NOTE ON CANONICAL PARTITIONS

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To B. L. van der Waerden on his eightieth birthday

1. For every set X and every cardinal number r we put

$$[X]^r = \{P \subseteq X : |P| = r\}.$$

Let $A = \{0, 1, 2, \ldots\}$ and $r \in \{1, 2, \ldots\}$. A partition, or colouring, of $[A]^r$ is a function $f: [A]^r \to F$, where F is a set. Let $L \subseteq \{0, 1, \ldots, r-1\}$. The partition f is called L-canonical on B if $B \subseteq A$ and, for

$$\{x_0, x_1, \ldots, x_{r-1}\}_<, \{y_0, \ldots, y_{r-1}\}_< \subseteq B,$$

we have $f\{x_0, \ldots, x_{r-1}\} = f\{y_0, \ldots, y_{r-1}\}$ if and only if $x_\lambda = y_\lambda$ for $\lambda \in L$.

In [1] the following result was proved:

THEOREM 1. *Given any partition $f: [A]^r \to F$, there is an infinite set $B \subseteq A$ and a set $L \subseteq \{0, \ldots, r-1\}$ such that f is L-canonical on B.*

The object of this note is (i) to give a new proof of Theorem 1 which is in some ways simpler than the proof in [1], (ii) to discuss connections between canonicity and some other properties of a partition. If $X \in [A]^r$ we sometimes write

$$X = \{X^0, X^1, \ldots, X^{r-1}\}_<,$$

and similarly for letters other than X.

2. We begin by showing that for every L there exists a L-canonical partition of $[A]^r$.

THEOREM 2. *Given any set $L \subseteq \{0, \ldots, r-1\}$, there exists a L-canonical partition of $[A]^r$.*

Proof. We define f by putting, for every $P \in [A]^r$,

$$fP = \{Q \in [A]^r : Q^\lambda = P^\lambda \text{ for } \lambda \in L\}.$$

We now show that f is L-canonical. We shall apply the definition of f repeatedly without referring to this fact.

(i) Let $fP = fQ$. Then $Q \in fQ = fP$; $Q^\lambda = P^\lambda$ for $\lambda \in L$.

(ii) Let $P^\lambda = Q^\lambda$ for $\lambda \in L$, for some $P, Q \in [A]^r$. Consider any set $R \in fP$. We have, for $\lambda \in L$,

$$R^\lambda = P^\lambda = Q^\lambda;\ R \in fQ.$$

Since R is arbitrary, we have $fP \subseteq fQ$. By symmetry, $fQ \subseteq fP$, and Theorem 2 follows.

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3. *Proof of Theorem* 1. For $\{x_0, \ldots, x_{2r-1}\}_< \subset A$ put

$$g\{x_0, \ldots, x_{2r-1}\} = \{(\alpha_0, \ldots, \alpha_{2r-1}): \alpha_0 < \ldots < \alpha_{r-1} < 2r; \alpha_r < \ldots < \alpha_{2r-1} < 2r; f\{x_{\alpha_0}, \ldots, x_{\alpha_{r-1}}\} = f\{x_{\alpha_r}, \ldots, x_{\alpha_{2r-1}}\}\}.$$

The range of the function g is finite. Hence, by Ramsey's theorem [2], there is an infinite set $B' \subseteq A$ such that g is constant on $[B']^{2r}$. Let $B' = \{b_0, b_1, b_2, \ldots\}_<$ and $B = \{b_0, b_2, b_4, \ldots\}$. Let L be the set of all numbers $\rho_0 < r$ such that, whenever

$$\{y_0, \ldots, y_{r-1}\}_<, \{y'_0, \ldots, y'_{r-1}\}_< \subset B',$$

$y_\rho = y'_\rho$ for $\rho \neq \rho_0$ and $y_{\rho_0} \neq y'_{\rho_0}$, then $f\{y_0, \ldots, y_{r-1}\} \neq f\{y'_0, \ldots, y'_{r-1}\}$. To complete the proof of Theorem 1 we show that f is L-canonical on B.

