NIM Games: Handout 1

Based on notes by William Gasarch

1 One-Pile NIM Games

Consider the following two-person game in which players alternate making moves.

- There are initially n stones on the board.
- During a move a player can remove either one, two, or three stones.
- The first player who cannot move loses (this only happens when there are 0 stones on the board).

Notation 1.1 We denote this game $(1, 2, 3)$-NIM.

Before reading on, think about how you should play this game to win if you go first starting with a pile of, say, 7 stones. How about 21 stones?

Strategy: It is clear that if there are only one, two, or three stones left on your turn, you can win the game by taking all of them. If, however, you have to move when there are exactly four stones you will lose, because no matter how many you take, you will leave one, two, or three, and your opponent will win by taking the remainder. If there are five, six, or seven stones, you can win by taking just enough to leave four stones. If there are eight stones, and your opponent plays optimally, you will again lose, because you must leave five, six, or seven.

If you go first and both you and your opponent play optimally, here is a table indicating whether you will win or lose $(1, 2, 3)$-NIM for up to 21 stones.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
L	W	W	W	L	W	W	W	L	W	W	W	L	W	W	W	L	W	W	W	L	W

In general, if there are a multiple of four stones you will lose. Otherwise you win by taking enough stones to leave a multiple of four for your opponent. So we see the pattern `LWWW`, which is repeated. Thus, 21 stones is a win: take one stone.

1.1 Modular Arithmetic

In this section we define notation for modular arithmetic, which will be useful in studying NIM games.

Def 1.2 We say that that two numbers a and b are *equivalent modulo* m if $a - b$ is a multiple of m. This is written $a \equiv b \pmod{m}$.

For example, $8 \equiv 2 \pmod 3$. (Some people also write $a \bmod m$ for the remainder obtained when a is divided by m. So in that notation, $8 \bmod 3 = 2$.)

We are used to working with modular arithmetic in everyday life. For example, starting from the midnight we could count the number of seconds, minutes, and hours exactly. In practice, this is too complicated so we take the seconds mod 60, the minutes mod 60, and the hours mod 12. (Sometimes we take hours mod 24 to distinguish a.m. and p.m.)

Consider the first game from the last section. Using the modular arithmetic notation, we say that a pile of n stones is a win for the first player if $n \equiv 1, 2, 3 \pmod 4$ and a loss if $n \equiv 0 \pmod 4$.

2 General One-Pile NIM

Def 2.1 Let $a_1, a_2, \ldots, a_k$ be k distinct positive integers. Then $(a_1, \ldots, a_k)$-NIM is the following game.

- There are initially n stones on the board.
- During a move a player can remove either $a_1, a_2, \ldots, a_{k-1}$, or a_k stones.
- The first player who cannot move loses. (This may happen even if there are a non-zero number of stones on the board. For example, if the game is $(2, 3)$-NIM and there is one stone on the board, then the player cannot move.)

Let's work out the best strategy for $(1, 4)$-NIM. Assume there are s stones left. If $n < 4$, you can take only one stone, leaving $n - 1$ stones. If $n - 1$ is a loss for your opponent, then n is a win for you, and if $n - 1$ is a win for your opponent, then n is a loss for you. If $n \geq 4$, you can remove one or four, leaving either $n - 4$ or $n - 1$ stones. If either $n - 4$ or $n - 1$ is a loss for your opponent, then n is a win for you. Otherwise $n-4$ and $n-1$ are both wins for your opponent, so n must be a loss for you.

We can build a win/loss table by looking for each n at the entries for $n-1$ and $n-4$ stones. Here it is up to 21 stones.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
L	W	L	W	W	L	W	L	W	W	L	W	L	W	W	L	W	L	W	W	L	W

We see that the pattern is `LWLWW`. How do we confirm that this pattern repeats forever? We can do this by showing that once the pattern `LWLWW` exists, it will keep repeating. The table clearly starts `LWLWW` (for zero to four stones). After that, the win/loss situation for any group of five starting numbers depends only on the previous group. So once the pattern `LWLWW` continues into the next group (five to nine), it has to repeat forever.

Let's turn this into a theorem. We'll call the player who moves first "player I" and the other one "player II". When we say that a player "wins", we mean that player has a strategy to win no matter what the other player does.

