c=\frac{k^3-k}{2}+1$ for odd $k\geq 5$. Guo [6] demonstrated that $R(c,4)=LB(c,4)$ for all even $c\geq 34$.

The authors of this paper computed $R(c,k)=\frac{1}{2}(k(k+1)-c+1)$ when both c and k are odd and $k\geq c+6$ [13]. In this case, $R(c,k)$ is significantly larger than $LB(c,k)$. In this paper, we study the reverse case for the same equation when c is sufficiently larger than k. We show that the Rado numbers agree with $LB(c,k)$, as predicted by Jones and Schaal [10]. The main theorem is as follows.

Theorem 1. *Assume that $k\geq 5$, $c\geq 2k^3+2k^2-k$ and that either k is odd or c is even. Then, the two-color Rado number $R(c,k)$ of $x+y+c=kz$ is equal to $LB(c,k)=\left\lceil \frac{2\lceil\frac{c+2}{k}\rceil+c}{k}\right\rceil$.*

For all $c\geq 13=\frac{3^3-3}{2}+1$, $R(c,3)=LB(c,3)$ was shown by Kézdy et al. [12], and for all even $c\geq 34=\frac{4^3-4}{2}+4$, $R(c,4)=LB(c,4)$ was shown by Guo [6]. Even when $k\geq 5$, the condition $c\geq 2k^3+2k^2-k$ in Theorem 1 is not sharp.

Remark. A consequence of Theorem 1 is that for each $k\geq 2$ there is the smallest positive integer $\nu(k)$ such that $R(c,k)=LB(c,k)$ whenever $c\geq \nu(k)$ under the assumption k is odd or c is even. It would also be interesting to determine $\nu(k)$ for all k. From Theorem 1 and the work of Kézdy et al. [12], we have $\nu(k)\leq 2k^3+2k^2-k$ for all $k\geq 5$ and $\nu(k)\geq \frac{k^3-k}{2}+1$ for odd $k\geq 3$. It is known that $\nu(2)=1$, $\nu(3)=13$ [12] and $\nu(4)\leq 34$ [6]. We can check that $R(30,4)=LB(30,4)=12$ and $R(32,4)=LB(32,4)=13$, and $12=R(28,4)>LB(28,4)=11$. Thus, $\nu(4)=30$. Theorem 1 indicates that $R(c,5)=LB(c,5)$ if $c\geq 295$. In the following example, we have $\nu(5)\geq 66$. The authors of this paper also checked that for all $c\in[66,100]$, $R(c,k)=LB(c,k)$. We conjecture that $\nu(5)=66$.

Example. When $k=5$ and $c=65$, let $\chi:[1,19]\to[0,1]$,

$$\chi(x)=\begin{cases}0, & \text{if } x=1,2,6,7,11,12,15,16,17\\ 1, & \text{if } x=3,4,5,8,9,13,14,18,19.\end{cases}$$

Then, no monochromatic solution for $x+y+65=5z$ exists. Thus, $R(65,5)>19=LB(65,5)$. In fact, $R(65,5)=20$. Consequently, we obtain $\nu(5)\geq 66$.

3. Proof of main theorem

Let m,r and N be integers such that

$$c=km+r\ (0\leq r\leq k-1) \text{ and } N=LB(c,k)=\left\lceil \frac{2\lceil\frac{c+2}{k}\rceil+c}{k}\right\rceil. \tag{1}$$

As $c\geq 2k^3+2k^2-k$, we obtain $m\geq 2k^2+2k-1$. As $N=m+\left\lceil \frac{2m+r+2\lceil\frac{r+2}{k}\rceil}{k}\right\rceil$, we have

$$m+\frac{2m+2}{k}\leq N\leq m+\frac{2m+r+k+3}{k}\leq m+\frac{2m+2k+2}{k}. \tag{2}$$

Suppose χ is a two-coloring of $[1,N]$ with colors 0 and 1. Let $A=\{x\in[1,N]|\chi(x)=0\}$ and $B=\{x\in[1,N]|\chi(x)=1\}$. We assume that no monochromatic solution exists for $x+y+c=kz$. Let $u\in[1,k]$ such that

$$r+2u=k\alpha_1 \tag{3}$$

for some $\alpha_1 \in \mathbb{Z}$. The existence of u and α_1 follows from the assumption that k is odd or c is even. As $2 \le k\alpha_1 = r + 2u \le 3k-1$, α_1 is 1 or 2. We may assume $\chi(u) = 0$. As $u \in A$, we have an integer $h_1 \ge 0$ such that

$$u, k+u, \cdots, kh_1 + u \in A \text{ and } k(h_1+1) + u \notin A. \tag{4}$$

Lemma 1. *Let h_1 be the number satisfying (4). Then, we have $h_1 < \frac{m+2}{k}$ and $k(h_1+1)+u \in B$.*

