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Graphs with Linearly Bounded Ramsey Numbers

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A graph G of order n is p-arrangeable if its vertices can be ordered as $v_1, v_2, ..., v_n$ such that $|N_{L_i}(N_{R_i}(v_i))| \leqslant p$ for each $1 \leqslant i \leqslant n-1$, where $L_i = \{v_1, v_2, ..., v_i\}$, $R_i = \{v_{i+1}, v_{i+2}, ..., v_n\}$, and $N_A(B)$ denotes the neighbors of B which lie in A. We prove that for each $p \geqslant 1$, there is a constant c (depending only on p) such that the Ramsey number $r(G, G) \leqslant cn$ for each p-arrangeable graph G of order n. Further we prove that there exists a fixed positive integer p such that all planar graphs are p-arrangeable. © 1993 Academic Press, Inc.

I. INTRODUCTION

If F, G, and H are graphs, we write $F \to (G, H)$ when the following condition is satisfied: If the edges of F are colored in any fashion with two colors, say red and blue, then either the red subgraph contains a copy of G or the blue subgraph contains a copy of H. It follows easily from Ramsey's theorem that for every pair (G, H) there is a least positive integer m such that $K_m \to (G, H)$. The integer m is called the Ramsey number and denoted by $r(G, H)$ and by $r(G)$ when $G = H$.

We are concerned in this paper with the following conjecture made by Burr and Erdős in 1973 [1].

Conjecture [1]. For each positive integer d, there exists a constant c (depending only on d) so that if G is a graph on n vertices for which for every subgraph G' of G the average degree of a vertex in G' is at most d, then $r(G) \leqslant cn$.

A weakened version of this conjecture was proved by Chvátal, Rödl, Szemerédi, and Trotter in 1983 [2]. They proved the following theorem using the Szemerédi Regularity Lemma [3].

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THEOREM A [2]. *For each positive integer d, there exists a constant c (depending only on d) so that if G is a graph on n vertices in which each vertex has at most d neighbors, then $r(G) \leqslant cn$.*

Our objective is to improve Theorem A by enlarging the family of graphs for which the Burr–Erdős conjecture holds. To present our improvement we need to introduce some terminology and give some definitions.

We say that a collection $\mathscr{G}$ of graphs is an L-set (linear set) if there exists a constant c such that $r(G) \leqslant c\,|V(G)|$ for each $G \in \mathscr{G}$. Further for a graph G define

$$\rho(G) = \max \left\{ \frac{1}{|V(H)|} \sum_{v \in V(H)} d_H(v) \mid H \text{ is a subgraph of } G \right\};$$

that is, $\rho(G)$ is the maximum average degree over all subgraphs of G. In these terms the Burr–Erdős conjecture says that for any positive integer d the set of graphs $\mathscr{G} = \{G \mid G \text{ is a graph with } \rho(G) \leqslant d\}$ is an L-set.

Given a graph G, let $N_G(a) = \{x \in V(G) \mid x \text{ is adjacent to } a\}$ and for $A, B \subseteq V(G)$, $N_B(A) = (\bigcup_{a \in A} N(a)) \cap B$. A graph G of order n is called p-arrangeable if there exists an ordering $v_1, v_2, ..., v_n$ of the vertices of G such that for each $1 \leqslant i \leqslant n-1$

$$|N_{L_i}(N_{R_i}(v_i))| \leqslant p,$$

where

$$L_i = \{v_1, v_2, ..., v_i\} \qquad \text{and} \qquad R_i = \{v_{i+1}, v_{i+2}, ..., v_n\}.$$

Clearly if G is p-arrangeable then $\rho(G) \leqslant 2p$.

We will show that the conjecture holds for all p-arrangeable graphs and that all planar graphs are p-arrangeable for some p.

II. RESULTS

The principal result of the paper is the following theorem.

THEOREM 1. *For each positive integer p, the set of p-arrangeable graphs forms an L-set.*

The proof of this result is lengthy and for that reason is placed in the next section. In fact, all longer proofs are placed in the next section, so that by reading this section one can learn what has been established without dealing with many intricate details.

PROPOSITION 2. *If G is a graph with maximum degree $\Delta(G) \leqslant p$, then G is $p(p-1)+1$-arrangeable.*

Proof. Let $v_1, v_2, ..., v_n$ be an arbitrary ordering of the vertices of G. Then $|N_{R_j}(v_j)| \leqslant p$ for each j so that $|N_{L_j}(N_{R_j}(v_j))| \leqslant p(p-1)+1$. ∎

Theorem A is thus a corollary to Proposition 2 and Theorem 1.