Theorem 2.2 *In the game* $(1,4)$*-NIM starting with* n *stones:*

1. *If* $n \equiv 1,3,4 \pmod 5$ *then player I wins.*
2. *If* $n \equiv 0,2 \pmod 5$ *then player II wins.*

Proof: We will prove by induction that the pattern `LWLWW` for player I in the table above always repeats. More formally, we will prove for each positive integer q that the pattern holds for the qth group of starting numbers: player II wins for $n = 5q-5$, player I wins for $n = 5q-4$, player II wins for $n = 5q-3$, and player I wins for $n = 5q-2$ and $n = 5q-1$.

Base Case: $q = 1$. For $n = 0$, player II wins because player I cannot move. For $n = 1, 4$, player I wins by removing all the stones. For $n = 2$, the only move player I can make is to remove one stone, leaving one. Then player II removes the last stone and wins. For $n = 3$, player I must remove one stone leaving two. Player II then removes one stone, and player I removes the last stone.

Inductive Step: Assume that the pattern holds for $q = r$, where $r \geq 1$. To complete the proof, we need to show that the pattern then also holds for $q = r+1$. In this case, the qth group of starting numbers is $5r, 5r+1, 5r+2, 5r+3$, and $5r+4$. Imagine that you are player I. Then:

- For $5r$ stones, you can move either to $5r-1$ and $5r-4$; each option leaves your opponent in a winning position, so it is a loss for you (and a win for player II).

- For $5r+1$ stones, you can take four stones and move to $5r-3$, leaving your opponent in a losing position, so $5r+1$ is a win for you. (You could also move to $5r$, which we just showed is a loss for your opponent too.)

- For $5r+2$ stones, both moves to $5r-2$ and to $5r+1$ leave your opponent in a winning position, so it is a loss for you (and a win for player II).

- For $5r+3$ stones, taking one stone leaves $5r+2$, which we just showed is a loss for your opponent, and thus a win for you.

- For $5r+4$ stones, taking four stones leaves $5r$, which is a loss for your opponent, and thus a win for you.

We thereby reproduce the pattern `LWLWW`.

We now present an alternative proof using induction on n directly. It is really the same proof just expressed differently. However, it is an example of what is often called "strong induction" (in the inductive step we'll assume that the theorem is true for all smaller values of n), whereas the previous induction on q used only the case $q=r$ to prove the case $q=r+1$.

Base Case: $n=0$. As before, player II wins because player I cannot move.

Inductive Step: Assume that the theorem is true for all $n<p$, where $p\geq 1$. To complete the proof, we need to show that the theorem is then also true for $n=p$. There are several cases. In each, we let m be the number of stones remaining after player I moves, and we use the induction hypothesis for $n=m$, which is necessarily less than p.

- $p \equiv 1 \pmod 5$. We need to show that player I wins. Hence we need to show a move from p stones that leaves $m \equiv 0$ or $2 \pmod 5$ stones. If player I removes one stone then $m = p-1 \equiv 1-1 \equiv 0 \pmod 5$.

- $p \equiv 2 \pmod 5$. We need to show that player II wins. Hence we show that every move from p stones leads to $m \equiv 1, 3$ or $4 \pmod 5$ stones. If player I removes one stone then $m = p-1 \equiv 2-1 \equiv 1 \pmod 5$. If player I removes four stones then $m = p-4 \equiv 2-4 \equiv 3 \pmod 5$. (If $p=2$, the second option is not available, but that's OK; player I must remove one stone and then lose.)

- $p \equiv 3 \pmod 5$. We need to show that player I wins. Hence we need to show a move that leaves $m \equiv 0$ or $2 \pmod 5$. If player I removes one stone then $m = p - 1 \equiv 3 - 1 \equiv 2 \pmod 5$.

- $p \equiv 4 \pmod 5$. We need to show that player I wins. Hence we need to show a move that leaves $m \equiv 0$ or $2 \pmod 5$. If player I removes four stones then $m = p - 4 \equiv 4 - 4 \equiv 0 \pmod 5$. (Notice that if $p \geq 1$ and $p \equiv 4 \pmod 5$, then in fact $p \geq 4$, so removing four stones is always possible in this case.)

- $p \equiv 0 \pmod 5$. We need to show that player II wins. Hence we show that every move leads to $m \equiv 1, 3$ or $4 \pmod 5$. If player I removes one stone then $m = p - 1 \equiv 0 - 1 \equiv 4 \pmod 5$. If player I removes four stones then $m = p - 4 \equiv 0 - 4 \equiv 1 \pmod 5$.

