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Title: Theoretical computer science /

Uniform Title:

Author: GI-Conference.

Edition:

Imprint: Berlin ; New York : Springer-Verlag,

Article: Nesetfil: Some Nonstandard Ramsey Like Applications

Vol: 34 **No.:** **Pages:** 3-15 **Date:** 1984

Dissertation:

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Theoretical Computer Science 34 (1984) 3–15
North-Holland

SOME NONSTANDARD RAMSEY LIKE APPLICATIONS

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Abstract. In this paper, a survey is given of some of the recent research which is related to a particular combinatorial principle namely the Ramsey theorem.

One uses very often elementary and less elementary combinatorial facts and it is not important whether one deserves the name 'principle' for them. As examples consider the principle of inclusion and exclusion, the pigeonhole principle, counting in two ways, several basic properties of trees etc. It is the author's aim here to show a large variety of applications of the Ramsey theorem. It should be stressed that none of the classical and 'standard' applications is mentioned. These can be found in several books and survey articles examples of which are the works of Graham et al. (1980) and Nešetřil and Rödl (1979).

1. Bounds of the Ramsey argument

We should start with the statement of the finite version of Ramsey theorem [11]:

(FRT) For every choice of positive integers p, k, n there exists an integer N with the following property: For every partition $\binom{N}{p} = C_0 \cup \cdots \cup C_{k-1}$ there exists an $i \in k$ and a set $X \subseteq N$ such that $\binom{X}{p} \subseteq C_i$ and $|X| \geqslant n$.

(A natural number is identified with the set of its predecessors and $\binom{X}{p}$ denotes the set $\{Y \subseteq X : |Y| = p\}$.)

The most standard interpretation of (FRT) is indicated by some special terminology; e.g., one usually refers to a partition as a *colouring*; the set Y is called *homogeneous*, etc.

Also note that some (but few) special cases of the theorem are simple. Most notably the case $p = 1$ is the continental Dirichlet's 'schubfach' principle and anglo-american 'pigeonhole' principle.

To shorten the above statement one can adopt the Erdös–Rado partition arrow, $N \to (n)_k^p$, by means of which the finite Ramsey theorem gets the following concise form:

(FRT) $\forall p \, \forall k \, \forall n \, \exists N \, (N \to (n)_k^p)$.

Also, this leads to the Ramsey number $r(p, k, n)$ which may be defined as follows:

$$r(p, k, n) = \min\{N : N \to (n)_k^p\}.$$

$r(1, k, n) = k(n-1)+1$, $r(p, 1, n) = n$ and $r(p, k, p) = p$ are the single classes of known exact values (see [3] for details).

0304-3975/84/$3.00

It is a well known fact that the Ramsey theorem is an example of a combinatorially complex and 'ineffective' statement. While this being a common feeling, not many exact results are known in this direction. Let us mention two more recent examples of this phenomenon by means of special games.

Consider the following game:

Two players I and II are playing on the board $\binom{\omega}{2}=\{\{i,j\}: i\neq j\}$. On each move, player I—the *constructor*—selects a previously unselected pair $\{i,j\}$. Player II—the *destructor*—assigns the colour either red or blue to the pair $\{i,j\}$. The constructor wins if he finds a monochromatic complete graph with n vertices. Otherwise, player II wins.

(FRT) when applied for $p=2$ (i.e., the graph case) implies that the constructor has a winning strategy. Moreover, as $r(2,2,n)\leqslant 4^{n-2}$, the constructor has a winning strategy which takes 'only' 4^n moves. (Because what player I can do is to restrict himself to the numbers $0,1,\ldots,4^{n-2}-1$ and to keep asking there. The above bound for the Ramsey number assures that he has to find a monochromatic complete subgraph of size n.) In fact, there is a simple procedure to do so, the so-called ramification procedure, which is a version of the 'divide-and-take-the-largest-one' heuristic.

Quite recently it has been shown by Beck [1] that the constructor cannot do much better.

Theorem 1.1 ([1]). *Destructor has a strategy such that the constructor is unable to win within $2^{n/2}$ moves.*

This theorem nicely complements a classic of Erdös who proved $r(2,2,n)>2^{n/2}$.

Another flavour of non-effectiveness of (FRT) stems from the recursion theory. By now it is well known that the functions related to Ramsey type questions grow very fast and in fact they may fail to be provably recursive.

This is not the case with the Ramsey function $r(p,k,n)$ which can be bounded from above by the tower function

$$\left.k^{\cdot^{\cdot^{\cdot^{k^{cn}}}}}\right\}p.$$

However, a small modification of (FRT) yields 'nonrecursive Ramsey numbers'. This was done first by Paris and Harrington [10]. Let us just briefly indicate their approach:

Denote by $N\xrightarrow{*}(n)^p_k$ the validity of the following statement:

For every partition $\binom{N}{p}=C_0\cup\cdots\cup C_{k-1}$ there exists an $i\in k$ and a set $X\subseteq N$ such that $\binom{X}{p}\subseteq C_i$, $|X|\geqslant n$ and $X\geqslant\min X$. (Here $\min X$ is the minimal element of X; the last condition is the only difference between 'star-arrow' and the Erdös-Rado arrow introduced above.)