PROPOSITION 3. *Every tree, hence every forest, is 1-arrangeable.*

Proof. The proof is an easy induction on the order n of the tree, being clear for $n=1$. Assume each tree T_n on n vertices is 1-arrangeable and consider a tree T_{n+1}. If v_{n+1} is any end vertex of T_{n+1} and $v_1, v_2, ..., v_n$ is a 1-arrangeable sequence for $T_{n+1} - v_{n+1}$, then $v_1, v_2, ..., v_n, v_{n+1}$ is a 1-arrangeable sequence for T_{n+1}. ∎

PROPOSITION 4. *Every outerplanar graph G is 3-arrangeable.*

Proof. It is well known that an outerplanar graph with a maximal number of edges is embeddable in the plane so that all vertices lie on the boundary of the outer region and each interior region is a triangle. Thus we may assume that G has a maximal number of edges and is so embedded. To find an appropriate ordering $v_1, v_2, ..., v_n$ of $V(G)$ which is 3-arrangeable let v_n be any vertex of degree 2 and if $v_n, v_{n-1}, ..., v_{n-i}$ have been selected let v_{n-i-1} be any vertex of degree 2 in $G - \{v_n, v_{n-1}, ..., v_{n-i}\}$. One can check that with this selection $v_1, v_2, ..., v_n$ is a 3-arrangeable sequence for G. ∎

THEOREM 5. *Every planar graph G is 761-arrangeable.*

The lengthy proof of this theorem appears in the next section. We shall see that the proof does not attempt to find the smallest p such that G is p-arrangeable but only shows that there exists a p with $p \leqslant 761$.

Theorems 1 and 5 imply that planar graphs form an L-set. We shall also see in the next section that a direct consequence of the proof of Theorem 5 is the following result.

THEOREM 6. *For each fixed positive integer d and $0 < \varepsilon < 1$ there is a constant c dependent only on d and ε, such that each graph G of order n with $\rho(G) \leqslant d$ contains an induced subgraph H of G with*

$$r(H) \leqslant c\,|V(H)| \qquad \text{and} \qquad |V(H)| \geqslant (1-\varepsilon)\,n.$$

There are other collections $\mathscr{G}$ of graphs which we wish to identify as p-arrangeable. To do so for each $G \in \mathscr{G}$ let $L(G) = \{x \in V(G) \mid d(x) \leqslant p\}$ and $H(G) = \{x \in V(G) \mid d(x) > p\}$.

THEOREM 7. *Each $G \in \mathscr{G}$ with $|H(G)| \leqslant p(p-1)+1$ is $(p(p-1)+1)$-arrangeable.*

Proof. Simply order the vertices of G so that the ordering $v_1, v_2, ..., v_n$ is such that $v_i \in H(G)$ and $v_j \in L(G)$ if and only if $i < j$. ∎

An immediate corollary of Theorems 1 and 7 is that the collection of graphs with at most a bounded number of vertices of degree greater than some fixed p forms an L-set.

THEOREM 8. *Let $G \in \mathscr{G}$ be such that each pair of vertices in $H(G)$ are at distance $\geqslant 3$. Then G is $(p(p-1)+1)$-arrangeable.*

Proof. Give the same ordering to G as that given in the proof of Theorem 7. ∎

Unfortunately there are many graphs G with $\rho(G) \leqslant p$ which are not p-arrangeable. The next result establishes this fact.

THEOREM 9. *Let p be a fixed positive integer and let G be a graph of minimum degree $\geqslant 2p$. Let G' be the graph obtained from G by subdividing each of its edges. Then $\rho(G') \leqslant 4$ and G' is not p-arrangeable.*

Proof. It is clear that $\rho(G') \leqslant 4$. Suppose G' is p-arrangeable with ordering $v_1, v_2, ..., v_n$. Split $V(G')$ into two sets T_1 and T_2, where T_1 are those vertices of degree 2 and $T_2 = V(G') - T_1$. Note that each vertex of T_2 has degree $\geqslant 2p$. Let l be the largest index of a vertex in T_2 under the ordering $v_1, v_2, ..., v_n$ and as done earlier let $R_l = \{v_{l+1}, v_{l+2}, ..., v_n\}$ and $L_l = \{v_1, v_2, ..., v_l\}$.

Set $|N_{R_l}(v_l)| = k$. If $k \geqslant p$, then $|N_{L_l}(N_{R_l}(v_l))| \geqslant k+1 > p$, which contradicts the fact that G' is p-arrangeable. Thus assume $k < p$ and let m be the largest index in the ordering such that $v_m \in T_1 \cap N_{L_l}(v_l)$ and $m < l$. Then $|N_{L_m}(N_{R_m}(v_m))| \geqslant |N_{L_l}(v_l) \cap T_1| \geqslant 2p - k > p$, again a contradiction. ∎

Since there exist many graphs of given fixed minimum degree (even with arbitrarily large girth), Theorem 9 provides infinitely many graphs G which have $\rho(G) \leqslant 4$ and fail to be p-arrangeable. Hence if the Burr–Erdős conjecture holds, then there are infinitely many graphs (even of large girth) which satisfy the conjecture, yet fail to be p-arrangeable. It is not known what portion of the graphs G with $\rho(G)$ bounded are p-arrangeable. Nevertheless our results show there are infinitely many new graphs (not previously known to satisfy the Burr–Erdős conjecture) which are p-arrangeable and thus satisfy the conjecture.

