Permutable Primes

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Permutable Primes

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Prime numbers and their properties have fascinated both novice and professional mathematicians in every age. Of particular interest in recent years are those primes whose digits can be rearranged to produce other primes. Such primes include palindromic primes [**5**] which are left unchanged by reversing their digits, and absolute primes [**3**], [**6**] which remain prime under all permutations of their digits. Using only two types of rearrangements, reversal of digits and cyclic permutations, we introduce the classes of reversible, cyclic, and symmetric primes. First we show the relationships between these five classes of permutable primes, and note the importance of cyclic primes to the study of absolute primes. Then, based on data from experiments, we propose an asymptotic formula for the number of reversible primes with n digits. If this formula is correct, it could lead to a proof of the conjecture of Gabai and Coogan [**4**] that there are infinitely many palindromic primes.

An **absolute prime** (base ten) is any prime such that every permutation of its digits produces a prime. Obviously each of the primes 2, 3, 5, 7, 11 is an absolute prime. We list in TABLE 1 all known absolute primes having two distinct digits, with primes having the same digits grouped in families.

$n=2$:	$\{13,31\},\{17,71\},\{37,73\},\{79,97\}$
$n=3$:	$\{113,131,311\},\{199,919,991\},\{337,373,733\}$.

TABLE 1. **Absolute primes with n digits**

It has been shown by Johnson [**6**] that if there is an absolute prime with two distinct digits that is not in TABLE 1, it will have more than nine billion digits. The only other absolute primes are the prime repunits. **Repunits** [**1**], [**2**], [**5**], [**8**], [**9**] are numbers all of whose digits are ones and are denoted by R_n where n is the number of digits. Besides $R_2=11$, it is known that R_{19}, R_{23}, and R_{317} are prime (see Yates [**9**]). The primality of the last of these, R_{317}, was discovered only recently and came as a surprise [**7**].

A slightly less restrictive condition on permutation of digits defines the set of cyclic primes. A **cyclic prime** is a prime such that every cyclic permutation of its digits is also prime. All absolute primes are cyclic primes. Using an Algol W program on a UNIVAC 90/80 computer we found the families of cyclic primes which are not absolute, shown in TABLE 2. Primality was tested using a list of primes generated by a subroutine based on a modified sieve of Eratosthenes. Using this program we also found that there are no cyclic primes with seven or eight digits.

$n=3$:	$\{197, 719, 971\}$
$n=4$:	$\{1193, 1931, 9311, 3119\}, \{3779, 7793, 7937, 9377\}$
$n=5$:	$\{11939, 19391, 93911, 39119, 91193\}$,
	$\{19937, 99371, 93719, 37199, 71993\}$
$n=6$:	$\{193939, 939391, 393919, 939193, 391939, 919393\}$,
	$\{199933, 999331, 993319, 933199, 331999, 319993\}$.

TABLE 2. **Cyclic primes with n digits that are not absolute primes**

An absolute or cyclic prime of two or more digits may only contain the digits 1, 3, 7, 9. Otherwise some permutation of the number ends in 0, 2, 4, 5, 6, or 8, and is divisible by 2 or 5. Bhargava and Doyle [**3**] have shown that no absolute prime may contain all four of the digits 1, 3, 7, 9. Johnson [**6**] has extended this result and proved (i) no absolute prime may contain three of the four digits 1, 3, 7, 9, and (ii) no absolute prime may contain two or more of each of two digits from 1, 3, 7, 9. Several cyclic primes in TABLE 2 (e.g., 11939) show that neither of Johnson's results holds for cyclic primes. The cyclic prime 19937 shows that a cyclic prime may contain all four of the digits 1, 3, 7, 9. The prime 19937 is also the twenty-fourth Mersenne exponent and is the largest known Mersenne exponent which is also cyclic.

A **symmetric** or **dihedral prime** is a prime such that every symmetric permutation of its digits also yields a prime. To obtain symmetric permutations of an n-digit number, imagine the digits attached in cyclic order to the vertices of an n-sided regular polygon and permute the digits according to the rotations and reflections that leave the polygon fixed. Rotations correspond to cyclic permutations of the digits. The reflection about the perpendicular bisector of the side of the polygon which connects the vertices corresponding to the first and last digits of the number reverses the digits of the number (see FIGURE 1). The symmetric permutations form the dihedral subgroup of the group of all permutations and are generated by the rotations and any single reflection. Thus *a number is a symmetric prime if and only if the number and its reverse are both cyclic primes*. The numbers 11939 and 193939 generate two families of symmetric primes that are not absolute primes. The only other known symmetric primes are the absolute primes.

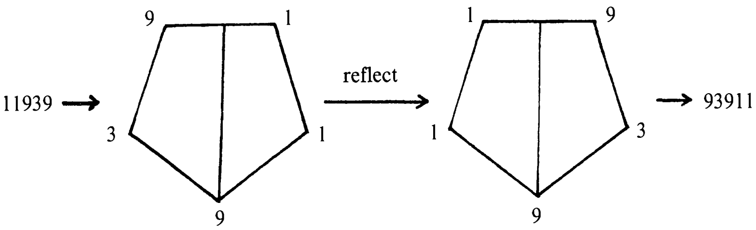


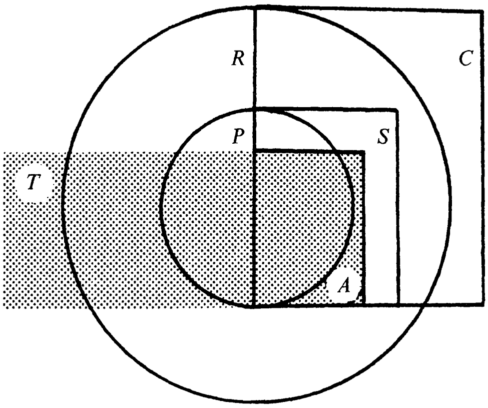
FIGURE 1

A **palindrome** is a string of characters which is the same when read from right to left as when read from left to right. A **palindromic prime** is a prime with this property. **Reversible primes** are primes which are also prime when read from right to left. Palindromic, symmetric, and absolute primes are all special classes of reversible primes. The relationships between the classes of permutable primes is illustrated in FIGURE 2. We note that the intersection of the class of reversible primes with the class of cyclic primes is larger than the class of symmetric primes since 37199 is cyclic and reversible, but its reverse is not cyclic. However, the intersection of the class of palindromic primes with the class of cyclic primes is contained in the class of symmetric primes. This is clear since the reverse of such a number is itself.

