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Extremal Uncrowded Hypergraphs

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Let G be a $(k+1)$-graph (a hypergraph with each hyperedge of size $k+1$) with n vertices and average degree t. Assume $k \ll t \ll n$. If G is uncrowded (contains no cycle of size 2, 3, or 4) then there exists an independent set of size $c_k(n/t)(\ln t)^{1/k}$.

Let G be a graph with n vertices and average valence t. Turan's theorem implies $\alpha(G) \geqslant n/(t+1)$. (See Section 1 for notation.) Fix $k \geqslant 1$. Let G be a $(k+1)$-graph with n vertices and average valence $t^k (k \ll t \ll n)$. It is known that $\alpha(G) \geqslant cn/t$. A hypergraph is called uncrowded if it contains no cycles of length 2, 3, or 4. We prove that if G is an uncrowded $(k+1)$-graph with n vertices and average degree t^k, then $\alpha(G) \geqslant (cn/t)(\ln t)^{1/k}$.

The case $k=1$ has been studied in [1, 2] and the case $k=2$ in [3]. In these papers various applications are discussed.

Notation, a description of the basic transformation, and two technical lemmas are given in Section 1. The reader is advised to skim this section and use it as a reference when examining the proof. The heart of the paper is Section 2. The proof is given with certain details left out. The formal proof is given in Section 3.

1. Preliminaries

Hypergraphs

A hypergraph is a pair (V, G), where G is a family of nonempty subsets of V. The $x \in V$ are called vertices, the $E \in G$ are called hyperedges. Throughout this paper k is an arbitrary but fixed positive integer. We assume (tacitly) that all $E \in G$ satisfy

$$2 \leqslant |E| \leqslant k+1.$$

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If all $|E| = k+1$ then (V, G) is a $(k+1)$-graph. When $k = 1$ this corresponds to the usual notion of graph. For $x \in V$ we set

$$\mathscr{F}_x = \{E \subseteq V - \{x\} : E \cup \{x\} \in G\}.$$

That is, $\mathscr{F}_x$ is the family of hyperedges containing x, with x deleted. We set

$$\deg_i(x) = |\{E \in \mathscr{F}_x : |E| = i\}|.$$

Note that $\deg_i(x)$ is the number of hyperedges of size $(i+1)$ containing x. We say (V, G) is regular if for $1 \leqslant i \leqslant k$, $\deg_i(x) = \deg_i(y)$ for all $x, y \in V$. The neighborhood of x, denoted by $N(x)$, is defined by

$$N(x) = \bigcup_{E \in \mathscr{F}_x} E.$$

Thus, $y \in N(x)$ iff x, y lie on a common hyperedge. For convenience, we set

$$N^+(x) = \{x\} \cup N(x).$$

The t-neighborhood $N^t(x)$ is defined inductively by

$$z \in N^t(x) \qquad \text{iff} \quad z \in N(y) \text{ for some } y \in N^{t-1}(x),$$

and the distance metric $\rho(x, y)$ is that minimal t so that $y \in N^t(x)$.

A set $I \subseteq V$ is *independent* if I contains no hyperedges $E \in G$. The restriction of (V, G) to a subset $W \subseteq V$, denoted by $G|_W$, is given by $(W, G|_W)$, where $G|_W = \{E \in G : E \subseteq W\}$. We call (V, G) uncrowded if for all $x \in V$, the following hold:

(i) The sets $E \in \mathscr{F}_x$ are pairwise disjoint (i.e., $|E_1 \cap E_2| \leqslant 1$ for all $E_1, E_2 \in G$).

(ii) Let $y \in N(x)$, $E \in \mathscr{F}_y$ with $x \notin E$. Then $E \cap N(x) = \varnothing$.

(iii) Let $y, z \in N(x)$, $E \in \mathscr{F}_y$, $F \in \mathscr{F}_z$, $x \notin E$, $x \notin F$. Then $E \cap F = \varnothing$.

In usual hypergraph terminology, (V, G) is uncrowded iff it has no cycles of length 2, 3, or 4. When (V, G) is uncrowded and regular, the restriction to $N^2(x)$ is known precisely. (See Fig. 1.)

When $\rho(x, y) = 1$, we let E_{xy} be the unique set in $\mathscr{F}_x$ that contains y (i.e., E_{xy} is the common hyperedge with x deleted).