III. Proofs

The first proof we consider is that of Theorem 1. This proof depends heavily on the Szemerédi Regularity Lemma [3] and as such requires some preliminary definitions. The reader acquainted with the proof of Theorem A will see that our proof of Theorem 1 has the same structure. Nevertheless our proof is enough different that we include it here in detail.

Let H be a graph. If $A, B \subseteq V(H)$, $A \cap B = \varnothing$, then the density $\delta(A, B)$ of the pair (A, B) is $e(A, B)/(|A| \cdot |B|)$, where $e(A, B)$ is the number of edges joining vertices of A to those of B. Clearly $0 \leqslant \delta(A, B) \leqslant 1$.

For a positive real number ε the pair of (A, B) is said to be *ε-regular* if whenever $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geqslant \varepsilon\, |A|$ and $|B'| \geqslant \varepsilon\, |B|$, then

$$\delta(A, B) - \varepsilon \leqslant \delta(A', B') \leqslant \delta(A, B) + \varepsilon.$$

Next, let $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$ be a partition of the vertex set $V(H)$ into pairwise disjoint sets. This partition is said to be *equipartite* if $||A_i| - |A_j|| \leqslant 1$ for all $1 \leqslant i \leqslant j \leqslant k$. With these definitions we can now state the Szemerédi Regularity Lemma.

Lemma 10 [3]. *For every $\varepsilon > 0$ and every integer $m \geqslant 0$, there exist integers N_1 and N_2 (depending only on ε and m) such that if H is a graph with $|V(H)| \geqslant N_2$, then there exists an equipartite partition $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$, where* (i) *$m \leqslant k \leqslant N_1$ and* (ii) *all but at most $\varepsilon\binom{k}{2}$ of the pairs (A_i, A_j) are ε-regular.*

In the proof of Theorem 1 we also use the following lemma.

Lemma 11. *Let p be a positive integer. If a graph G of order n is p-arrangeable, then the vertex set $V(G)$ can be partitioned into $p+1$ sets $V_1, V_2, ..., V_{p+1}$ such that for each i the subgraph induced by V_i is the empty graph.*

Proof. Let $v_1, v_2, ..., v_n$ be a p-arrangeable sequence for G. Then since $|N_{L_i}(v_i)| \leqslant p$ for all i, the result follows by induction on n. ∎

Proof of Theorem 1

Let p be any positive integer. Choose t as the least positive integer such that $1/3^p > p2^p/t$ and $\frac{1}{2}\log_2(t/2) > p+1$ hold. Next set $m = 1/\varepsilon = t$ so that the above equalities hold if and only if $m = t = 1/\varepsilon > \max\{p6^p, 2^{2p+3}\}$. Let N_1 and N_2 be the values determined by ε and m in Lemma 10 and set $c = \max\{N_2, 2N_1/\varepsilon\}$. Note that c is a constant depending only on p and ε.

Let G be a p-arrangeable graph under the ordering $v_1, v_2, ..., v_n$ on its vertex set and let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_{p+1}$ be the partition guaranteed by Lemma 11.

We show $r(G) \leqslant cn$. Consider an arbitrary coloring of the edges of K_{cn} with colors red and blue. Let H be the subgraph of the colored K_{cn} induced by its red edges, so that $\bar{H}$ is the subgraph induced by its blue edges. Since for disjoint sets A and B of vertices $\delta_H(A, B) = 1 - \delta_{\bar{H}}(A, B)$, the pair (A, B) is ε-regular in H if and only if it is ε-regular in $\bar{H}$.

Since $|V(H)| = cn \geqslant N_2$, we know by Lemma 10 that there exists an equipartition $V(H) = A_1 \cup A_2 \cup \cdots \cup A_k$. Let H^* be the graph with vertex set $\{1, 2, ..., k\}$ and edges (i, j), where (A_i, A_j) is ε-regular in H, $1 \leqslant i < j \leqslant k$. The graph H^* has at least $(1-\varepsilon)\binom{k}{2}$ edges and thus by Turan's theorem has a complete subgraph H^{**} of size at least $1/(2\varepsilon)$. Without loss of generality, we assume the subsets in the partition have been labelled so that (A_i, A_j) is ε-regular whenever $1 \leqslant i < j \leqslant 1/(2\varepsilon)$. Next we two color the edges of H^{**} with colors green and white as follows: color (i, j) green if $\delta_H(A_i, A_j) \geqslant \frac{1}{2}$ and color it white if $\delta_H(A_i, A_j) < \frac{1}{2}$. Since by choice $\frac{1}{2}\log_2(1/2\varepsilon) > p+1$, it follows from Ramsey's theorem (being generous) that we have a monochromatic complete subgraph H^{***} of H^{**} of order $p+1$.