This covers all the cases for p and completes the proof. ▮

Note 2.3 The two sets $\{0, 2\}$ and $\{1, 3, 4\}$ have the following properties:

1. If the number of stones is $p \equiv 1, 3$ or $4 \pmod 5$, then some move will leave $m \equiv 0$ or $2 \pmod 5$ stones.

2. If the number of stones is $p \equiv 0$ or $2 \pmod 5$, then all moves will leave $m \equiv 1, 3$ or $4 \pmod 5$ stones.

This kind of structure is common in proofs that certain values lead to player I or player II wins in NIM.

Def 2.4 If there is a pattern of wins and losses (as a function of the starting number of stones n) that repeats after some initial segment (which does not have to fit the pattern), the game is *periodic*. The length of a minimum repeating pattern is the *period*.

Using this notation $(1, 2, 3)$-NIM has period 4, and $(1, 4)$-NIM has period 5.

Consider the game $(2, 4, 7)$-NIM (where each player can remove 2, 4, or 7 stones). It has the following win/loss table.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
L	L	W	W	W	W	L	W	W	L	W	W	L	W	W	L	W	W	L	W	W	L

It has an initial segment `LLWW`. Then the pattern `WWL` repeats forever, so the game is periodic with period 3.

Theorem 2.5 *For each $h_1, \ldots, h_k$, the game $(h_1, \ldots, h_k)$-NIM is periodic.*

Proof: Let m be the maximum of $h_1, \ldots, h_k$. Consider the first $m(2^m+1)$ entries in the win/loss table. Group them into 2^m+1 groups of m contiguous entries. There are only 2^m possible patterns for each group. (Why?) Since there are 2^m+1 groups, by the pigeon hole principle, two groups must be the same. Whatever pattern occurs between those two groups must repeat after that forever. So the period must be at most $m2^m$. ∎

3 Easy Two-Pile NIM

Consider the following NIM-type game:

- The game begins with two piles, one of a stones, and one of b stones. We denote this position (a, b).
- During a move a player must remove one or more stones from one of the piles. (Any number can be removed, up to the number of stones in the pile.)
- The first player who cannot move loses. (This only happens when the position is $(0, 0)$.)

Here is an example of a play of the game.

1. Starting position is $(20, 14)$.
2. Player I removes 6 from pile 1. Position is now $(14, 14)$.
3. Player II removes 4 from pile 1. Position is now $(10, 14)$.
4. Player I removes 4 from pile 2. Position is now $(10, 10)$.
5. Player II removes 8 from pile 2. Position is now $(10, 2)$.
6. Player I removes 8 from pile 1. Position is now $(2, 2)$.
7. Player II removes 2 from pile 1. Position is now $(0, 2)$.

8. Player I removes 2 from pile 2. Position is now $(0, 0)$.

9. Player II loses.

Notice that Player I's strategy was to always even out the piles. We'll prove that this always works. Let ONE be the set of ordered pairs where I wins, and let TWO be the set of ordered pairs where II wins.

Theorem 3.1 *$(a, b) \in ONE$ if and only if $a \neq b$.*

Proof: We prove this by induction on $n = a + b$.

Base Case: $n = 0$. If $a + b = 0$ then $a = b = 0$ and II wins.

Inductive Step: Assume that $n \geq 1$ and that for all (c, d) where $c + d < n$, we have $(c, d) \in ONE$ if and only if $c \neq d$, where. We must show that the theorem is then true for (a, b) if $a + b = n$. There are two cases.

- $a = b$. We take the position to be (a, a). We show that *any* move that player I makes leads to a position in ONE. By symmetry, we can assume without loss of generality that player I removes from pile 1. Let x be the number of stones removed; hence the position is now $(a - x, a)$. Since $a - x + a < a + a$, we can use the induction hypothesis on $(a - x, a)$. Then because $a - x \neq a$, we know that $(a - x, a) \in ONE$. Since every possible move leads to a position in ONE, we've shown that $(a, a) \in TWO$.

- $a \neq b$. Without loss of generality, assume $a < b$. We claim that player I has a winning move: remove $b - a$ stones from pile 2. Now the position is (a, a). Since $a + a < a + b = n$, we can use the induction hypothesis on (a, a). It says that $(a, a) \in TWO$, and hence $(a, b) \in ONE$.

This completes the induction. ∎

NIM Games: Handout 2

Based on notes by William Gasarch

4 General Two-Pile NIM

Consider the following game.