The Transformation

Fix a hypergraph (V, G) and a vertex set $C \subseteq V$. The $v \in C$ are called chosen. We set

$$D = \{v \in V : E \subseteq C \text{ for some } E \in \mathscr{F}_v\}.$$

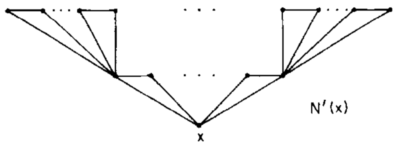

FIG. 1. $N^2(x)$.

The $v \in D$ are called discarded. We set $I = \bar{D} \cap \bar{C}$ and call the $v \in I$ isolated. We set $V^* = \bar{D} \cap \bar{C}$ and call the $v \in V^*$ remaining. We define a hypergraph (V^*, G^*) by letting $E' \in G^*$ iff $E' = E \cap V^*$, where $E \in G$ and $E \subseteq V^* \cup C$ and $E \cap V^* \neq \emptyset$. Our definitions ensure

$$\alpha(G) \geqslant |I| + \alpha(G^*). \tag{$*$}$$

For if $E \in G$ and $E \subseteq D = I \cup V^*$, then $E \cap V^* \in G^*$. Thus if J is independent in (V^*, G^*), there can be no $E \in G$ with $E \subseteq I \cup J$ so $I \cup J$ is independent. (G^* was defined so as to give the forbidden sets on V^* and to give $(*)$.) (See Fig. 2.)

Remark. G^* was defined so that $(*)$ would be satisfied. In fact G^* contains more hyperedges than we need. Suppose $\{x, y, z\} \in G$ and $y, z \in V^*$ and $x \in C \cap D$. Then $\{x, y\} \in G^*$ even though x has been discarded. These "additional" hyperedges allow us to check if $\{y, z\} \in G^*$ without examining the neighbors of x to see if $x \in D$.

Remark. One could similarly replace $(*)$ by $\alpha(G) \geqslant \alpha(G|_C) + \alpha(G^*)$.

Remark. (V^*, G^*) includes $G|_{V^*}$. Moreover, (V^*, G^*) contains no additional hyperedges of size $(k+1)$. (V^*, G^*) contains no singletons $E' = \{v\}$ since then $v \in E \subseteq \{v\} \cup C$ so $v \notin V^*$.

Let $x \in V$, $E \in \mathscr{F}_x$. If $E \subseteq C \cup V^*$ we say E is *transformed* into $E' = E \cap V^*$. If $E \subseteq C \cup V^*$ we say E is *discarded.* We set $\mathscr{F}_x^*$ equal to those $E' = E \cap V^*$ where $E \subseteq C \cup V^*$ and $E \in \mathscr{F}_x$. We set $\deg_i^*(x)$ equal to the number of $E' \in \mathscr{F}_x^*$ with $|E'| = i$. We set $\deg_{ij}^*(x)$ equal to the number of $E \in \mathscr{F}_x$ with $|E| = j$ such that E is transformed to $E' \in \mathscr{F}_x^*$ with $|E'| = i$. When $x \in V^*$, $\mathscr{F}_x^*$ and $\deg_i^*(x)$ give analogues to $\mathscr{F}_x$ and $\deg_i(x)$ for the graph (V^*, G^*). (When $x \notin V^*$ we might think of $\mathscr{F}_x^*$ as being what $\mathscr{F}_x^*$

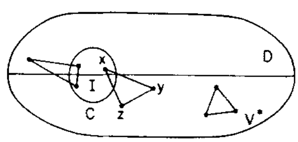

FIG. 2. The transformation.

would have been had x been in V^*. The defining of $\mathscr{F}_x^*$ and $\deg_i^*(x)$ in these cases is only a technical convenience.)

Remark. Essentially we will find an independent set in (V, G) by selecting C so that I is reasonably large and then examining (V^*, G^*).

We give a technical lemma here.

ALMOST REGULAR LEMMA. *Let (V, G) be uncrowded, $|V| = n$ such that for all $x \in V$*

$$\deg_i(x) \leqslant a_i, \qquad 1 \leqslant i \leqslant k.$$

Then there exists $G^+ \supseteq G$ so that (letting $\deg^+$ *represent degree in (V^+, G^+))*

(i) *(V, G^+) is uncrowded.*

(ii) *$\deg_i^+(x) \leqslant a_i$, $1 \leqslant i \leqslant k$, all x.*

(iii) *Set $B = \{x: \deg_i^+(x) \neq a_i$, some $i\}$. Then $|B| \leqslant k^2 b^2$, where $b = 1 + \sum_{i=1}^k ia_i$.*