Without loss of generality we may assume that H^{***} is a complete green graph. Then relabelling the sets in the partition (if necessary) we have (i) (A_i, A_j) is ε-regular and (ii) $\delta_H(A_i, A_j) \geqslant \frac{1}{2}$ for all $1 \leqslant i < j \leqslant p+1$. We will show that the red subgraph then contains a copy of G. (If the edges of H^{***} are white then by placing H with $\bar{H}$ in (ii) one can show that the blue subgraph $\bar{H}$ contains a copy of G.)

To construct a copy of G in H we proceed inductively to choose vertices $w_1, w_2, ..., w_n$ from H so that the map $v_i \to w_i$ is an isomorphism. Furthermore, we will choose these vertices w_i so that for each $i = 1, 2, ..., n$ the following conditions hold.

(a) If $1 \leqslant \alpha \leqslant i$ and $v_\alpha \in V_\beta$ for some $1 \leqslant \beta \leqslant p+1$, then $w_\alpha \in A_\beta$.

(b) If $1 \leqslant \alpha_1 < \alpha_2 \leqslant i$ and $v_{\alpha_1} v_{\alpha_2} \in E(G)$, then $w_{\alpha_1} w_{\alpha_2} \in E(H)$.

(c) If $i < \alpha' \leqslant n$, $V(\alpha', i) = \{w_\alpha \mid 1 \leqslant \alpha \leqslant i,\ v_\alpha v_{\alpha'} \in E(G)\}$, $x = |V(\alpha', i)|$, and $v_{\alpha'} \in V_\beta$ for some $1 \leqslant \beta \leqslant p+1$, then A_β contains a subset A'_β having at least $|A_\beta|/3^x$ vertices so that every vertex in A'_β is adjacent to every $w_\alpha \in V(\alpha', i)$.

Assume that for some nonnegative integer i with $i < n$, the vertices w_α, $1 \leqslant \alpha \leqslant i$, have been chosen so that conditions (a)–(c) are satisfied. We show how to choose w_{i+1} suitably. (Observe that the choice allows $i = 0$ because the rule for choosing w_1 is the same as that for choosing all other values of i.)

Assume $v_{i+1} \in V_{\beta_0}$, so that A_{β_0} does not contain a vertex from $V(i+1, i)$.

Let A'_{β_0} be the subset of A_{β_0} consisting of those vertices adjacent to every w_α in $V(i+1, i)$. By condition (c), $|A'_{\beta_0}| \geqslant |A_{\beta_0}|/3^x$, $x = |V(i+1, i)|$. Also, $x = |V(i+1, i)| = |N_{L_i}(v_{i+1})| \leqslant p$ so that $1/3^x \geqslant 1/3^p \geqslant \varepsilon$.

With the choice of any vertex of A'_{β_0} as w_{i+1} conditions (a) and (b) clearly hold. However, more care is needed to insure that condition (c) is satisfied. It is clear that in choosing w_{i+1} from A'_{β_0} we need only be concerned with those values $\alpha' > i+1$ for which v_{i+1} is adjacent to $v_{\alpha'}$. Since G is p-arrangeable, $|N_{L_{i+1}}(N_{R_{i-1}}(v_{i+1}))| \leqslant p$ and therefore

$$\left| \bigcup_{v'_\alpha \in N_{R_{i+1}}(v_{i+1})} N_{L_{i+1}}(v_{\alpha'}) - \{v_{i+1}\} \right| \leqslant p-1.$$

Thus there exists $v_{\alpha'_1}, v_{\alpha'_2}, ..., v_{\alpha'_q} \in N_{R_{i-1}}(v_{i+1})$ with $q \leqslant p2^{p-1}$ so that for each $v_{\alpha'} \in N_{R_{i+1}}(v_{i+1})$, there is $1 \leqslant j \leqslant q$ such that $N_{L_{i+1}}(v_{\alpha'}) = N_{L_{i+1}}(v_{\alpha'_j})$ and $v_{\alpha'}, v_{\alpha'_j} \in V_\beta$ for some β.

