Fractional Chromatic Number of Graphs and Hypergraphs
Writeup by Gasarch

1 Introduction

It is easy to define the chromatic number of a graph or hypergraph. These quantities are always integers. We will define the fractional chromatic number.

To do this we will first define covering and fractional covering a a hypergraph.

Def 1.1 Let $H = (V, E)$ be a hypergraph.

1. An *edge covering of* H is a set $X \subseteq E$ such that, for all $v \in V$, there exists $x \in x$ with $v \in s$.
2. A *minimal edge covering of* H is the edge covering with a minimal number of edges in it.

Note 1.2

1. If H is an ordinary graph then the edge covering is what is usually called a vertex covering of a graph.
2. Let $H = (V, E)$ be a hypergraph. Let $H' = (V, E')$ where E' is the set of ind sets of vertices in H (sets of vertices that contain no edges). Note that an edge covering of H' can be interpreted as a proper coloring of H.

2 Two Definitions

We present two definitions of fractional covering number and then show that they are equivalent.

Lemma 2.1 *The minimal covering problem an be formulated as an integer programming problem.*

Proof:

Let $H = (V, E)$ be a hypergraph. For each edge e we have a variable y_e. The y_e's are 0-1 valued and indicate if e is in the cover or not.

MINIMIZE $\sum_{e \in E} y_e$

SUBJECT TO the following contraints.

For every $v \in V$ we have the constraint that it belongs to some edge that is chosen. Formally this is

$$\sum_{e: v \in e} y_e \geq 1$$

We also have that $0 \leq y_e \leq 1$. And of course we already said that y_e is integer valued. ∎

PUT IN MATRIX NOTATION HERE.

Def 2.2 Let H be a hypergraph. $FRACCOV_1(H)$ is the value of $\sum_{e \in E} y_e$ in the relaxed LP version of the IP program in Lemma 2.1.

Def 2.3 Let H be a hypergraph.

1. Let $t \in \mathsf{N}$. A *t-fold edge covering of H* is a *multiset* of edges $F = \{f_1, \ldots, f_L\}$ such that, for every $v \in V$, there are t edges in F that contain v.

2. Let $MIN_t(H)$ be the least L such that there is a multiset of size L. Let

$$FRACCOV_2(H) = \inf_{t \to \infty} MIN_t(H)/t = \lim_{t \to \infty} MIN_t(H)/t.$$

(We prove that the limit exists and equals the inf in a later section.)