1. Initially there are two piles of stones. As before, we denote a position by (a, b).
2. During a turn a player can do one of the following.
   - Remove 1,2, or 3 stones from pile 1.
   - Remove 1,3, or 4 stones from pile 2.
3. The first player who cannot move loses. (Again, this will only happen if the position is $(0, 0)$.)

More generally, consider the following game.

Def 4.1 Let G_1 and G_2 be one-pile NIM games. Then $G = G_1 \oplus G_2$ is defined as follows.

1. The game begins with two piles, one of a stones, and one of b stones. We denote this position (a, b).
2. During a turn a player can do one of the following.
   - Remove from pile 1 an amount allowed by G_1.
   - Remove from pile 2 an amount allowed by G_2.
3. The first player who cannot move loses. (This may happen even if the position is not $(0, 0)$. For example, if G_1 is $(2, 3)$-NIM and G_2 is $(1, 2)$-NIM then there is no move from position $(1, 0)$.)

Recall the following philosophies from the previous games studied.

In $(1, 2, 3)$-NIM you try to make your opponent move from a position a where $a \equiv 0 \pmod 4$, while never facing such a position yourself. Notice that the losing position 0 has the property that $0 \equiv 0 \pmod 4$. Hence you are trying to make your opponent face a position that has a property also shared by the losing position, while never facing a position with that property yourself. Since the position's numeric value keeps decreasing, and you never face a position with that property, you must win.

In "easy" two pile-NIM you try to make your opponent face a position (a, b) where $a = b$, while never facing such a position yourself. Notice that the losing position $(0, 0)$ has the property that $a = b$. Hence you are trying to make your opponent face a position that has a property also shared by the losing position, while never facing a position with that property yourself. Since the total number of stones $a + b$ keeps decreasing, and you never face a position (a, b) with $a = b$, you must win.

So in general, we would like to find some property of the losing position(s) such that we can make sure we never face a position with that property (and our opponent always does).

4.1 Grundy numbers

Before analyzing two-pile NIM games, we need to analyze one-pile NIM games in more depth. We will assign numbers to all the positions in a one-pile NIM game. These numbers would not be needed if all we wanted to do was win the one-pile game; however, they help to study many-pile games.

Def 4.2 A game in which two players take turns making moves is *impartial (or nonpartisan)* if from each position, exactly the same moves are available to both players.

All the games of the form $(a_1, \ldots, a_k)$-NIM are impartial. Chess is not impartial because one player can only move the white pieces and the other player the black pieces. Similarly, checkers and go are not impartial.

We will see that we can "solve" all impartial games where the last player to move wins. We give a recursive definition for the Grundy number of a position in an impartial game.

Def 4.3 Let G be an impartial game. The *Grundy number* of a position P in G is

- If P is a final position (from which no further move is possible), it has Grundy number 0.

- Otherwise the Grundy number is the minimum nonnegative integer that is *not* a Grundy number of any position that can be attained by making one move from P.

The *Grundy function* of G is the function g that takes each position P to its Grundy number $g(P)$.

Notice that if $g(P) > 0$, then there must be a move to a position P' with $g(P') = 0$. And if $g(P) = 0$, then every move must lead to a position P' with $g(P') > 0$. These observations prove the following proposition.

Proposition 4.4 *Let G be an impartial game, and let g be its Grundy function. Then P is a winning position for player I whenever $g(P) > 0$, and P is a winning position for player II whenever $g(P) = 0$.*

The following lemma will also be useful later.

Lemma 4.5 *Let g be the Grundy function of an impartial game G. Let P, P' be positions in G such that you can get from P to P' in one move. Then $g(P) \neq g(P')$.*

Proof: Assume that from position P you can get to, in one move, the positions $P_1, P_2, \ldots, P_k$. Then $g(P)$ is the least number that is not in the set $\{g(P_1), \ldots, g(P_k)\}$. In particular $g(P)$ cannot equal any number in $\{g(P_1), \ldots, g(P_k)\}$. Since P' is one of the P_i, we have $g(P) \neq g(P')$. ∎

Example 4.6 Here are Grundy numbers for $(1, 2, 3)$-NIM for up to 21 stones.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1

The sequence is periodic with repeating pattern $0, 1, 2, 3$.

We can write the Grundy function as

$$g(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4; \\ 1 & \text{if } n \equiv 1 \pmod 4; \\ 2 & \text{if } n \equiv 2 \pmod 4; \\ 3 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Example 4.7 Here are Grundy numbers for $(1,3,4)$-NIM for up to 21 stones.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	1	0	1	2	3	2	0	1	0	1	2	3	2	0	1	0	1	2	3	2	0

The sequence is periodic with repeating pattern $0,1,0,1,2,3,2$.