For any $v_{\alpha'_j}$, $1 \leqslant j \leqslant q$, let $v_{\alpha'_j} \in V_\beta$. Then since $v_{\alpha'_j} v_{i+1} \in E(G)$, $\beta \neq \beta_0$. Set $x' = |V(\alpha'_j, i+1)| = 1 + |V(\alpha'_j, i)|$. We already know that A_β contains a subset A'_β containing at least $|A_\beta|/3^{x'-1}$ vertices so that each vertex in A'_β is adjacent to each vertex of $V(\alpha'_j, i)$. Also, since $1/3^{x'-1} \geqslant 1/3^p \geqslant \varepsilon$, it follows that $|A_\beta| \geqslant \varepsilon\, |A_\beta|$. In addition since $\delta(A_{\beta_0}, A_\beta) \geqslant \frac{1}{2}$ at most $\varepsilon\, |A_{\beta_0}|$ of the vertices in A'_{β_0} are adjacent to less than one-third of the vertices in A'_β. This means that we will eliminate at most $\varepsilon\, |A_{\beta_0}|$ of the vertices of A'_{β_0} as candidates for w_{i+1} when considering the neighbor $v_{\alpha'_j}$ of v_{i+1}.

If we range over all possible values for α'_j, $1 \leqslant j \leqslant q \leqslant p \cdot 2^{p-1}$, we will eliminate at most $p2^{p-1} \varepsilon\, |A'_{\beta_0}|$ vertices in A'_{β_0} as candidates for w_{i+1}. For any of the remaining candidates, say w^*_{i+1} for w_{i+1}, and any α'_j, $1 \leqslant j \leqslant q$, if $v_{\alpha'_j} \in V_\beta$ there is a subset $A'_\beta \subseteq A_\beta$ such that every vertex in A'_β is adjacent to every vertex of $V(\alpha'_j, i) \cup \{w^*_{i+1}\}$ with $|A'_\beta| \geqslant |A_\beta|/3^{x'}$.

Consider any $v_{\alpha'} \in N_{R_{i+1}}(v_{i+1})$. Then there are an α'_j and a β such that $v_{\alpha'}, v_{\alpha'_j} \in V_\beta$ and $N_{L_{i+1}}(v_{\alpha'}) = N_{L_{i+1}}(v_{\alpha'_j})$. Consequently $V(\alpha', i+1) = V(\alpha'_j, i+1)$ and there is a subset $A'_\beta \subseteq A_\beta$ of the appropriate cardinality such that every vertex in A'_β is adjacent to every vertex of

$$V(\alpha', i+1) \cup \{w^*_{i+1}\} = V(\alpha'_j, i) \cup \{w^*_{i+1}\}.$$

We cannot select for w_{i+1} any of the vertices in A'_{β_0} which have been previously selected. This eliminates at most n additional vertices from V'_{β_0}. Since the number k of sets in the partition of $V(H)$ satisfies $k \leqslant N_1$ and $c \geqslant 2N_1/\varepsilon$ we know that $|A_{\beta_0}| + 1 \geqslant cn/N_1$ so $n \leqslant \frac{1}{2}(|A_{\beta_0}| + 1)\,\varepsilon \leqslant \varepsilon\, |A_{\beta_0}|$. Therefore in order to select vertex w_{i+1} from the set A'_{β_0} appropriately we need that $\varepsilon\, |A_{\beta_0}| + p2^{p-1}\varepsilon\, |A_{\beta_0}| \leqslant p2^p\varepsilon\, |A_{\beta_0}| < |A'_{\beta_0}|$ and that $|A_\beta|/3^{x'} \geqslant 1$. But $|A_\beta|/3^{x'} \geqslant \varepsilon\, |A_\beta| \geqslant 1$ follows from $c \geqslant 2N_1/\varepsilon$ while $p2^p\varepsilon\, |A_{\beta_0}| < |A'_{\beta_0}|$ follows from $|A'_{\beta_0}|/|A_{\beta_0}| \geqslant 1/3^p > p2^p\varepsilon$.

Thus whenever $w_1, w_2, ..., w_i$ can be chosen such that conditions (a)–(c) hold, then w_{i+1} can also be chosen such that these conditions hold. By induction G is a subgraph of the red subgraph of H, completing the proof of the theorem. ∎

Before proceeding to the proof of Theorem 5 we introduce a greedy algorithm which will be used in the proof. This algorithm will attempt to find (in a greedy way) a p-arrangeable sequence for a graph G with $\rho(G) \leqslant d$. The algorithm we describe will be called the (d, p)-arrangeable algorithm and will be applied only to graphs G of order n with $\rho(G) \leqslant d \leqslant p$.

THE (d, p)-ARRANGEABLE ALGORITHM

Step 1. Choose any vertex $v_n \in V(G)$ such that $d(v_n) \leqslant d$, set $R_{n-1} = \{v_n\}$, $L_{n-1} = V(G) - R_{n-1}$ and $i = n-1$.