Proof. Suppose that $h_1 \ge \frac{m+2}{k}$. As $m \ge 2k^2 + 2k - 1$, we have $h_1 \ge \frac{m+2}{k} > 2k+2$, and thus, $h_1 \ge 2k+3$. Hence, for all $i \in [0, k-1]$, $ki + u \in A$. As

$$u + (ki + u) + c = ki + 2u + km + r = k(m + \alpha_1 + i),$$

$m + \alpha_1 + i \notin A$. As $\alpha_1 + i \le 2 + k$, we have $m + \alpha_1 + i \le m + 2 + k \le m + \frac{2m+2}{k} \le N$, and thus, $m + \alpha_1 + i \in B$. Choose $i \in [0, k-1]$ such that $m + \alpha_1 + i - u = kj$ for some $j \in \mathbb{Z}$. Then, $kj + u \in B$, and thus, $h_1 + 1 \le j$. As $m + \alpha_1 \le kj + u \le m + \alpha_1 + k - 1$, we have

$$\frac{m+2}{k} \le h_1 \le j - 1 \le \frac{m + \alpha_1 - u - 1}{k} \le \frac{m}{k}.$$

This is a contradiction. Thus, $h_1 < \frac{m+2}{k}$.

As $k(h_1+1) + u < m + 2 + 2k \le m + \frac{m+2}{k} \le N$, we obtain $k(h_1+1)+u \in B$. □

From Lemma 1, there exists an integer $h_2 \ge h_1 + 1$ such that

$$k(h_1+1)+u, k(h_1+2)+u, \cdots, kh_2 + u \in B \text{ and } k(h_2+1)+u \notin B. \tag{5}$$

Lemma 2. *Let h_1 be the number satisfying (4). Then, we have $h_1 > \frac{m-k^2}{k}$.*

Proof. We assume, to the contrary, that $h_1 \le \frac{m-k^2}{k}$.

If $h_2 < \frac{m+k+1}{k}$, then since $k(h_2+1)+u \le m+k+1+2k \le m + \frac{2m+2}{k} \le N$, we have $k(h_2+1)+u \in A$. From inequality (2), we note that $m + \frac{2m+2}{k} \le N$. Since

$$(kh_1 + u) + (k(h_2+1)+u) + c = k(m + \alpha_1 + h_1 + h_2 + 1)$$

and

$$m + \alpha_1 + h_1 + h_2 + 1 < m + 2 + \frac{m-k^2}{k} + \frac{m+k+1}{k} + 1 \le m + \frac{2m+2}{k} \le N,$$

we have $m + \alpha_1 + h_1 + h_2 + 1 \in B$. Thus, this presents a contradiction because

$$\Big(k(h_1+1)+u\Big) + (kh_2 + u) + c = k(m + \alpha_1 + h_1 + h_2 + 1).$$

Thus, $h_2 \ge \frac{m+k+1}{k}$.

Additionally,

$$m + \alpha_1 + 2h_1 + 2 \le m + 4 + 2\left(\frac{m-k^2}{k}\right) \le m + \frac{2m+2}{k} \le N.$$

As

$$N - m - \alpha_1 \le \frac{2m+2k+2}{k} - \alpha_1 \le 2\left(\frac{m+k+1}{k}\right) \le 2h_2,$$

for all i such that $2h_1 + 2 \le i \le N - m - \alpha_1$, we have that there exist $i_1, i_2 \in [h_1+1, h_2]$ satisfying $i = i_1 + i_2$. Since $ki_1 + u, ki_2 + u \in B$, $m + \alpha_1 + i \le N$ and

$$(ki_1 + u) + (ki_2 + u) + c = k(m + \alpha_1 + i_1 + i_2) = k(m + \alpha_1 + i),$$

we have $m + \alpha_1 + i \in A$. Consequently, we have $[m + \alpha_1 + 2h_1 + 2, N] \subset A$. Since

$$\begin{aligned} N - (m + \alpha_1 + 2h_1 + 2) &\ge m + \frac{2m+2}{k} - \Big(m + 4 + 2(\frac{m-k^2}{k})\Big) \\ &= \frac{2k^2 - 4k + 2}{k} \ge k - 1, \end{aligned}$$

there exists j_1 such that $m+\alpha_1+2h_1+2 \le kj_1+u \le N$. Thus, $kj_1+u \in A$, and hence, $h_2 < j_1$. Additionally, $k(h_2+1)+u \in A$. Furthermore, since