Proof. Let G^+ be a maximal set so that $G^+ \supseteq G$ and (i), (ii) are satisfied. Set $B_i = \{x: \deg_i^+(x) \leqslant a_i\}$ so that $B = \bigcup_{i=1}^k B_i$. Suppose $|B_i| > ib^3$. Select $x_1, \ldots, x_{i+1} \in B$ inductively, letting x_j be an arbitrary element not in $N^3(x_s)$ for $s < j$. (This is possible as $|N^3(x)| \leqslant b^3$.) Add $\{x_1, \ldots, x_{i+1}\}$ to G^+. Condition (ii) remains satisfied since all $\deg_i(x_j) < a_i$. No crowding is created since that would imply $x_s \in N^3(x_t)$ for some s, t. This is a contradiction. Thus $|B_i| \leqslant ib^3$ and $|B| \leqslant \sum |B_i|$ so (iii) is satisfied.

Remark. In application, the a_i will be functions of a parameter t and we will write $|B| \leqslant \bar{t}$, where $\bar{t}$ is a function of t. We will need only $\bar{t} \ll n$.

Probability

The statement "C has distribution $V(p)$" means that C is a random variable whose values are subsets of V such that for every $v \in V$

$$\Pr\{v \in C\} = p$$

and these probabilities are mutually independent. We may think of C as the result of a coin flipping experiment. Note that I, V^*, G^*, $\deg_i^*(x)$, $\deg_{ij}^*(x)$, being dependent on C, also become random variables.

We give one more technical lemma here.

ALMOST INDEPENDENT LEMMA. *Let $X_1,\ldots,X_m$ be random variables assuming values* 0, 1 *with* $\Pr\{X_i=1\}=p$. *Assume that for each i, X_i and X_j are independent for all but at most s j's. Set $Y=\sum_{i=1}^m X_i$. Then*

$$P\{|Y-mp|>\varepsilon mp\}\leqslant\frac{1}{mp}\left(\frac{s}{\varepsilon^2}\right).$$

Proof. $\operatorname{Var}(Y)=\sum_{i,j=1}^m \operatorname{Cov}(X_i,X_j)$.
The covariance is zero for all but at most ms pairs. In those cases

$$\operatorname{Cov}(X_i,X_j)=E[X_iX_j]-p^2\leqslant p(1-p).$$

Thus,

$$\operatorname{Var}(Y)\leqslant ms\,p(1-p)\leqslant ms\,p$$

and the lemma follows from Chebyschev's Inequality.

Remark. In applications s will be small relative to m and we will say, roughly, $T\sim mp$ almost always. When $s=1$, Y is the Binomial Distribution. In this case Chebyschev's Inequality is quite weak but we do need stronger methods.

2. THE PROOF (DETAILS OMITTED)

We assume $k\ll t\ll n$ tacitly throughout this section.

Let (V,G) be a regular uncrowded $(k+1)$-graph with n vertices and valence t^k. Let $C\subseteq V$ have distribution $V(p)$, where $p=1/t$. The set C defines I, (V^*,G^*) by the transformation of Section 1.

Let $x\in V$. For each $E\in\mathscr{F}_x$, $\Pr\{E\subseteq C\}=p^k$. Since the sets E are disjoint, these events are mutually independent and

$$\begin{aligned}\Pr\{x\notin D\}&=\Pr\{E\nsubseteq C \text{ for all } E\in\mathscr{F}_x\}\\&=\prod_{E\in\mathscr{F}_x}\Pr\{E\nsubseteq C\}\\&=(1-p^k)^{t^k}\sim\exp[-p^kt^k]=e^{-1}.\end{aligned}$$

The events $x\in D$, $x\in C$ are independent so

$$\Pr\{x\in I\}=\Pr\{x\notin D\}\Pr\{x\in C\}\sim 1/et,$$
$$\Pr\{x\in V^*\}=\Pr\{x\notin D\}\Pr\{x\notin C\}\sim(1-p)\,e^{-1}\sim e^{-1}.$$

If $\rho(x, y) > 2$, the events $x \in I$, $y \in I$ are independent, as are the events $x \in V^*$, $y \in V^*$. Apply the Almost Independent Lemma,

$$|I| \sim n/\text{et} \qquad \text{almost always,}$$

$$|V^*| \sim n/e \qquad \text{almost always.}$$

Note we have already shown that for some specific C, $|I| \sim n/et$ so that $\alpha(G) > n/et$.

