Ergodic Proofs of VDW Theorem-Handout

1 Definitions from Topology

Def 1.1

1. X is a *metric space* if there exists a function d (called a metric) with the following properties. (1) $d(x, y) = 0$ iff $x = y$, (2) $d(x, y) = d(y, x)$, (3) $d(x, y) \le d(x, z) + d(z, y)$ (this is called the triangle inequality).

2. X, Y metric space. If $x \in X$ and $\epsilon \in R^+$ then $B(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$. Sets of this form are called *balls*.

3. Any union of balls is an open set.

4. If A is the complement of an open set then A is closed.

5. Let $A \subseteq X$ and $x \in X$. x is a *limit point of* X if $(\forall \epsilon > 0)(\exists y)[y \in B(x, \epsilon) \cap A]$.

6. If $x_1, x_2, \ldots \in X$ then $\lim_i x_i = x$ means $(\forall \epsilon > 0)(\exists i)(\forall j)[j \ge i \Rightarrow x_j \in B(x, \epsilon)]$.

7. $T : X \to X$. T is *continuous* if for all $x, x_1, x_2, \ldots \in X$ $\lim_i x_i = x \Rightarrow \lim_i f(x_i) = f(x)$.

8. $T : X \to X$ is *unif-continous* if $(\forall \epsilon)(\exists \delta)(\forall a, b \in X)[d(a, b) < \delta \Rightarrow d(T(a), T(b)) < \epsilon]$.

9. $T : X \to X$ is *bi-unif-continous* if T is a bijection, T is uniformily continous, and T^{-1} is uniformily continous.

10. If $A \subseteq X$ then *the closure of* X, denoted $cl(A)$, is the intersection of all closed sets containing X.

11. X is *Barg* if every infinite subset of X has a limit point.

Fact 1.2

1. *$cl(A)$ is the smallest closed set containing A.*

2. *If a set is closed then it contains all of its limit points.*

3. *In a metric space $cl(A)$ is the union of A and the limit points of A.*

4. *If X is Barg and $X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$ are nonempty closed sets then $\cap_i X_i \ne \emptyset$.*

Def 1.3 Let X be a metric space and $T : X \rightarrow X$ be continous. Let $x \in X$. The point x is *Recurrent for* T if

$$(\forall \epsilon)(\exists n)[d(T^{(n)}(x), x) < \epsilon].$$

We prove a theorem about Recurrent points and then apply it to get VDW theorem.

Def 1.4 A metric space S is *minimal* if, for every $x \in S$,

$$S = cl(\{\ldots, T^{-3}(x), T^{-2}(x), T^{-1}(x), T^{0}(x), T^{1}(x), T^{2}(x), T^{3}(c), \ldots\})$$

We show the following by a multiple induction.

1. A_r: $(\forall \epsilon > 0)(\exists x, y \in S, n \in \mathsf{N})$
   $d(T^{(n)}(x), y) < \epsilon$, $d(T^{(2n)}(x), y) < \epsilon$, $\ldots$, $d(T^{(rn)}(x), y) < \epsilon$.
2. B_r: $(\forall \epsilon > 0)(\forall z \in S)(\exists x \in S, n \in \mathsf{N})$
   $d(T^{(n)}(x), z) < \epsilon$, $d(T^{(2n)}(x), z) < \epsilon$, $\ldots$, $d(T^{(rn)}(x), z) < \epsilon$.
3. C_r: $(\forall \epsilon > 0)(\forall z \in S)(\exists x \in S, n \in \mathsf{N}, \epsilon' > 0)$
   $T^{(n)}(B(x, \epsilon') \subseteq B(z, \epsilon)$, $T^{(2n)}(B(x, \epsilon') \subseteq B(z, \epsilon)$, $\ldots$, $T^{(rn)}(B(x, \epsilon') \subseteq B(z, \epsilon)$.
4. $(\forall \epsilon > 0)(\exists w \in S, n \in \mathsf{N})$
   $d(T^{(n)}(w), w) < \epsilon$, $d(T^{(2n)}(w), w) < \epsilon$, $\ldots$, $d(T^{(rn)}(w), y) < \epsilon$.

Def 1.5 Let X be a metric space, $T : X \rightarrow X$ be a bijection, and $x \in X$.

1. $$\mathrm{CLT(x)} = \mathrm{cl}(\{\ldots, T^{(-3)}(x), T^{(-2)}(x), T^{(-1)}(x), T^{(0)}(x), T^{(1)}(x), T^{(2)}(x), T^{(3)}(x), \ldots)$$
2. x is *homogenous* if
   $$(\forall y \in \mathrm{CLT(x)})[\mathrm{CLT(x)} = \mathrm{CLT(y)}].$$
3. X is *barg* if every infinite subset of X has a limit point in X.

Lemma 1.6 *Let $(X, \preceq)$ be a partial order. If every chain has an upper bound then there exists a maximal element*

Lemma 1.7 *Let X be a metric space, $T : X \rightarrow X$ be bi-continous, and $x \in X$. If $y \in \mathrm{CLT(x)}$ then $\mathrm{CLT(y)} \subseteq \mathrm{CLT(x)}$.*

Theorem 1.8 *Let X be a barg metric space. Let $T : X \rightarrow X$ be a bijection then there exists a homogenous point $x \in X$.*

Theorem 1.9 *For all c, for all k, for every c-coloring of Z there exists a monochromatic arithmetic sequence of length k.*