$$u+\Big(k(h_2+1)+u\Big)+c = k(m+\alpha_1+h_2+1)$$

and

$$m+\alpha_1+h_2+1 \le m+\alpha_1+j_1 \le m+\frac{N-u}{k}+2 \le m+\frac{2m+2}{k} \le N,$$

we have $m+\alpha_1+h_2+1 \in B$. As $[m+\alpha_1+2h_1+2, N] \subset A$, we have $m+\alpha_1+h_2+1 \le m+\alpha_1+2h_1+1$, which implies $2h_1 \ge h_2$. Let $j_2 = 2h_1-h_2+1$. As $1 \le j_2 = 2h_1-h_2+1 \le h_1$, we have $kj_2+u \in A$, and because $kj_2+u, k(h_2+1)+u \in A$ and

$$(kj_2+u)+\Big(k(h_2+1)+u\Big)+c = k(m+\alpha_1+j_2+h_2+1) = k(m+\alpha_1+2h_1+2),$$

we have $m+\alpha_1+2h_1+2 \in B$. However, this is a contradiction, as

$$\Big(k(h_1+1)+u\Big)+\Big(k(h_1+1)+u\Big)+c = k(m+\alpha_1+2h_1+2).$$

Thus, $h_1 > \frac{m-k^2}{k}$. $\square$

Lemma 3. *If $M = \min\{N, m+\alpha_1+2h_1\}$, then $[m+\alpha_1, M] \subset B$.*

Proof. If $0 \le i \le 2h_1$, then $i = i_1+i_2$ for some $i_1, i_2 \in [0, h_1]$. Considering

$$(ki_1+u)+(ki_2+u)+c = k(m+\alpha_1+i_1+i_2) = k(m+\alpha_1+i),$$

we have $m+\alpha_1+i \notin A$. Thus, $[m+\alpha_1, m+\alpha_1+2h_1] \cap A = \emptyset$, and hence, $[m+\alpha_1, M] \subset B$. $\square$

Proof of Theorem 1. Choose $i_1 \in \mathbb{Z}$ such that $m+\alpha_1-2u+k \le ki_1 \le m+\alpha_1-2u+2k-1$. For all $i \in [1,k]$, as $2h_1 \ge \frac{2m-2k^2}{k} \ge 2k-1$, we have

$$m+\alpha_1 \le ki_1+2u-i \le m+\alpha_1+2k-1 \le \min\{N, m+\alpha_1+2h_1\}.$$

Thus, from Lemma 3, $ki_1+2u-i \in B$.

As

$$0 \le i_1 \le \frac{m+\alpha_1+2k-1-2u}{k} \le \frac{2m-2k^2}{k}+\frac{2k^2-m+2k-1}{k} \le 2h_1,$$

by Lemma 3, we have $m+\alpha_1+i_1 \in B$. Since

$$i+(ki_1+2u-i)+c = k(m+\alpha_1+i_1),$$

we have $i \in A$ for all $i \in [1,k]$. Let $\alpha_2 = \left\lceil \frac{r+2}{k} \right\rceil$. As $r+2 \le k\alpha_2 < r+2+k$, we have $k\alpha_2 = r+i_3+i_4$ for some $i_3, i_4 \in [1,k] \subset A$. Since $m+\alpha_2 \le m+2 \le m+\frac{2m+2}{k} \le N$ and

$$i_3+i_4+c = k(m+\alpha_2),$$

we have $m+\alpha_2 \in B$.

Moreover, as $\alpha_1, \alpha_2 \in \{1,2\}$ and $2h_1 \ge 2k-1 \ge k$, by Lemma 3 we have $[m+\alpha_2, m+\alpha_2+k] \subset B$. If we let $\alpha_3 = N-m$, then $\alpha_3 = \left\lceil \frac{2m+r+2\alpha_2}{k} \right\rceil$. Thus, there is $i_5 \in [0, k-1]$ such that $k\alpha_3 = 2m+r+2\alpha_2+i_5$. As $[m+\alpha_2, m+\alpha_2+k] \subset B$, we have $m+\alpha_2, m+\alpha_2+i_5 \in B$, and because

$$(m+\alpha_2)+(m+\alpha_2+i_5)+c = k(m+\alpha_3) = kN,$$

we have $N \in A$. Since

$$\begin{aligned}
1 &\le (k-1)(m+\frac{2m+2}{k})-km-r \\
&\le (k-1)N-c \\
&\le (k-1)(m+\frac{2m+2k+2}{k})-km-r \\
&= m+2k-\frac{2m+2}{k}-r \le m+\frac{2m+2}{k} \le N,
\end{aligned}$$