We can write the Grundy function as

$$g(n) = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 2 \pmod 7; \\ 1 & \text{if } n \equiv 1 \text{ or } 3 \pmod 7; \\ 2 & \text{if } n \equiv 4 \text{ or } 6 \pmod 7; \\ 3 & \text{if } n \equiv 5 \pmod 7. \end{cases}$$

Example 4.8 Here are the Grundy numbers for $(2,4,7)$-NIM up to 21 stones.

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
0	0	1	1	2	2	0	3	1	0	2	1	0	2	1	0	2	1	0	2	1	0

The sequence is eventually periodic with repeating pattern $1,0,2$. Notice that it has an initial segment of length 8, as compared to length 4 when we looked only at win/loss information.

We can write the Grundy function as

$$g(n) = \begin{cases} 0 & \text{if } n = 0,1,6, \text{ or } n \geq 8 \text{ and } n \equiv 0 \pmod 3; \\ 1 & \text{if } n = 2,3, \text{ or } n \geq 8 \text{ and } n \equiv 2 \pmod 3; \\ 2 & \text{if } n = 4,5, \text{ or } n \geq 8 \text{ and } n \equiv 1 \pmod 3; \\ 3 & \text{if } n = 7. \end{cases}$$

4.2 Using Grundy Numbers for Two-pile NIM Games

We can now state the philosophy of two-pile NIM games. Let G_1 be a one-pile NIM game with Grundy function g_1. Let G_2 be a one-pile NIM game with Grundy function g_2. Let $G = G_1 \oplus G_2$. The position $(0,0)$ has the property that $g_1(0) = g_2(0)$. To win we make our opponent always face a position (a,b) with $g_1(a) = g_2(b)$.

Theorem 4.9 *Let G_1 be a one-pile NIM game with Grundy function g_1. Let G_2 be a one-pile NIM game with Grundy function g_2. Let $G = G_1 \oplus G_2$. Let ONE be the set of initial positions (a,b) from which player I wins. Then $(a,b) \in ONE$ if and only if $(g_1(a) \neq g_2(b))$.*

Proof: We prove this by induction on $n = a + b$. Let TWO be the complement of ONE, namely the set of initial positions (a, b) from which player II wins.

Base Case: $a + b = 0$. This is only possible if $a = b = 0$, and the theorem is true in this case: $g_1(0) = g_2(0) = 0$ and $(0, 0) \in TWO$.

Inductive Step: Assume that $n \geq 1$ and that for all (c, d) such that $c+d < n$, we have $(c, d) \in ONE$ if and only if $g_1(c) \neq g_2(d)$. We must show that the theorem is then true for (a, b) if $a + b = n$. There are two cases.

Case 1: $g_1(a) = g_2(b)$. To show that $(a, b) \in TWO$, we show that any move on pile 1 creates a position in ONE (the argument for pile 2 is analagous by symmetry). Let the new position be (a', b). By Lemma 4.5, $g_1(a) \neq g_1(a')$. Since $a' + b < a + b$, we can apply the induction hypothesis to (a', b). Then since $g_1(a') \neq g_1(a) = g_2(b)$, we know that $(a', b) \in ONE$.

Case 2: $g_1(a) \neq g_2(b)$. To show that $(a, b) \in ONE$, we show that there is a move that creates a position in TWO. Without loss of generality, assume $g_1(a) < g_2(b)$. Consider the positions that can be reached in one move from b in G_2. By the definition of Grundy numbers, for every number $x < g_2(b)$ there is move in G_2 to a position b' such that $g_2(b') = x$. In particular, with $x = g_1(a)$, we can make $g_2(b') = g_1(a)$. The winning move is to make that move in G_2. This creates position (a, b') in G. Since $a + b' < a + b$ we can apply the induction hypothesis to (a, b'). Then since $g_1(a) = g_2(b')$, we know that $(a, b') \in TWO$. ∎

We end this section where we began: with a particular two-pile NIM game.

