Discrete Comput Geom 25:629–634 (2001)
DOI: 10.1007/s00454-001-0009-z

Discrete & Computational
Geometry

Distinct Distances in the Plane*

J. Solymosi and Cs. D. Tóth

Institut für Theoretische Informatik, ETH Zürich,
CH-8092 Zürich, Switzerland
{solymosi,toth}@inf.ethz.ch

Abstract. It is shown that every set of n points in the plane has an element from which there are at least $cn^{6/7}$ other elements at distinct distances, where $c > 0$ is a constant. This improves earlier results of Erdős, Moser, Beck, Chung, Szemerédi, Trotter, and Székely.

1. Introduction

The following well-known problem is due to Erdős [6] (see also [9]): Given n distinct points in the plane, what is the minimum number of distinct distances determined by them?

Denoting this minimum by $G(n)$, Erdős [6] conjectured that $G(n) = \Omega(n/\sqrt{\log n})$, and showed that this bound is attained by the $\sqrt{n} \times \sqrt{n}$ grid. Moser [8] proved that the number of distinct distances is at least $\Omega(n^{2/3})$. Chung [3] and Chung et al. [4] improved this bound to $\Omega(n^{5/7})$ and $\Omega(n^{4/5}/\log^c n)$, respectively, where c is a small positive constant. Finally, Székely [11] proved that $G(n) = \Omega(n^{4/5})$, and that there always exists at least one point from which there are at least $\Omega(n^{4/5})$ distinct distances. Our main result is

Theorem 1. *Any set P of n points in the plane has an element from which the number of distinct distances to the other points of P is at least $\Omega(n^{6/7})$.*

The proof relies on three results: (a) Beck's theorem [2] on the minimum number of lines connecting points in a planar point set; (b) the Szemerédi–Trotter theorem [12] on the number of incidences between points and lines; and (c) Székely's method [11] for

* Supported by the joint Berlin/Zürich graduate program "Combinatorics, Geometry, Computation", financed by the German Science Foundation (DFG) and ETH Zürich.

estimating the number on incidences using a previously known upper bound on crossing numbers of graphs drawn in the plane. It is worth noting that these three ingredients are related to each other. Beck's result follows from the Szemerédi–Trotter theorem, which, in turn, has a very elegant proof using Székely's method.

A topological (multi)graph is a (multi)graph $G(V, E)$ drawn in the plane such that the vertices of G are represented by distinct points in the plane, and its edges by continuous arcs between the corresponding point pairs. Any two arcs representing distinct edges have finitely many points in common. We make no notational distinction between the vertices (resp., edges) and the points (resp., arcs) representing them.

Unlike in the standard definition of topological graphs, we allow arcs representing edges of G to pass through other vertices. Such topological graphs were first employed by Pach and Sharir [10]. Two edges of a topological graph are said to form a *crossing* if they have a common point which is not an endpoint of both curves. Thus, two crossing edges together have four distinct endpoints. The *crossing number* of a *topological* graph or multigraph is the total number of crossing pairs of edges. The *crossing number* of an *abstract* graph or multigraph G is the minimum crossing number over all possible representations (i.e., drawings) of G as a topological graph.

2. Proof of Theorem 1

Let P be a set of n points in general position in the plane. Let t be the maximal number of distinct distances measured from one point, that is $t = \max_{p\in P} |\{|pq|\colon q \in P\}|$. Suppose that $t = o(n)$ and $t \geq c_1 n^{3/4}$, for some constant $c_1 > 0$. This latter assumption follows from the earlier results [4], [5]. We apply Beck's theorem to the point set P.

Theorem 2 [2]. *Given n points in the plane, at least one of the following two statements holds*:

1. *There is a line incident to at least* $n/100$ *points*.
2. *There are at least* $\Omega(n^2)$ *lines incident to at least two points*.

Let $\mathcal{L}$ denote the set of all lines connecting at least two points of P. We have

Corollary 3. *There is an absolute constant* $c_0 > 0$ *with the property that the number of points in* P *incident to at least* $c_0 n$ *distinct lines of* $\mathcal{L}$ *is at least* $c_0 n$, *provided that* $t < n/100$.

Let B denote the set of points in P incident to at least $c_0 n$ lines of $\mathcal{L}$. By Corollary 3, $|B| \geq c_0 n$.

Fix a point $a \in B$. Let $P_a \subset P\backslash\{a\}$ be a maximal set such that for each point $q \in P_a$ the line aq contains no other point of P_a. Consider the set $\mathcal{C}_a$ of all circles centered at $a \in B$ that contain at least three points of P_a. Let P'_a denote the set of all elements of P_a which belong to a circle in $\mathcal{C}_a$. Clearly, $|P'_a| = \Omega(n)$, because $|P_a| = \Omega(n)$, and $t = o(n)$, by assumption.