Remark. We may modify this argument so as not to use the assumption that (V, G) is uncrowded. Set $p = \frac{1}{2}t$,

$$\Pr\{x \in D\} \leqslant \sum_{E \in \mathscr{F}_x} \Pr\{E \subseteq C\} \leqslant t^k p^k = 2^{-k},$$

so

$$\Pr\{x \in I\} = \Pr\{x \notin D\} \Pr\{x \in C\} \geqslant (1 - 2^{-k})/2t$$

and

$$\alpha(G) \geqslant E[|I|] \geqslant cn/t,$$

where $c = (1 - 2^{-k})/2$. This result is, up to a constant factor, best possible. Let (V, G) be the union of n/t disjoint omplete $(k+1)$-graphs, each on t points. Then each point has valence ct^k $(c = 1/k!)$ and $\alpha(G) \sim c_1 n/t$ $(c_1 = k - 1)$.

Let $x \in V$. The event $x \in V^*$ depends only on which points of $N(x)$ are chosen. If $\rho(x, y) > 2$, then $N(x) \cap N(y) = \varnothing$ and the events $x \in V^*$, $y \in V^*$ are independent.

Let $z_1, \ldots, z_t$ be distinct elements of $N(x)$. The events $z_i \in V^*$ depend only on which points of $N^+(z_i)$ are chosen. Here is the central idea of the proof: Since (V, G) is uncrowded, the sets $N(z_i)$ are "nearly" disjoint so the events $z_i \in V^*$ are nearly mutually independent.

We remove the "nearly" by conditioning on $x \notin C$. Then, for $z \in N(x)$, the event $z \in V^*$ depends on which points of $N(x) - E_{zx}$ are chosen. These sets are disjoint so the events $z_i \in V^*$ are mutually independent. The event $z \in I$ also depends only on $N(z) - E_{zx}$. In general, when $y_1, \ldots, y_s$, $z_1, \ldots, z_t$ are distinct elements of $N(x)$, the events $y_i \in I$, $1 \leqslant i \leqslant s$; $z_j \in V^*$, $1 \leqslant j \leqslant t$, are, conditional on $x \notin C$, mutually independent.

Let $x \in V$, $E \in \mathscr{F}_x$, $E' \subseteq E$, $|E'| = i$. Then $E' \in \mathscr{F}_x^*$ iff $y \in C$ for all $y \in E - E'$ and $z \in V^*$ for all $z \in E'$. Then mutual independence gives

$$\Pr\{E' \in \mathscr{F}_x^* | x \notin C\} = \prod_{y \in E - E'} \Pr\{y \in C | x \notin C\} \prod_{z \in E'} \Pr\{z \in V^* | x \notin C\}.$$

Clearly,

$$\Pr\{y \in C | x \notin C\} = \Pr\{y \in C\} = p.$$

Now

$$\Pr\{z \notin D | x \notin C\} = \prod_{E \in \mathscr{F}_z} \Pr\{E \nsubseteq C | x \in C\}.$$

For $E \neq E_{zx}$, $\Pr\{E \nsubseteq C | x \notin C\} = \Pr\{E \nsubseteq C\}$. For $E = E_{zx}$, $\Pr\{E \nsubseteq C\} = 1 - p^k$ but $\Pr\{E \nsubseteq C | x \notin C\} = 1$. Thus

$$\Pr\{z \notin D | x \notin C\} = \Pr\{z \notin D\}/(1 - p^k)$$

and so

$$\Pr\{z \in V^* | x \notin C\} = \Pr\{z \in V^*\}/(1 - p^k) \sim \Pr\{z \in V^*\} \sim e^{-1}.$$

(That is, the effect of $x \notin C$ is negligible.) Hence

$$\Pr\{E' \in \mathscr{F}_x^* | z \notin C\} \sim p^{|E - E'|}(e^{-1})^{|E'|} = p^{k-i}e^{-i}.$$

Let $E \in \mathscr{F}_x$. The events $E' \in \mathscr{F}_x^*$ for $E' \subseteq E$ are mutually disjoint so

$$\Pr\{E \text{ is transformed to an } i\text{-set}\} \sim \binom{k}{i} p^{k-i}e^{-i},$$

there being $\binom{k}{i}$ possible i-sets. Now $\deg_i^*(x)$ is simply the number of $E \in \mathscr{F}_x$ which transform to an i-set. There are t^k potential E and the events "E is transformed to an i-set" are mutually independent. (Another critical use of the uncrowdedness assumption.) Thus $\deg_i^*(x)$ has binomial distribution

$$\deg_i^*(x) \approx B[t^k, \binom{k}{i} p^{k-i}e^{-i}],$$

and therefore, applying Chebyschev's Inequality

$$\deg_i^*(x) \approx t^k \binom{k}{i} p^{k-i}e^{-i} = \binom{k}{i}(t/e)^i,$$

with probability almost unity.