Step 2. If there is a vertex $v \in L_i$ such that the properties $|N_{L_i}(v)| \leqslant d$ and $|N_{L_i}(N_{R_i}(v) \cup \{v\})| \leqslant p$ hold, then set $v_i = v$, $R_{i-1} = R_i \cup \{v\}$, $L_{i-1} = L_i - \{v\}$, and go to Step 3, otherwise stop.

Step 3. Set $i = i-1$. If $i = 0$ then stop, otherwise go to Step 2.

Observe that when this algorithm is applied to a graph G and stops for $i = k$, then the subgraph induced by R_k is p-arrangeable. In particular if the algorithm stops for $i = k = 0$, then $R_k = V(G)$, G is p-arrangeable, and we say the algorithm *generates* G. The reader will notice that the second property in Step 2 of the algorithm requires something stronger than $|N_{L_i}(N_{R_i}(v))| \leqslant p$. This is a technical requirement needed for the algorithm to apply to the proof of Theorem 5.

We next give a lemma which has several useful applications, one which is in the proof of Theorem 5.

LEMMA 12. *Let G be a graph of order n with $\rho(G) \leqslant d$. If the (d, p)-arrangeable algorithm applied to G stops when $i = k > 0$, then $|L_k| \leqslant [d^2(d+1)\,|R_k|]/[p-d+1]$.*

Proof. Set $L'_k = \{v \in L_k \mid |N_{L_k}(v)| \leqslant d\}$ and $L''_k = \{v \in L_k \mid |N_{L_k}(v)| \geqslant d+1\}$. Since $\sum_{v \in L_k} d(v) \leqslant d\,|L_k|$ (remember $\rho(G) \leqslant d$) and $\sum_{v \in L_k} d(v) \geqslant \sum_{v \in L''_k} d(v) \geqslant (d+1)\,|L''_k|$, it follows that $|L''_k| \leqslant d\,|L'_k|$. Also, by assumption for each $v \in L'_k$, $|N_{L_k}(N_{R_k}(v) \cup \{v\})| \geqslant p+1$ and $|N_{L_k}(v)| \leqslant d$. Thus $|N_{L_k}(N_{R_k}(v))| \geqslant p+1-d$, which implies that $|e(N_{R_k}(v), N_{L_k}(N_{R_k}(v)))| \geqslant p+1-d$. Therefore

$$\sum_{v \in L'_k} |e(N_{R_k}(v), N_{L_k}(N_{R_k}(v)))| \geqslant (p+1-d)\,|L'_k|.$$

Since each edge e with end vertex $u \in R_k$ contributes at most $|N_{L_k}(u)| \leqslant d$ times to the left-hand side of the above inequality, it follows that $d\,|e(L_k, R_k)| \geqslant (p+1-d)\,|L_k'|$. But $|e(L_k, R_k)| = \sum_{v \in R_k} N_{L_k}(v) \leqslant d\,|R_k|$. Thus $d^2\,|R_k| \geqslant (p+1-d)\,|L_k'|$ and $d\,|L_k'| \geqslant |L_k''|$, so that $|L_k| \leqslant ([d^2(d+1)\,|R_k|)/[p+1-d]$. ∎

Proof of Theorem 5

It is well known that any planar graph G with q edges and $n \geqslant 3$ vertices satisfies $q \leqslant 3n-6$. Hence for any planar graph G, $\rho(G) \leqslant 6$. We thus prove the following proposition, which has Theorem 5 as an immediate corollary.

PROPOSITION 13. *If G is a planar graph of order n then the* (6, 761)-*arrangeable algorithm applied to G generates G.*

A consequence of this proposition is that a 761-arrangeable sequence can be found algorithmically for a planar graph by a greedy approach.

Proof of Proposition 13

The proof is by induction on n, the number of vertices G. Since the result is trivial for $n=1$, we assume $n>1$ and that the result holds for any planar graph of order $<n$. Further suppose that when the (6, 761)-arrangeable algorithm is applied to G the algorithm stops for $i=k>0$. Then by Lemma 12

$$|L_k| \leqslant \frac{6^2(6+1)}{761-6+1}\,|R_k| = \frac{1}{3}\,|R_k|.$$

Throughout the remainder of the proof we will assume that G has been embedded in the plane. By the algorithm $R_k = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ and $L_k = V(G) - R_k$.

We first show there does not exist a vertex $u \in R_k$ with $|N_{L_k}(u)| \leqslant 3$. To see that this is the case suppose the contrary, that $u \in R_k$ with $|N_{L_k}(u)| \leqslant 3$. Let $A = \{(z, w) \mid z, w \in N_{L_k}(u),\ z \neq w\}$. Further let H be the graph of order $n-1$ obtained from G by deleting vertex u and adding as new edges each nonadjacent pair of vertices of G which belong to A. Thus H may have as many as three new edges (not in G) while losing those edges of G incident to u. It is easy to see that H is planar, with its embedding in the plane obtained from that of G.