Put also $r^*(p, k, n) = \min\{N : N \xrightarrow{*} (n)^p_k\}$.

It follows by a standard (compactness) argument that the number $r^*(p, k, n)$ is well defined for every choice of p, k, n. Moreover, as was shown in [10], for every p the function $r^*(p, \cdot, \cdot)$ is provably recursive while the diagonal function $r^*(n, n, n+1)$ fails to be so. In other words, the statement

$$(\text{FRT})^* \forall p\ \forall k\ \forall n\ \exists N\ (N \xrightarrow{*} (n)^p_k)$$

while being true, is an example of an unprovable statement (in the theory of finite sets).

By now there are many examples of combinatorial undecidable results (see [9]). However, most examples are so far related to particular statements of Ramsey type. Let us mention another recent example. As was mentioned above this will be a particular game. The results reported here are due to Kirby and Paris [4] and to my student M. Loebl. The game in question is a 2-person game—one person is Hercules, the other is called Hydra. Hydra is any finite rooted tree, the endpoints of which different from the root are called *heads*. A look at Fig. 1 may be helpful. (Hercules is then Hercules.)

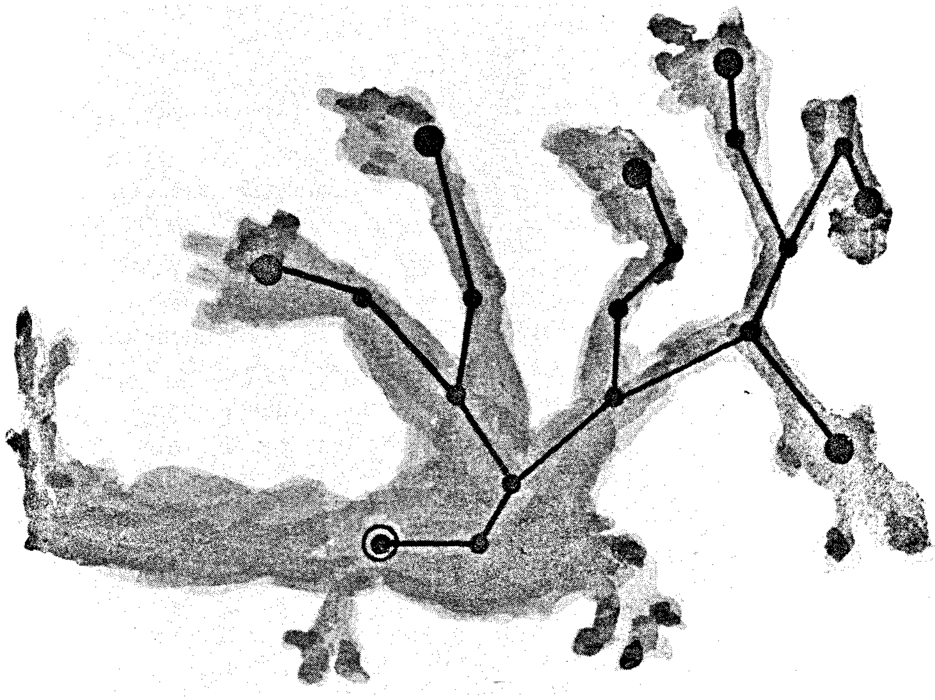


Fig. 1.

The game (one should better use the term battle) between Hercules and a given Hydra proceeds as follows: At stage n $(n \geq 1)$ Hercules chops off one head from the Hydra. As a revenge, Hydra grows at least n new heads in the following manner:

Consider the 2-predecessor (i.e., the grandfather) of a given head (which was just removed), then from this point x sprout n replicas of the part of the Hydra which form a branch at x originally containing the removed head.

If there is no grandfather, then nothing happens.

Similarly we may define a *k-predecessor game* $(k \geq 2)$ as the game where the replicas sprout from the k-predecessor of a given head; the above game is then the 2-predecessor game. Also, the ∞-predecessor game is the game where the replicas sprout from the root.

One can easily see that while Hercules certainly has to work a bit, he has a winning strategy. For example, the strategy which is removing heads with the largest distance from the root is a winning strategy.

Slightly surprisingly one has the following results which mean that these games have a bad moral (unconvenient as a fairy tale for children).

Theorem 1.2 ([4]). *Every strategy of Hercules is winning for the* 2*-predecessor game. I.e., a* 2*-predecessor game is a finite game for every Hydra.*

Theorem 1.3 ([4]). *The statement "every recursive strategy is winning" is unprovable in the theory of finite sets (or in* PA*).*

Loebl extended the Kirby–Paris results for $k > 2$ and proved the following theorems.

Theorem 1.4 (PA). *An ∞-predecessor game is a finite game (in the theory of finite sets).*

Moreover, the finiteness of all k-predecessor games is equivalent (in the theory of finite sets).

Given a Hydra H, denote by $b(H)$ the number of steps in the longest battle between Hercules and Hydra H (in the 2-predecessor game). Theorem 1.3 is equivalent to saying that $b(H)$ is not a provable recursive function. Yet these numbers may be investigated by finite means. We have the following theorem.