We may discard from (V^*, G^*) the few points x for which $\deg_i^*(x)$ is much more than expected. Set $n_1 = ne^{-1}$, $t_1 = te^{-1}$. Then (V^*, G^*) has at least $n_1(1 - \varepsilon)$ points and

$$\deg_i^*(x) \leqslant \binom{k}{i} t_1^i(1 + \varepsilon), \qquad 1 \leqslant i \leqslant k,$$

for all $x \in V^*$. Here ε is a small error which we will ignore for the remainder of this section.

Remark. In some sense we have shown that V^* behaves like a random set with distribution $V(e^{-1})$.

Remark. Let us ignore the $(i+1)$-sets, $i < k$, of G^*. Then G^* has the same edge density as G. Repeating our argument, we find $n_1/et_1 = n/et$ additional independent points and this continues at each iteration. In fact, the $(i+1)$-sets, $i < k$, of G^* cannot be ignored. At the sth iteration we find $(n/et)f(s)$ independent points where $f(s)$ approaches zero. "Fortunately," $\sum f(s)$ diverges to prove our theorem.

Now consider a more general stuation. Let (V, G) be a regular uncrowded hypergraph with n vertices and (applying with foresight a convenient parameterization)

$$\deg_i(x) = \alpha_i \binom{k}{i} t^i, \qquad 1 \leqslant i \leqslant k,$$

where $\alpha_n = 1$. Let $p = w/t$ and let C have distribution $V(p)$. Now

$$\begin{aligned}\Pr\{x \notin D\} &= \prod_{E \in \mathscr{F}_x} \Pr\{E \not\subseteq C\} \\ &= \prod_{i=1}^{k} (1-p^i)^{\alpha_i \binom{k}{i} t^i} \sim \beta,\end{aligned}$$

where we define

$$\beta = \exp\left[-\sum_{i=1}^{k} \alpha_i \binom{k}{i} w^i\right].$$

We again deduce that almost always

$$|I| \sim np\beta = (n/t)\, w,$$

$$|V^*| \sim n\beta.$$

Let $E \in \mathscr{F}_x$, $|E| = j$ and let $E' \subseteq E$, $|E'| = i$. The probability that E is transformed to E' is approximately $p^{j-i}\beta^i$.

Then $\deg^*_{ij}(x)$, the number of such pairs E, E', satisfies

$$\deg^*_{ij}(x) \sim \left[\alpha_j \binom{k}{j} t^j\right] \binom{j}{i} p^{j-i} \beta^i$$

almost always and so

$$\begin{aligned}\deg_i^* &\sim \sum_{v=i}^{k} \alpha_j \binom{k}{j}\binom{j}{i}(\beta t)^i (pt)^{j-i} \\ &= \binom{k}{i}(\beta t)^i \sum_{j=i}^{k} \alpha_j \binom{k-i}{j-i} w^{j-i}.\end{aligned}$$

Set $n^* = n\beta$, $t^* = t\beta$. Then (V^*, G^*) has n^* vertices and

$$\deg_i^*(x) \sim \alpha_i^* \binom{k}{i}(t^*)^i,$$

where

$$\alpha_i^* = \sum_{j=i}^{k} \alpha_j \binom{k-i}{j-i} w^{j-i}, \qquad 1 \leqslant i \leqslant k.$$

The analysis is considerably simplified by the special nature of the above transformation. Observe that if

$$\alpha_i = v^{k-i}, \qquad 1 \leqslant i \leqslant k,$$

then

$$\alpha_i^* = (v+w)^{k-i}, \qquad 1 \leqslant i \leqslant k,$$

and

$$\beta = \exp\left[-\sum_{i=1}^{k} \alpha_i \binom{k}{i} w^i\right] = \exp[-((v+w)^k - v^k)].$$

Let $(V, G) = (V_0, G_0)$ be an uncrowded $(k+1)$-graph with $n = n_0$ vertices and $\deg_k(x) = t^k$, where $t = t_0$. We find a large independent set by iteration of the above procedure, giving a sequence of hypergraphs (V_s, G_s). At each iteration there is a choice of the parameter $w = w_s$. We shall, for convenience, always choose w so that $\beta = \beta_s \sim e^{-1}$. At the sth stage there are parameters

$$n_s \sim ne^{-s}, \qquad t_s \sim te^{-s}, \qquad v_s \sim s^{1/k}$$

so that $|V_s| \sim n_s$ and all $x \in V_s$ have degrees

$$\deg_i(x) = \alpha_i \binom{k}{i} t_s^i, \qquad \text{where } \alpha_i = v_s^{k-i}.$$