After deleting at most two points from each circle in $\mathcal{C}_a$, partition the remaining points into pairwise disjoint consecutive triples (q_1, q_2, q_3). Clearly, the number of such triples over all circles around a is $\Omega(n)$.

A line ℓ is called *rich* if ℓ is incident to at least m points in P, where m is a number to be specified later. A triple (q_1, q_2, q_3) is said to be *good* if the bisector of at least one of the segments q_1q_2, q_1q_3, or q_2q_3 is not rich; otherwise it is called *bad*. A point $p \in B$ is *good*, if at least half of the triples associated with it are good, otherwise it is *bad*. Denote by g the number of good points in B.

The outline of the rest of the proof is as follows. First, we choose an m so that at least half of the points in B will be bad. Then we deduce a lower bound on the number of rich lines incident to a fixed bad point, which then implies a *lower* bound for the overall number of incidences between bad points and rich lines. On the other hand, a theorem of Szemerédi and Trotter, mentioned in the Introduction, provides an *upper* bound on the same quantity. Comparing the two bounds, the desired inequality for t will follow.

Define a topological multigraph G on the vertex set $V = P$, as follows. If a triple (q_1, q_2, q_3) is good, add to the graph one edge between a pair of points from $\{q_1, q_2, q_3\}$ whose bisector is not rich. We generate exactly one edge for each good triple. Draw each such edge along the circle determined by the triple.

The number of vertices of G is $|V| = n$; the number of edges of G is $E = g \cdot \Omega(n) = \Omega(gn)$. The graph G may have multiple edges when two points, u and v, happen to belong to more than one good triple, associated with different points of B (as centers of the corresponding circles). However, the multiplicity of each edge is at most m, because all of these points of B must lie on the perpendicular bisector of u and v, which, by assumption, is not rich.

The following lemma of [11] is a straightforward extension of a result of Ajtai et al. [1] and of Leighton [7] to topological *multi*graphs. As we pointed out in the Introduction, we use a slightly nonstandard definition of topological multigraphs, which allows edges to pass through vertices, but Székely's proof applies *verbatim* to this case as well.

Lemma 4 [11]. *Let $G(V, E)$ be a topological multigraph, in which every pair of vertices is connected by at most m edges. If $|E| \geq 5|V|m$, then the crossing number of G is*

$$cr(G) \geq \frac{\beta |E|^3}{m|V|^2},$$

for an absolute constant $\beta > 0$.

Apply Lemma 4 to the graph G defined above, with

$$m = \frac{c_2 n^2}{t^2},$$

where $c_2 > 0$ is a small constant. We distinguish two cases. If the condition in the lemma is not satisfied, then

$$\Omega(gn) = |E| < 5|V|m = \frac{5c_2 n^3}{t^2} \leq \left(\frac{5c_2}{c_1^2}\right) n^{3/2},$$

and, by choosing c_2 sufficiently small, we have $g \leq (c_0/2)n$. Otherwise, according to the statement,

$$cr(G) \geq \frac{\beta g^3 n^3}{(c_2 n^2/t^2) \cdot n^2} = \frac{\beta g^3 t^2}{c_2 n}.$$

On the other hand, since the edges of G are constructed along at most nt circles (at most t concentric circles around each point), and two circles have at most two common points, we clearly have

$$cr(G) \leq 2 \cdot \binom{nt}{2} \leq n^2 t^2.$$

Comparing the last two inequalities, we obtain, just as in the previous case, that $g \leq (c_0/2)n$, provided that c_2 is chosen sufficiently small.

Therefore, we can conclude that B has at least $\Omega(n)$ bad points.

Next, we estimate the number of rich lines incident to each bad point. For this, we need the following simple lemma.

Lemma 5. *Let T be a set of N triples (a_i, b_i, c_i) of distinct real numbers such that $a_i < b_i < c_i$ for $i = 1, \dots, N$, and assume that $c_i < a_{i+1}$ for all but at most $t - 1$ indices i. Let $W = \{a_i + b_i, a_i + c_i, b_i + c_i \mid i = 1, \dots, N\}$. Then*

$$|W| = \Omega\left(\frac{N}{t^{2/3}}\right).$$

This bound cannot be improved.

Proof. Let the *range* of a triple $(a, b, c) \in T$ be defined as the interval $[a, c]$. By assumption, the sequence $(a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_N, b_N, c_N)$ can be partitioned into at most t contiguous monotone increasing subsequences. Partition the real axis into $N/(2t)$ open intervals so that each interval fully contains the ranges of t triples. (These intervals are constructed from left to right. Let x denote the right endpoint of the rightmost interval constructed so far. Discard the at most t triples whose ranges contain x, and move to the right until we reach a point y that lies to the right of exactly t new ranges. We add (x, y) as a new open interval, and continue in this manner until all triples are processed.)