Example 4.10 Let G_1 be $(1, 2, 3)$-NIM game with Grundy function g_1. Let G_2 be $(1, 3, 4)$-NIM game with Grundy function g_2. Let $G = G_1 \oplus G_2$. Earlier we found that

$$g_1(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4; \\ 1 & \text{if } n \equiv 1 \pmod 4; \\ 2 & \text{if } n \equiv 2 \pmod 4; \\ 3 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

and

$$g_2(n) = \begin{cases} 0 & \text{if } n \equiv 0 \text{ or } 2 \pmod 7; \\ 1 & \text{if } n \equiv 1 \text{ or } 3 \pmod 7; \\ 2 & \text{if } n \equiv 4 \text{ or } 6 \pmod 7; \\ 3 & \text{if } n \equiv 5 \pmod 7. \end{cases}$$

Hence $(a, b) \in TWO$ if and only if one of the following conditions is true:

- $a \equiv 0 \pmod 4$ and $b \equiv 0, 2 \pmod 7$.
- $a \equiv 1 \pmod 4$ and $b \equiv 1, 3 \pmod 7$.
- $a \equiv 2 \pmod 4$ and $b \equiv 4, 6 \pmod 7$.
- $a \equiv 3 \pmod 4$ and $b \equiv 5 \pmod 7$.

NIM Games: Handout 3

Based on notes by William Gasarch

5 Many-Pile NIM Games

We define many-pile NIM games similarly to two-pile NIM games.

Def 5.1 Let $G_1, G_2, \ldots, G_k$ be one-pile NIM games. $G = \oplus_{i=1}^{k} G_i$ is defined as follows.

1. Initially there are k piles of stones. We denote a position by $(a_1, a_2, \ldots, a_k)$, where a_i is the number of stones in pile i for $i = 1, 2, \ldots, k$.
2. During a turn a player can do one of the following.
   - Remove from pile 1 what is allowed by G_1.
   - Remove from pile 2 what is allowed by G_2.
   - $\vdots$
   - Remove from pile k what is allowed by G_k.
3. The first player who cannot move loses. (This may happen even if the position is not $(0, 0, 0, \ldots, 0)$. For example, if G_1 is $(2, 3)$-NIM and all the rest are $(1, 2)$-NIM, then from position $(1, 0, 0, \ldots, 0)$ there is no move.)

We need the following definition to just state our theorem.

Def 5.2 Let $G_1, G_2, \ldots, G_k$ be one-pile NIM games and $G = \oplus_{i=1}^{k} G_i$. Let $(a_1, \ldots, a_k)$ be a position in G. Write $g_1(a_1)$, …, $g_k(a_k)$ in base 2. Add zeros to the left of the numbers so that all the numbers have the same length. Write the numbers down in a table. This table is called $T(G, a_1, \ldots, a_k)$.

Example 5.3 Let G_1 be $(1, 2, 3, 4)$-NIM. Let G_2 be $(1, 3, 4)$-NIM. Let G_3 be $(2, 4, 7)$-NIM. Let $G = G_1 \oplus G_2 \oplus G_3$. Let g_1, g_2, g_3 be the Grundy functions of G_1, G_2, G_3. We write the Grundy Functions of G_1, G_2, and G_3 in both base 10 and base 2. In base 2, we use 3 bits per number since that is the most we need for any of the numbers. The Grundy function of G_1 is

$$g_1(n) = \begin{cases} 0 = (000)_2 & \text{if } n \equiv 0 \pmod 5; \\ 1 = (001)_2 & \text{if } n \equiv 1 \pmod 5; \\ 2 = (010)_2 & \text{if } n \equiv 2 \pmod 5; \\ 3 = (011)_2 & \text{if } n \equiv 3 \pmod 5; \\ 4 = (100)_2 & \text{if } n \equiv 4 \pmod 5. \end{cases}$$

The Grundy function of G_2 is

$$g_2(n) = \begin{cases} 0 = (000)_2 & \text{if } n \equiv 0 \text{ or } 2 \pmod 7; \\ 1 = (001)_2 & \text{if } n \equiv 1 \text{ or } 3 \pmod 7; \\ 2 = (010)_2 & \text{if } n \equiv 4 \text{ or } 6 \pmod 7; \\ 3 = (011)_2 & \text{if } n \equiv 5 \pmod 7. \end{cases}$$

The Grundy function of G_3 is

$$g_3(n) = \begin{cases} 0 = (000)_2 & \text{if } n = 0, 1, 6, \text{ or } n \ge 8 \text{ and } n \equiv 0 \pmod 3; \\ 1 = (001)_2 & \text{if } n = 2, 3, \text{or } n \ge 8 \text{ and } n \equiv 2 \pmod 3; \\ 2 = (010)_2 & \text{if } n = 4, 5, \text{or } n \ge 8 \text{ and } n \equiv 1 \pmod 3; \\ 3 = (011)_2 & \text{if } n = 7. \end{cases}$$