We then select

$$w = w_s = (s+1)^{1/k} - s^{1/k}$$

and continue by induction. In (V_s, G_s) we find an independent set I_s with

$$|I_s| \sim (n_s/t_s)\beta_s w_s = (n/et)\, w_s.$$

We continue this procedure until $t_s = te^{-s}$ becomes small. We stop at $s = 0.01 \ln t$, yielding an independent set

$$I = \bigcup_{s=0}^{0.01 \ln t} |I_s|$$

with

$$\begin{aligned} \alpha(G) \geqslant |I| = \sum |I_s| &= (n/et) \sum w_s \\ &= c(n/t)(\ln t)^{1/k}. \end{aligned}$$

All that remains is a careful examination of the "error terms" and a more formal proof.

3. A More Formal Proof

Please, dear reader, read the previous sections first! We assume T, N are sufficiently large so that the inequalities we give (which are generally quite rough) will hold but we do not explicitly define sufficiently large. We will not concern ourselves with the nonintegrality of certain expressions.

Lemma. *For T sufficiently large (dependent on k) and N sufficiently large (dependent on k, T) the following holds. Let $0 \leqslant s \leqslant 0.01 \ln T$, s integral. Let*

$$w = (s+1)^{1/k} - s^{1/k},$$

$$\varepsilon = 10^{-6}/\ln T \qquad (\textit{the error term}).$$

Let

$$e^{-s}N/2 \leqslant n \leqslant e^{-s}N,$$

$$e^{-s}T/2 \leqslant t \leqslant e^{-s}T.$$

Let (V, G) be an uncrowded hypergraph with n vertices and

$$\deg_i(x) \leqslant \binom{k}{i} s^{(k-i)/k} t^i, \qquad 1 \leqslant i \leqslant k, \textit{ all } x.$$

Then there exists a set I and a hypergraph (V^{**}, G^{**}) *such that*

(P1) $\alpha(G) \geqslant |I| + \alpha(G^{**})$,

(P2) $|I| \geqslant (n/et)(0.99)$,

(P3) $|V^{**}| \geqslant n^*$,

(P4) $\deg_i^*(x) \leqslant \binom{k}{i}(s+1)^{(k-i)/k}(t^*)^i$, $1 \leqslant i \leqslant k$, *all* $x \in V^*$.

Remark. Since $t \geqslant e^{-s}T/2 \geqslant T^{0.99}/2$, $t \gg k$. Since $n \geqslant e^{-s}N/2 \geqslant NT^{-0.01}/2$, $n \gg t$. Also $\varepsilon = 10^{-6}/\ln T < 2 \times 10^{-6}/\ln t$ and $w < 0.01 \ln T < 0.01 \ln t$. The variables N, T are useful only in proving the theorem following.

Proof. Apply the Almost Regular Lemma and replace (V, G) by its extension. (This cannot increase the independence number.) Let B be those (bad) points $x \in V$ so that some $y \in N^+(x)$ does not have full degree. Then $|B| < \bar{T}$, where $\bar{T}$ is independent of N.

Remark. While adding edges to (V, G) would appear counterproductive we have not been able to remove the Almost Regular Lemma from our proof.

Let C have distribution $V(p)$, where $p = w/t$. Then D, I, V^*, G^*, $\deg_i^*(x)$, $\deg_{ij}^*(x)$ are defined as in Section 1. For every $x \in V$

$$\Pr\{x \notin D\} = \prod_{i=1}^{k} (1-p^i)^{\binom{k}{i}s^{(k-i)/k}t^i}.$$

Since p is sufficiently small

$$\exp[-p^i(1+\varepsilon/10)] \leqslant 1 - p^i \leqslant \exp[-p^i].$$

Applying the upper estimate

$$\Pr\{x \notin D\} \leqslant \exp\left[-\sum_{i=1}^{k} \binom{k}{i} s^{(k-i)/k} t^i p^i\right]$$
$$= e^{-1}$$

by the judicious choice of w. Thus

$$e^{-1}e^{\epsilon/10} \leqslant \Pr\{x \notin D\} \leqslant e^{-1} \tag{$*$}$$

and therefore

$$(w/et)\, e^{-\epsilon/10} \leqslant \Pr\{x \in I\} \leqslant w/et$$

and

$$E[|I|] \geqslant (n - \bar{T})(w/et)\, e^{-\epsilon/10}$$
$$\geqslant (nw/et)\, e^{-0.11\epsilon}.$$