One relationship illustrated in FIGURE 2 merits special attention. From Johnson's results [**6**], every absolute prime must be a repunit (called **type A** by Johnson) or a number of the form $aR_n + (b-a)10^m$ (called **type B**) where a and b are distinct digits selected from 1, 3, 7, 9 and $0 \leqslant m \leqslant n$. Thus a type B number consists of $n-1$ digits a and one digit b. All distinct permutations of a type B number may be obtained as cyclic permutations. Hence,

THEOREM. *If x is a type* A *or a type* B *number, then the following are equivalent*:
(i) *x is a cyclic prime*,
(ii) *x is a symmetric prime*,
(iii) *x is an absolute prime*.

The results of our computer search suggest that the classes of absolute, symmetric, and cyclic



R-reversible primes
P-palindromic primes
C-cyclic primes
S-symmetric primes
A-absolute primes
T-type A or B numbers

R and P are bounded by circles. C, S and A are bounded by rectangles. T is shaded.

FIGURE 2

primes may all be finite. Establishing this result is likely to be a difficult problem. An easier problem would be to prove that there are only a finite number of twin primes one or both of which are cyclic or absolute. The pair $\{197, 199\}$ is the largest pair of twin primes each of which is known to be cyclic.

Gabai and Coogan [4] suggest that there may be infinitely many palindromic primes, and that there may be a larger relative percentage of primes in the set of palindromic numbers than in the set of positive integers. This behavior could be a special case of a more general property, namely, that the reverse of a number is more likely to be a prime if the original number is prime. To test this conjecture, consider the following experiments.

EXPERIMENT 1: Choose at random a number with n digits that does not end in the digit 0 (so that the reverse also has n digits), and reverse the digits. The probability of obtaining a prime is $p(n)/(81\times 10^{n-2})$ where $p(n)$ is the number of primes with n digits. One can increase the chances of obtaining a prime by modifying Experiment 1.

EXPERIMENT 2: Choose a prime number with n digits and reverse the digits. The probability of obtaining a prime is $r(n)/p(n)$ where $r(n)$ is the number of reversible primes with n digits. By a computer search, we obtained values of $r(n)$ for $n\leqslant 7$. By comparing columns 4 and 5 of TABLE 3 we see that for $n\leqslant 7$ the reverse of a number is more likely to be prime if the original number is prime. This behavior may be partially explained by a property of multiples of 3. A number is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Thus a number is divisible by 3 if and only if its reverse is divisible by 3. It is also known that a number is divisible by 11 if and only if its reverse is divisible by 11. Experiment 2 eliminates any multiple of 3 or 11, and hence the reverse is not divisible by 3 or 11 and is more likely to be prime. In order to test whether or not this completely explains the observed behavior of $r(n)$ we modify Experiment 1 again.

EXPERIMENT 3: Choose a number with n digits that is not divisible by 3, 10, or 11 and reverse its digits. Of the possible choices of numbers in Experiment 1, only $(20/33)(81\times 10^{n-2})$ of these are possible choices in this experiment. Thus, the probability of obtaining a prime in Experiment 3 is $p(n)/(54/11\times 10^{n-1})$. By comparing columns 5 and 6 of TABLE 3 we see that (except for $n=2$) column 5 has larger values when n is odd, and that column 6 has larger values when n is even. This variation is most likely due to the fact that, except for 11, there are no palindromic primes with an even number of digits [4].

The results of our computations suggest an asymptotic formula for $r(n)$, namely

$$r(n)\approx 11p^2(n)/(54\times 10^{n-1}).$$

n	$p(n)$	$r(n)$	$\dfrac{p(n)}{(81\times 10^{n-2})}$	$\dfrac{r(n)}{p(n)}$	$\dfrac{p(n)}{(54/11\times 10^{n-1})}$
2	21	9	.259	.429	.428
3	143	43	.177	.301	.291
4	1061	204	.131	.192	.216
5	8363	1499	.103	.179	.170
6	68906	9538	.085	.138	.141
7	586081	71142	.074	.121	.119

TABLE 3

If this formula is correct, it would prove the existence of an infinite number of reversible primes, and might lead to the same result for palindromic primes.

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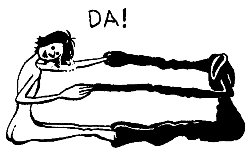
∀ and ∃

Said an upside-down A to an inside-out E,
"*Universal*'s the epithet measuring me.
Your scope is so small
Compared with *For all*—
There is is no more than a form of *To be*."

Said the inside-out E to the upside-down A,
"*To be* is the question! Avoid it? No way!
My role's existential—
I'm basic, essential—
Of equal importance the parts that we play!"

Said the upside-down A *and* the inside-out E,
"Let's drop the dispute and agree to agree!
A nifty notation
Reversed in negation—
Our tandem performance is something to see!"



—KATHARINE O'BRIEN