Let s be one of these open intervals. Each triple in T, whose range is fully contained in s, contributes three elements to $W \cap s$, and no two triples of T contribute the same triple to $W \cap s$, as can be easily verified. It follows that $|W \cap s| \geq t^{1/3}$, or else the number of distinct triples of its elements will be smaller than t. Since the number of intervals s is $N/(2t)$, the inequality in the lemma follows. The easy construction showing that this result is tight is left to the reader. □

Apply Lemma 5 to the system of $\Omega(n)$ disjoint bad triples along the circles centered at a fixed bad point $a \in B$. Each point u which participates in such a bad triple is mapped to the *orientation* of the ray $\vec{au}$, i.e., to the counterclockwise angle between the positive x-axis and $\vec{au}$. By the construction of P_a, this mapping is an injection of P_a into $\mathbb{R}$. There are at most t bad triples whose ranges are mapped into an interval containing 0, and we discard all of them. The images of the remaining bad triples form a set of $N = \Omega(n)$ triples meeting the requirements of Lemma 5. Notice that there are at most two orientations in W that correspond to the same rich line trough a. Hence, Lemma 5 implies that, for each bad point a, the number of rich lines incident to a is at least $\Omega(n/t^{2/3})$.

Therefore, the number I of incidences between bad points and rich lines, satisfies

$$I = \Omega\left(\frac{n^2}{t^{2/3}}\right). \tag{1}$$

The same number can be estimated from above, using the following theorem of Szemerédi and Trotter, which comes in two equivalent formulations, both stated below.

Theorem 6 [12].

(a) *Given n distinct points in the plane, the number L_m of lines incident to at least $m > 2$ points is*

$$L_m = O\left(\frac{n^2}{m^3} + \frac{n}{m}\right).$$

(b) *Given n distinct points and ℓ distinct lines in the plane, the number of point–line incidences is*

$$I(n, \ell) = O(n^{2/3}\ell^{2/3} + n + \ell).$$

Both of these bounds are asymptotically tight.

Since $t \geq c_1 n^{3/4}$, we have $m = c_2 n^2/t^2 \leq (c_2/c_1^2)n^{1/2} = O(n^{1/2})$. Thus, by Theorem 6(a), the number L_m of rich lines satisfies $L_m = O(n^2/m^3) = O(t^6/n^4)$. This, in turn, implies, by part (b) of the same theorem, that the number I of incidences between bad points and rich lines satisfies

$$I = O(n^{2/3}L_m^{2/3} + n + L_m) = O\left(\frac{t^4}{n^2} + \frac{t^6}{n^4} + n\right) = O\left(\frac{t^4}{n^2}\right). \tag{2}$$

Comparing (1) and (2), we obtain that $t = \Omega(n^{6/7})$, as required. This completes the proof of Theorem 1. □

It is very likely that, if we use k-tuples in place of triples, the above argument can be modified to give a better lower bound on the number of distinct distances determined by a point set.

Acknowledgments

We thank János Pach and Micha Sharir for useful discussions and moral encouragement, and for their help in bringing the paper to its current form.

References

[1] M. Ajtai, V. Chvátal, M. Newborn, and E. Szemerédi, Crossing-free graphs, *Ann. Discrete Math.* **12** (1982), 9–12.

[2] J. Beck, On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry, *Combinatorica* **3** (1983), 281–297.

[3] F. R. K. Chung, The number of different distances determined by n points in the plane, *J. Combin. Theory Ser. A* **36** (1984), 342–354.

[4] F. R. K. Chung, E. Szemerédi, and W. T. Trotter, The number of different distances determined by a set of points in the Euclidean plane, *Discrete Comput. Geom.***7** (1992), 1–11.

[5] K. L. Clarkson, H. Edelsbrunner, L. J. Guibas, M. Sharir, and E. Welzl, Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete Comput. Geom.***5** (1990), 99–160.

[6] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.

[7] F. T. Leighton, *Complexity Issues in VLSI*, MIT Press, Cambridge, MA, 1983.

[8] L. Moser, On the different distances determined by n points, *Amer. Math. Monthly* **59** (1952), 85–91.

[9] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, Wiley, New York, 1995.

[10] J. Pach and M. Sharir, On the number of incidences between points and curves, *Combin. Probab. Comput.* **7** (1998), 121–127.

[11] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.* **6** (1997), 353–358.

[12] E. Szemerédi and W. T. Trotter, Extremal problems in discrete geometry, *Combinatorica* **3** (1983), 381–392.

Received November 15, 2000, *and in revised form December* 13, 2000. *Online publication April* 6, 2001.