If $\rho(x, y) > 2$ the events $x \in I$, $y \in I$ are independent. The Almost Independent Lemma implies

$$\Pr\{|I| \geqslant 0.99nw/et\} > 0.99. \tag{R1}$$

Set Z equal to those $x \in V$ such that $x \notin B$, $x \notin C$ and

$$\deg_i^*(x) > \binom{k}{i}(s+1)^{(k-i)/k}\,(t^*)^i$$

for some $1 \leqslant i \leqslant k$. (As a convenience we have allowed Z to contain points in D. However, Z will still be small.) Set

$$V^{**} = V - B - C - D - Z$$

and let (V^{**}, G^{**}) be the restriction of (V^*, G^*) to V^{**}. This construction ensures (P1), (P4), leaving only (P3) in doubt. We have immediately

$$\begin{aligned} |B| &\leqslant \bar{T} \leqslant 10^{-10}\varepsilon n, \\ \Pr\{|C| &\leqslant 2nw\} > 0.99. \end{aligned} \tag{R2}$$

For $\rho(x, y) > 4$ the events $x \notin D$, $y \notin D$ are mutually independent. The Almost Independent Lemma and $(*)$ yield

$$\Pr\{|V - B - D| \geqslant |V - B|\, e^{-1} e^{-.11\epsilon}\} > 0.99. \tag{R3}$$

For $1 \leqslant i \leqslant j \leqslant k$ let Z_{ij} equal those $x \in V$ such that $x \notin B$, $x \notin C$ and

$$\deg_{ij}^*(x) > (1+\varepsilon)\left[\binom{k}{j} s^{(k-j)/k} t^j\right]\left[\binom{j}{i} p^{j-i} e^{-i}\right].$$

If $x \notin Z_{ij}$ for all i, j then

$$\begin{aligned} \deg_i(x) &\leqslant (1+\varepsilon) \sum_{j=1}^{k} \binom{k}{k-j} s^{j/k-j} p^{j-i} e^{-i} \\ &\leqslant (1+\varepsilon)\binom{k}{i}(s+1)^{i/k}\,(t/e)^i \\ &\leqslant \binom{k}{i}(s+1)^{i/k}\,(t^*)^i \end{aligned}$$

for all i and thus $x \notin Z$. That is,

$$Z \subseteq \bigcup Z_{ij} \qquad \text{so} \quad |Z| \leqslant \sum |Z_{ij}|.$$

Fix i,j with $1 \leqslant i \leqslant j \leqslant k$. Let $x \in V - B$. If $z \in E \in \mathscr{F}_x$ and $|E| = j$, then

$$\Pr\{z \in C | x \notin C\} = p,$$
$$\Pr\{z \in V^* | x \notin C\} = \Pr\{z \in V^*\}/(1 - p^j)$$

as argued in Section 2. Applying $(*)$

$$\Pr\{z \in V^* | z \notin C\} \leqslant e^{-1}/(1 - p^j) < e^{-1} e^{\varepsilon/10k},$$

since p is sufficiently small. These probabilities are independent over distinct $z \in N(x)$. For a given $E' \subseteq E$, $|E'| = i$.

$$\Pr\{E' \in \mathscr{F}_x | x \notin C\} = \Pr\{z \in C | x \notin C\}^{j-i} \Pr\{z \in V^* | x \notin C\}^i$$
$$< p^{j-i} e^{-i} e^{\varepsilon/10}$$

and thus

$$\Pr\{E' \in \mathscr{F}_x^* \text{ for some } E' \subseteq E \text{ with } |E'| = i | x \notin C\} < \binom{j}{i} p^{j-i} e^{-i} e^{\varepsilon/10}.$$