we have $(k-1)N-c \in [1,N]$. As

$$\Big((k-1)N-c\Big)+N+c=kN,$$

we have $(k-1)N-c \in B$. Choose $i_6 \in [0,k-1]$ such that $(k-1)N-c+m+\alpha_2+i_6=kj_1+2u$ for some $j_1 \in \mathbb{Z}$. Note that $m+\alpha_2+i_6 \in [m+\alpha_2, m+\alpha_2+k] \subset B$. Since

$$\begin{aligned} m+\alpha_1+j_1 &\leq m+2+\frac{(k-1)N-c+m+\alpha_2+i_6-2u}{k} \\ &\leq m+2+\frac{(k-1)\left(\frac{(k+2)m+r+k+3}{k}\right)-km-r+m+2+k-1}{k} \\ &\leq m+\frac{2m+2}{k} \leq N \end{aligned}$$

and $m+\alpha_1+j_1 \leq m+\alpha_1+2h_1$, we have $m+\alpha_1+j_1 \in B$ by Lemma 3. Then,

$$((k-1)N-c)+(m+\alpha_2+i_6)+c=k(m+\alpha_1+j_1),$$

and thus we obtain a contradiction. Therefore, $R(c,k) \leq N$. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgement

The authors thank the reviewers for their constructive and valuable suggestions regarding Revisions of the article. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (Grant Nos. NRF-2018R1D1A3B07048195) (B.M.K.). This result was supported by "Regional Innovation Strategy (RIS)" through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (MOE) (2021RIS-004) (W.H.). This work was supported by Korea University grant (W.H.).

References

[1] A. Beutelspacher, W. Brestovansky, Generalized Schur number, Lect. Notes Math. 969 (1982) 30–38.
[2] S. Burr, S. Loo, D. Schaal, on Rado numbers I, preprint.
[3] W. Deuber, Developments based on Rado's dissertation 'Studien zur Kombinatorik', in: Survey Combin., Cambridge University Press, 1989, pp. 52–74.
[4] S. Dwivedi, A. Tripathi, On the two-color Rado number for $x_1+ax_2-x_3=c$, Integers 20 (2020), #A36.
[5] S. Dwivedi, A. Tripathi, On the two-color Rado number for $\sum_{i=1}^{m-2} x_i+ax_{m-1}-x_m=c$, Integers 22 (2022), #A90.
[6] S. Guo, Two-color Rado number for $x+y+c=4z$, Ars Comb. 94 (2010) 257–264.
[7] S. Guo, Z. Sun, Determination of the two-color Rado number for $a_1x_1+\cdots+a_mx_m=x_0$, J. Comb. Theory, Ser. A 115 (2008) 345–353.
[8] M. Heule, Schur number five, in: The Thirty-Second AAAI Conference on Artificial Intelligence (AAAI-18), 2018.
[9] B. Hopkins, D. Schaal, On Rado numbers for $\sum_{i=1}^{m-1} a_ix_i=x_m$, Adv. Appl. Math. 35 (2005) 433–441.
[10] S. Jones, D. Schaal, Two-color Rado numbers for $x+y+c=kz$, Discrete Math. 289 (2004) 63–69.
[11] W. Kosek, D. Schaal, Rado numbers for the equation $\sum_{i=1}^{m-1} x_i+c=x_m$ for the negative values of c, Adv. Appl. Math. 27 (2001) 805–815.
[12] A. Kézdy, H. Snevily, S. White, Generalized Schur numbers for $x_1+x_2+c=3x_3$, Electron. J. Comb. 16 (2009) 1–13.
[13] B. Kim, W. Hwang, B. Song, Two-color Rado number of $x+y+c=kz$ for odd c and k with $k \geq c+6$, Discrete Math. 345 (3) (2022) 112750.
[14] B. Martinelli, D. Schaal, On generalized Schur numbers for $x_1+x_2+c=kx_3$, Ars Comb. 85 (2007) 33–42.
[15] R. Rado, Studien zur Kombinatorik, Math. Z. 36 (1933) 424–480.
[16] D. Schaal, On generalized Schur numbers, Congr. Numer. 98 (1993) 178–187.
[17] I. Schur, Uber die Kongruenz $x^m+y^m \equiv z^m \pmod p$, Jahresber. Dtsch. Math.-Ver. 25 (1916) 114–117.
[18] W. Wallis, A. Street, J. Wallis, Combinatorics: Room Squares, Sum-Free Sets, Hadamard Matrices, Lecture Notes in Math., vol. 48 and, Springer, 1972.