For each j, $n \geqslant j \geqslant k+1$ with $v_j \neq u$, set $R_j^* = \{v_{j+1}, v_{j+2}, \ldots, v_n\} - \{u\}$ and $L_j = V(G) - R_j$. Then letting $L_j(H) = L_j \cap V(H)$ and $R_j^*(H) = R_j^* \cap V(H)$ we obtain

$$|N_{L_j(H)}(N_{R_j^*(H)}(v_j) \cup \{v_j\})| \leqslant |N_{L_j}(N_{R_j}(v_j) \cup \{v_j\})| \leqslant 761$$

and

$$|N_{L_j(H)}(v_j)| \leqslant |N_{L_j}(v_j)| \leqslant 6.$$

Also, by the induction assumption the (6, 761)-arrangeable algorithm generates H so there is a $v \in L_k$ such that $|N_{L_k(H)}(v)| \leqslant 6$ and $|N_{L_k(H)}(N_{R_k^*(H)}(v) \cup \{v\})| \leqslant 761$. But $|N_{L_k}(v)| \leqslant |N_{L_k(H)}(v)| \leqslant 6$, and

$$|N_{L_k}(N_{R_k}(v) \cup \{v\})| = |N_{L_k(H)}(N_{R_k^*(H)}(v) \cup \{v\})| \leqslant 761$$

unless $N_{R_k}(v) = \{u\}$, in which case

$$|N_{L_k}(N_{R_k}(v) \cup \{v\})| \leqslant |N_{L_k}(\{u, v\})| \leqslant 3 + 6 < 761.$$

This contradicts the fact that the (6, 761)-arrangeable algorithm stops at $i = k > 0$. Hence for each $u \in R_k$, $|N_{L_k}(u)| \geqslant 4$.

Next it is shown that for each $v_i \in R_k$ there is a pair of vertices $w_{i_1}, w_{i_2} \in N_{L_k}(v_i)$ such that for $v_l \neq v_j$ in R_k the sets $\{w_{l_1}, w_{l_2}\} \neq \{w_{j_1}, w_{j_2}\}$. To prove this fact assume we have found the pairs $\{w_{i_1}, w_{i_2}\}$ for all v_i $(j < i \leqslant n)$ such that the condition holds for the selected pairs and that we wish to select a pair $\{w_{j_1}, w_{j_2}\}$ such that the condition continues to hold for all selected pairs. We consider three separate cases.

Case 1. There exists a $v_l \in R_k$ $(l \neq j)$ such that the four element set $\{w_1, w_2, w_3, w_4\} \subseteq N_{L_k}(v_j) \cap N_{L_k}(v_l)$. Assume the subgraph induced by $\{v_j, v_l, w_1, w_2, w_3, w_4\}$ in the planar embedding of G contains the subgraph shown in Fig.1.

Then for each $v \in R_k - \{v_j, v_l\}$ the set $\{w_1, w_3\} \nsubseteq N(v)$ and $\{w_2, w_4\} \nsubseteq N(v)$. Hence if $l < j$ or if $l > j$ and $\{w_{l_1}, w_{l_2}\} \neq \{w_1, w_3\}$, then set $w_{j_1} = w_1$, $w_{j_2} = w_3$, while if $l > j$ and $\{w_{l_1}, w_{l_2}\} = \{w_1, w_3\}$, then set $w_{j_1} = w_2$, $w_{j_2} = w_4$.

Case 2. There exists a $v_l \in R_k$ $(l \neq j)$ such that the three element set $\{w_1, w_2, w_3\} = N_{L_k}(v_j) \cap N_{L_k}(v_l)$ and $w_4 \in N_{L_k}(v_j) - N_{L_k}(v_l)$. Then the sub-

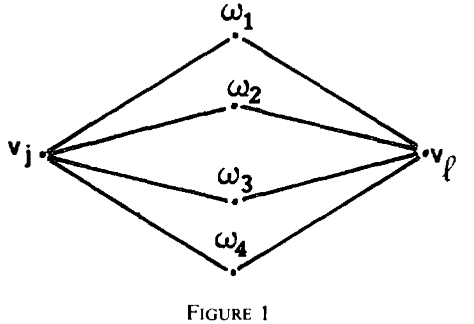

FIGURE 1

graph induced by $\{v_l, v_j, w_1, w_2, w_3, w_4\}$ contains one of the embeddings in the plane as shown in Fig. 2.

Then set $\{w_{j_1}, w_{j_2}\} = \{w_2, w_4\}$ when the embedding is as shown in Fig. 2a, $\{w_{j_1}, w_{j_2}\} = \{w_1, w_4\}$ when as in Fig. 2b, and $\{w_{j_1}, w_{j_2}\} = \{w_3, w_4\}$ when as in Fig. 2c.