These events are independent over the $E \in \mathscr{F}_x$, $|E| = j$ and $\deg_{ij}^*(x)$ counts the number of such events that occur so $\deg_{ij}^*(x)$ has distribution at most

$$B\left[\binom{k}{j} s^{(k-j)/k} t^j, \binom{j}{i} p^{j-i} e^{-i} e^{\varepsilon/10}\right]$$

so that

$$\Pr\{x \in Z_{ij} | x \notin C\} < \varepsilon 10^{-10}/k^2.$$

Allowing i,j to range over $1 \leqslant i \leqslant j \leqslant k$

$$\Pr\{x \in Z | x \notin C\} < \varepsilon 10^{-10}.$$

We remove the annoying condition by noting that if $x \in C$ then $x \notin Z$ by definition so

$$\Pr\{x \in Z\} < \Pr\{x \in Z | x \notin C\} < \varepsilon 10^{-10}.$$

The events $x \in Z$, $y \in Z$ are independent when $\rho(x, y) > 4$ so using the Almost Independent Lemma

$$\Pr\{|Z| < 2\varepsilon 10^{-10} n\} \geqslant 0.99. \tag{R4}$$

Now (returning to reality) select a specific C so that the events (R1), (R2), (R3), (R4) simultaneously occur. In that case (P1), (P2), (P4) hold and

$$|V^{**}| \geqslant |V - B - D| - |C| - |Z| \geqslant n^*$$

giving (P3) as well and completing the lemma.

THEOREM. *For T sufficiently large (dependent on k) and N sufficiently large (dependent on k, T) the following holds: If (V, G) is an uncrowded $(k+1)$-graph with N vertices and*

$$\deg(x) \leqslant T^k$$

for all $x \in V$, then

$$\alpha(G) \geqslant c(N/T)(\ln T)^{1/k}.$$

Proof. Applying the lemma, we find for $0 \leqslant s \leqslant 10^{-5} \ln T$ graphs (V_s, G_s) so that

$$\alpha(G_s) \geqslant |I| + \alpha(G_{s+1}),$$

where

$$e^{-s}N(1-\varepsilon)^s \leqslant n_s \leqslant e^{-s}N, \qquad w_s = (s+1)^{1/k} - s^{1/k},$$
$$e^{-s}T(1-\varepsilon)^s \leqslant t_s \leqslant e^{-s}T$$

and

$$|I| \geqslant \frac{n_s}{t_s}\frac{w_s}{e}(0.99) \geqslant \frac{N}{T}\frac{0.98}{e} w_s$$

(since $n_s/t_s \geqslant (N/T)(0.99)$). Thus

$$\alpha(G) \geqslant \sum_{s=0}^{10^{-5}\ln T} \left(\frac{N}{T}\frac{0.98}{e}\right) w_s$$
$$= \frac{N}{T}\frac{0.98}{e}(10^{-5} \ln T)^{1/k}$$

proving the theorem for $c = (0.98/e)\, 10^{-5/k}$.

COROLLARY. *For T sufficiently large (dependent on k) and N sufficiently* *large (dependent on k, T) the following holds: If (V, G) is an uncrowded $(k+1)$-graph with N vertices and at most NT^k hyperedges, then*

$$\alpha(G) \geqslant c'(N/T)(\ln T)^{1/k}.$$

Proof. The average valence is at most $(k+1)\,T^k$. Deletion of those points of valence at most $2(k+1)\,T^k$ yields a set V' with $|V'| \geqslant N/2$. Set $G' = G|_{V'}$. Then for all $x \in V'$

$$\deg'(x) \leqslant 2(k+1)\,T^k = (c_1 T)^k,$$

where $c_1 = [2(k+1)]^{1/k}$. Then

$$\begin{aligned} \alpha(G) \geqslant \alpha(G') &\geqslant c[(N/2)/(c_1 T)] \ln(c_1 T)^{1/k} \\ &\geqslant c'(N/T)(\ln T)^{1/k}. \end{aligned}$$

Remark. It would be interesting to prove the corollary without first elimination of the vertices of high degree.

Remark. A simple random graph theory argument shows that the corollary is best possible. Consider a random $(k+1)$-graph G with n vertices and hyperedge probability $p = (t/n)^k$ $(k \ll t \ll n)$. Let $x = c(n/t)(\ln t)^{1/k}$. The probability that a particular I, $|I| = x$ is independent is approximately $\exp[-px^{k+1}/(k+1)!]$. The expected number of independent x-sets is then

$$\begin{aligned} \binom{n}{x} \exp[-px^{k+1}/(k+1)!] &< [(ne/x)\exp[-px^k/(k+1)]]^x \\ &\ll 1, \end{aligned}$$

for appropriately large c. On the other hand the expected number of cycles of order $\leqslant 4$ in G is about $(cn^{tk})\,p^{tk} \sim ct^4$. Eliminating points in small cycles gives a hypergraph G' with no small cycles and $\alpha(G') \leqslant x$.

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