Consider the position $(24, 2, 0)$. The Grundy numbers are $(100, 010, 000)$. Hence $T(G, 24, 2, 0)$ is:

```
100
010
000
```

Theorem 5.4 *Let $G_1, \ldots, G_k$ be one-pile NIM games. Let $g_1, \ldots, g_k$ be the associated Grundy functions. Let $G = \oplus_{i=1}^{k} G_i$, and let $(a_1, \ldots, a_k)$ be a position in G. Then $(a_1, \ldots, a_k)$ is a winning position for player I if and only if some column of $T(G, a_1, \ldots, a_k)$ has an odd number of 1's.*

Proof:

We prove this by induction on $\sum_{i=1}^{k} a_i$. As before, let ONE be the set of initial positions from which player I wins, and let TWO be the set of initial positions from which player II wins.

Base Case: $\sum_{i=1}^{k} a_i = 0$. This only happens when $a_1 = a_2 = \cdots = a_k = 0$. Notice that then $(a_1, \ldots, a_k) \in TWO$ and $g(a_1) = \cdots = g(a_k) = 0$. Clearly all of the columns in $T(G, a_1, \ldots, a_k)$ have an even number of 1's (0 is an even number), so the theorem is correct in this case.

Inductive Step: Assume that for all positions $p = (c_1, \ldots, c_k)$ with $\sum_{i=1}^{k} g(c_i) < n$, we have that $(c_1, \ldots, c_k) \in ONE$ if and only if some column of $T(G, c_1, \ldots, c_k)$ has an odd number of 1's. Let $(a_1, \ldots, a_k)$ be a position where $\sum_{i=1}^{k} a_i = n$. We denote $T(G, a_1, \ldots, a_k)$ by

$$\begin{array}{lccccccc}
g(a_1) = & b_{1,m} & b_{1,m-1} & \cdots & b_{1,i} & \cdots & b_{1,1} \\
g(a_2) = & b_{2,m} & b_{2,m-1} & \cdots & b_{2,i} & \cdots & b_{2,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g(a_j) = & b_{j,m} & b_{j,m-1} & \cdots & b_{j,i} & \cdots & b_{j,1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
g(a_k) = & b_{k,m} & b_{k,m-1} & \cdots & b_{k,i} & \cdots & b_{k,1}
\end{array}$$

There are two cases.

Case 1: There is a column in $T(G, a_1, \ldots, a_k)$ with an odd number of 1's. We need to show that some move creates a position in TWO. Let i be the left-most column such that the number of 1's in $b_{1,i}, \ldots, b_{k,i}$ is odd. Choose j such that $b_{j,i} = 1$. Player I's winning move will involve removing stones from pile j. By the definition of Grundy numbers, for any number $x < g(a_j)$, there is a position a'_j such that $g(a'_j) = x$. In particular, for any bit sequence $b'_{j,i-1} \cdots b'_{j,1}$ there is a position a'_j such that $g(a'_j) = b_{j,m} b_{j,m-1} \cdots b_{j,i+1} 0 b'_{j,i-1} \cdots b'_{j,1}$. Notice that for each of these moves the ith column will have an even number of 1's. We can find $b'_{j,i-1} \cdots b'_{j,1}$ such that the columns $(i-1), \ldots, 1$ also have an even number of 1's. The columns indexed higher than i are untouched so they already have an even number of 1's. This move will produce a position of the form $(a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k)$. Since $a'_j < a_j$ the sum of these numbers is less than $\sum_{i=1}^{n} a_i = n$. Hence we can apply the induction hypothesis. By the choice of a'_j, in $T(G, a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k)$ every column has an even number of 1's. By the induction hypothesis $(a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k) \in TWO$.