Case 3. For each $v_l \in R_k$ $(l \neq j)$, $|N_{L_k}(v_j) \cap N_{L_k}(v_l)| \leq 2$. Let $B=$

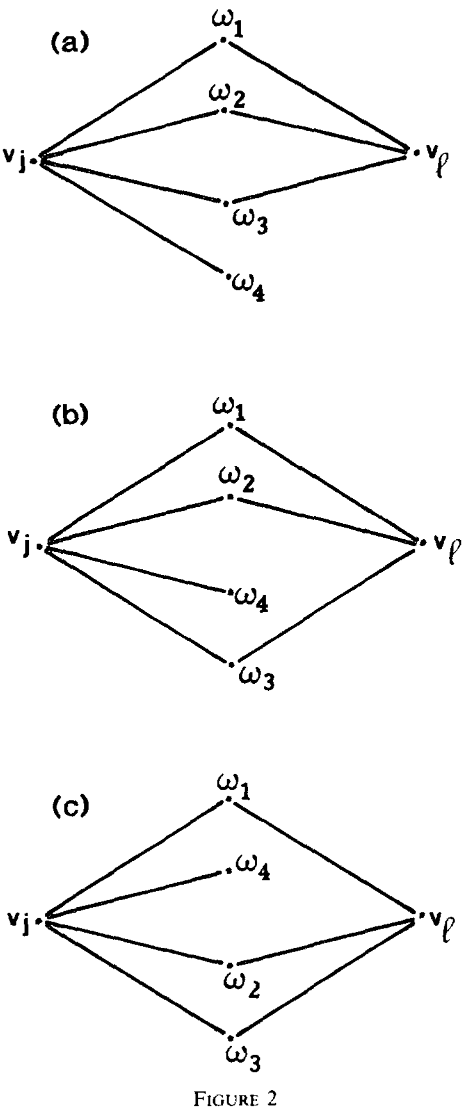

FIGURE 2

$\{w_1, w_2, w_3, w_4\} \subseteq N_{L_k}(v_j)$. For each pair $w_{t_1}, w_{t_2} \in B$ there is a vertex $v_{l(t_1, t_2)} \in R_k$ so that $N_{L_k}(v_{l(t_1, t_2)}) \cap N_{L_k}(v_j) = \{w_{t_1}, w_{t_2}\}$. Therefore the subgraph induced by $\{v_j, w_1, w_2, w_3, w_4, v_{l(1,2)}, v_{l(1,3)}, v_{l(1,4)}, v_{l(2,3)}, v_{l(2,4)}, v_{l(3,4)}\}$ contains a subdivision of K_5, contradicting the fact that G is planar. Hence this case cannot occur.

Since for each $v_i \in R_k$ there is a pair of vertices $w_{i_1}, w_{i_2} \in N_{L_k}(v_i)$ such that for $v_i \neq v_j$ in R_k the sets $\{w_{i_1}, w_{i_2}\} \neq \{w_{j_1}, w_{j_2}\}$, we form the graph H with vertex set $V(H) = L_k$ and edge set $E(H) = \bigcup_{i=k+1}^{n} \{w_{i_1}, w_{i_2}\}$. Since G is planar and all pairs $\{w_{i_1}, w_{i_2}\}$ for $k+1 \leqslant i \leqslant n$ are distinct, H is easily seen to be planar. But

$$|E(H)| = \left| \bigcup_{i=k+1}^{n} \{w_{i_1}, w_{i_2}\} \right| = |R_k| \geqslant 3\,|L_k| = 3\,|V(H)|,$$

a contradiction to the planarity of H. Hence the original supposition that the (6, 761)-arrangeable algorithm applied to G stops for $i = k > 0$ is false, completing the proof of Proposition 13 and Theorem 5. ∎

Proof of Theorem 6

For each positive integer d and $0 < \varepsilon < 1$ set $p = \lceil d^2(d+1)/\varepsilon \rceil + d - 1$. The result follows as a direct application of Lemma 12. ∎

IV. Concluding Remarks

The principal results of the paper, Theorems 1 and 5, give further support to the truth of the Burr–Erdős conjecture. Also, Theorems 8 and 9 suggest that those graphs which fail to be p-arrangeable might well each contain an induced subgraph isomorphic to a graph of the type G' as described in Theorem 9. It would be interesting to classify the graphs which fail to be p-arrangeable. Further if $\mathscr{G}$ denotes the collection of all graphs G with $\rho(G) \leqslant d$ (d fixed), what portion of $\mathscr{G}$ are p-arrangeable (p fixed)? An answer to the latter question would give some indication as to how much Theorem 1 improves Theorem A.

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