(a) Let $\{y_0, \ldots, y_{r-1}\}_<, \{y'_0, \ldots, y'_{r-1}\}_< \subset B$ and

$$y_\lambda = y'_\lambda \quad \text{for} \quad \lambda \in L. \tag{1}$$

We have to show that $f\{y_0, \ldots, y_{r-1}\} = f\{y'_0, \ldots, y'_{r-1}\}$. To this end we define an operator T thus: Let

$$\{y_0, \ldots, y_{r-1}\}_<, \{y'_0, \ldots, y'_{r-1}\}_< \subset B.$$

If $y_\rho = y'_\rho$ for $\rho < r$ then put

$$T(\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}) = (\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}).$$

Now let $y_\rho \neq y'_\rho$ for at least one ρ. Let $\rho_0 = \min\{\rho: y_\rho \neq y'_\rho\}$. Then, by our assumption (1), $\rho_0 \notin L$. Put $z_\rho = y_\rho$ and $z'_\rho = y'_\rho$ for $\rho \neq \rho_0$, and $z_{\rho_0} = z'_{\rho_0} = \min\{y_{\rho_0}, y'_{\rho_0}\}$. It follows from $\rho_0 \notin L$ and the definition of L that

$$f\{z_0, \ldots, z_{r-1}\} = f\{y_0, \ldots, y_{r-1}\},$$
$$f\{z'_0, \ldots, z'_{r-1}\} = f\{y'_0, \ldots, y'_{r-1}\}.$$

We put

$$T(\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}) = (\{z_0, \ldots, z_{r-1}\}, \{z'_0, \ldots, z'_{r-1}\}).$$

We iterate T r times and obtain

$$T^r(\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}) = (\{w_0, \ldots, w_{r-1}\}_<, \{w_0, \ldots, w_{r-1}\}).$$

Then $f\{y_0, \ldots, y_{r-1}\} = f\{w_0, \ldots, w_{r-1}\} = f\{y'_0, \ldots, y'_{r-1}\}$, as required.

(b) Let $\{x_0, \ldots, x_{r-1}\}_<, \{x'_0, \ldots, x'_{r-1}\}_< \subset B$; $\rho_0 \in L$;

$$x_{\rho_0} < x'_{\rho_0}. \tag{2}$$

To complete the proof of Theorem 1, we now proceed to deduce that $f\{x_0, \ldots, x_{r-1}\} \neq f\{x'_0, \ldots, x'_{r-1}\}$. Let us assume that $f\{x_0, \ldots, x_{r-1}\} = f\{x'_0, \ldots, x'_{r-1}\}$. We have to deduce a contradiction.

For $P, P', Q, Q' \in [B]^r$ let $(P, P') \equiv (Q, Q')$ mean that there is an order preserving bijection $\phi: P \cup P' \to Q \cup Q'$ such that $\phi P = Q$ and $\phi P' = Q'$.

Lemma: *Let* $P, P', Q, Q' \in [B]^r$; $fP = fP'$; $(P, P') \equiv (Q, Q')$. *Then* $fQ = fQ'$.

Proof of the Lemma. There is a set $E \in [B]^{2r - |P \cup P'|}$ such that $x < y$ whenever $x \in P \cup P' \cup Q \cup Q'$ and $y \in E$. Then

$$P \cup P' \cup E, \quad Q \cup Q' \cup E \in [B]^{2r}$$

and hence $g(P \cup P' \cup E) = g(Q \cup Q' \cup E)$. It follows from the definition of g that $fQ = fQ'$, and the Lemma is proved.

To continue the proof of Theorem 1 put, for $t \in \{1, 2, 3, \ldots\}$,

$$B(t) = \{b_0, b_t, b_{2t}, \ldots\}.$$

Let $r < s \in \{2, 3, \ldots\}$. There are sets $X_0, X_1 \in [B(r^s)]^r$ such that