Case 2: All of the columns of $T(G, a_1, \ldots, a_k)$ have an even number of 1's. We need to show that every move results in a position in ONE. If Player I's move is in pile j, moving from a_j to a'_j, then $g(a'_j) \neq g(a_j)$. Let i be a bit so that the ith bit of $g(a_j)$ and $g(a'_j)$ differ. Then the ith column of

$T(G, a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k)$ has an odd number of 1's. Since $a'_j < a_j$ we can apply the induction hypothesis to $(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k)$. By the induction hypothesis, since a column of $T(G, a_1, a_2, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k)$ has an odd number of 1's, we know that $(a_1, \ldots, a_{j-1}, a'_j, a_{j+1}, \ldots, a_k) \in ONE$. ▮

Example 5.5 We revisit this example from before. Let G_1 be $(1, 2, 3, 4, 5)$-NIM. Let G_2 be $(1, 3, 4)$-NIM. Let G_3 be $(2, 4, 7)$-NIM. Let $G = G_1 \oplus G_2 \oplus G_3$. Let g_1, g_2, g_3 be the Grundy functions of G_1, G_2, G_3. Their formulas are given before the theorem.

It is possible to write down a statement like "$(a_1, a_2, a_3) \in TWO$ if and only if XXX", but it would be quite complicated involving many cases. We give a couple examples of cases that determine what $T(G, a_1, a_2, a_3)$ is and hence which player wins.

1. If $a_1 \equiv 0 \pmod 5$, $a_2 \equiv 4$ or $6 \pmod 7$, and $a_3 \geq 8$ and $a_3 \equiv 1 \pmod 3$, then $g_1(a_1) = 0 = (00)_2$, $g_2(a_2) = 2 = (10)_2$, and $g_3(a_3) = 2 = (10)_2$. So $T(G, a_1, a_2, a_3)$ is

   ```
   00
   10
   10
   ```

   Notice that every column has an even number of 1's. Hence $(a_1, a_2, a_3) \in TWO$.

2. If $a_1 \equiv 3 \pmod 5$, $a_2 \equiv 1$ or $3 \pmod 7$, and $a_3 \geq 8$ and $a_3 \equiv 2 \pmod 3$, then $g_1(a_1) = 3 = (11)_2$, $g_2(a_2) = 1 = (01)_2$, and $g_3(a_3) = 2 = (01)_2$. $T(G, a_1, a_2, a_3)$ is

   ```
   11
   01
   01
   ```

   Notice that the last column has an odd number of 1's. Hence $(a_1, a_2, a_3) \in ONE$.

6 Dynamic Programming

Consider the following game

- Initially there are 3 piles of stones. We denote a position by (a, b, c).
- A player may remove a prime from the first pile, or a square from the second pile, or a Fibonacci number from the third pile.
- If a player cannot move then they lose.

This is a rather complicated game it is doubtful that it has a succinct statement about when player I wins. However, we can still (with a computer program) calculate who wins which positions rather easily. The key is that we never try to calculate who wins (a_1, a_2, a_3) until we know ALL of the lower positions.

$$W(a,b,c) = \begin{cases} I & \text{if player I wins when the game starts with } (a,b,c); \\ II & \text{if player II wins when the game starts with } (a,b,c). \end{cases}$$

Let PR be the set of primes, SQ be the set of squares and FIB be the set of Fibonacci numbers. Then the following holds:
$W(0,0,0) = II$
$W(a,b,c) =$

- I if
  - $(\exists p \in PR)[a \geq p \wedge W(a-p,b,c) = II]$ OR
  - $(\exists s \in SQ)[b \geq s \wedge W(a,b-s,c) = II]$ OR
  - $(\exists f \in FIB)[c \geq f \wedge W(a,b,c-f) = II]$
- II otherwise.

If by the time we are looking at (a,b,c) we have already computed W of all (a',b',c') with $a'+b'+c' < a+b+c$ then we can carry out this calculation easily. In fact, the problem we are really solve here is not "What is $W(a,b,c)$?' but instead "What is $W(a,b,c)$ for all (a,b,c) $a+b+c \leq n$?"

Here is psuedocode for the problem. It is not very efficient; however, it can be made alot more efficient.

```
Input(n).
W(0,0,0)=II. (This sets W(0,0,0) to the value II.)
for i=1 to n
    for a=0 to n
        for b=0 to n-a
            c=n-(a+b) (So now a+b+c=n.)
            W(a,b,c)=II (Will set this to I if find a good move.)
            for p=1 to a, p PRIME
                if (W(a-p,b,c)=II) then W(a,b,c)=I
            for s=1 to b, s SQUARE
                if (W(a,b-s,c)=II) then W(a,b,c)=I
            for f=1 to c, c FIB
                if and (W(a,b,c-f)=II) then W(a,b,c)=I
```

Even though this is not a nice formula, it is a calculation that gives the answer. It can be made alot more efficient by putting in a condition to stop when you find the p, s, or f that works.