$$(X_0, X_1) \equiv (\{x_0, \ldots, x_{r-1}\}, \{x_0', \ldots, x_{r-1}'\}).$$

Then there is a set $X_2 \in [B(r^{s-1})]^r$ such that $[X_0, X_1) \equiv (X_1, X_2)$. There is a set $X_3 \in [B(r^{s-2})]^r$ such that $(X_1, X_2) \equiv (X_2, X_3)$, and so on until there is a set $X_s \in [B(r)]^r$ such that $(X_{s-2}, X_{s-1}) \equiv (X_{s-1}, X_s)$. We have $X_\sigma \{X_\sigma^0, \ldots, X_\sigma^{r-1}\}_<$ for $\sigma \leqslant s$. Then, by (2) and the definition of $\equiv$, we have

$$X_0^{\rho_0} < X_1^{\rho_0} < \ldots < X_s^{\rho_0}.$$

In view of $s > r$ there is σ_0 with $1 \leqslant \sigma_0 \leqslant s$ such that

$$X_{\sigma_0}^{\rho_0} \neq X_0^{\rho} \quad \text{for} \quad \rho < r. \tag{3}$$

There is a number π such that $X_{\sigma_0}^{\rho_0} = b_{2\pi}$. Put $Z_{\sigma_0} = \{Z_{\sigma_0}^0, \ldots, Z_{\sigma_0}^{r-1}\}_<$, where $Z_{\sigma_0}^{\rho} = X_{\sigma_0}^{\rho}$ for $\rho \neq \rho_0$ and $Z_{\sigma_0}^{\rho_0} = b_{2\pi+1}$. Since $\rho_0 \in L$ we have $fX_{\sigma_0} \neq fZ_{\sigma_0}$.

On the other hand, we have, by choice of σ_0 and the definition of Z_{σ_0}, that

$$(X_0, X_1) \equiv (X_{\sigma_0 - 1}, X_{\sigma_0}); \quad (X_0, X_{\sigma_0}) \equiv (X_0, Z_{\sigma_0}).$$

We have $fX_0 = fX_1 = \ldots = fX_{\sigma_0}$. Hence, by (3) and the Lemma, $fX_{\sigma_0} = fX_0 = fZ_{\sigma_0}$, which yields the required contradiction. This proves Theorem 1.

4. We now consider connections between canonicity and some other properties of partitions. Let A and B denote infinite subsets of $\{0, 1, \ldots\}$. Consider a partition $f: [A \cup B]^r \to F$. We require some definitions.

(f, A) is called invariant if, whenever $P, Q, P', Q' \in [A]^r$ and $(P; Q) \equiv (P', Q')$, then $fP = fQ$ if and only if $fP' = fQ'$.

(f, A) is called isomorphic to (f, B) [in symbols $(f, A) \cong (f, B)$] if, whenever $P, Q \in [A]^r$ and $\phi: A \to B$ is an order preserving bijection, then $fP = fQ$ if and only if $f\phi P = f\phi Q$.

(f, A) is called stationary if, whenever $B \subseteq A$ then $(f, B) \cong (f, A)$.

THEOREM 3. *The following three conditions are equivalent*:

(i) (f, A) *is invariant*,

(ii) (f, A) *is stationary*,

(iii) (f, A) *is L-canonical for some L*.

Proof of (i) $\Rightarrow$ (ii). (f, A) is invariant. Let $B \subseteq A$. There is an order preserving bijection $\phi: A \to B$. Let $P, Q \in [A]^r$. Then $(P, Q) \equiv (\phi P, \phi Q)$. By invariance we have $fP = fQ$ if and only if $f\phi P = f\phi Q$, and (ii) holds.

Proof of (ii) $\Rightarrow$ (iii). (f, A) is stationary. By theorem 1 there is an infinite set $B \subseteq A$ such that (f, B) is L-canonical for some L. Then, by (ii), $(f, B) \cong (f, A)$ which implies that (f, A) is L-canonical, and (iii) holds.

Proof of (iii) $\Rightarrow$ (i). (f, A) is L-canonical for some L. Let $P, Q, P', Q' \in [A]^r$ and $(P, Q) \equiv (P', Q')$, Then we have

$$(fP = fQ) \Leftrightarrow (P^\lambda = Q^\lambda \text{ for } \lambda \in L) \Leftrightarrow (P'^\lambda = Q'^\lambda \text{ for } \lambda \in L) \Leftrightarrow (fP' = fQ'),$$

and (i) holds. This proves Theorem 3.

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