Theorem 1.5 (PA). *If H has n points, then*

$$b(H) \leq b(P_n) = b(\underbrace{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet \cdots \bullet\!\!-\!\!\bullet}_{n}).$$

Theorem 1.6 (PA). *The following strategy gives the longest battle for P_n:*

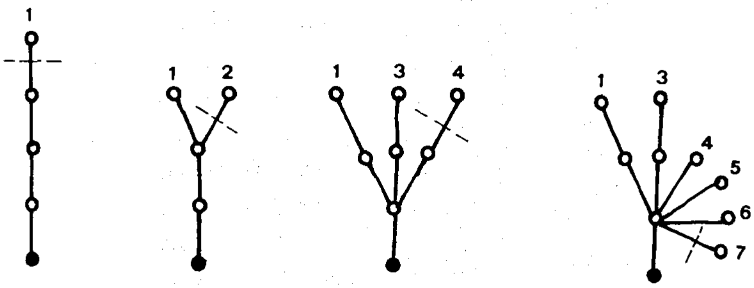
"always chop off the head with the largest number".

Here the numbering of heads of a hydra is a simple process which may be described by induction as follows:

The unique head of P_n gets number 1.

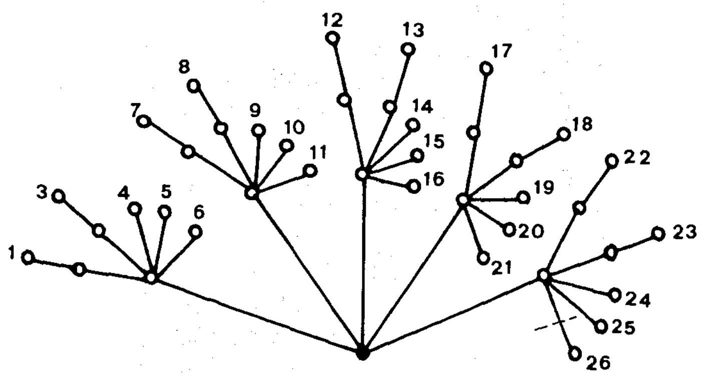
Suppose that the heads of a Hydra H are numbered. If the chopping of a head h leads to a new head, then this head gets the number of h. Moreover, all replicas which sprout are numbered successively by larger numbers in the same way as the original branch.

The diagram of Fig. 2(a) shows an initial segment of a battle of Hercules versus P_5:



P_5

(a)



(b)

Fig. 2.

Finally let us note that the same strategy is valid for a k-predecessor game (see Fig. 2(b).

2. Structural applications

There are two main directions in Ramsey applications.

The above two examples are related to the negative part of the Ramsey theorem. A (good) lower bound for the Ramsey theorem establishes the existence of large complex graphs and set systems which may be in turn used to produce large and complex examples (counterexamples). In other words, these applications use the fact that Ramsey like functions grow fast.

There are other results which use the positive part of the theorem namely the fact that a Ramsey function exists and also an upper bounds for it. These structural applications have often one common patter: In order to establish a bound for an invariant related to a large object one first proves that every large object contains a regular ('homogeneous') sub-object of a given size. If the invariant related to this regular sub-object is easier to determine, then we obtain a lower bound on the invariant of every large object.

An example of this technique may be found in [14]: Yao proves there that if we consider storing of n distinct keys from a set (the key space) of N keys by means of tables, then for large N the storing by means of sorted tables is optimal. This follows from the Ramsey theorem. If $N \geqslant r(n, 2n-1, n!)$, then for every table structure (i.e., a map $\binom{X}{n} \to X^n$) there exists a $Y \subseteq X$, $|Y| = 2n-1$, such that the table structure on Y is a sorted table (with respect to a convenient ordering of Y). It follows that any search strategy when applied to $\binom{Y}{n}$ uses at least $\lceil \log n + 1 \rceil$ probes (see [14] for details).

A similar application yields a simple (and combinatorial) proof of the following number-theoretical result of Erdös.

Theorem 2.1 ([2]). *Let A be a set of integers with the following property: for every n there are $a, b \in A$ such that $n = a \cdot b$. Then for every k there exists an n such that $n = a \cdot b$ has at least k distinct solutions in A.*

Proof ([7]). Consider square-free integers only. Every such integer x may be regarded as a set $M(x)$ of primes ($p \in M(x)$ iff $p \mid x$). By the assumption, for every finite set M of primes (i.e., for the number $\prod M$) there exists a partition $M' \cup M''$ such that $\prod M'$ and $\prod M''$ belong to A. By the Ramsey theorem there exists a set X of natural numbers, $|X| \geqslant 2k^2$, such that for every $M \in \binom{X}{k}$ the partition $M' \cup M''$ is of the same 'type'. This in turn means that there exists a $Y \subseteq X$, $|Y| \geqslant 2k$, such that $\binom{Y}{m} \subseteq A$ for an $m \geqslant \frac{1}{2}$. Thus, for every $M \in \binom{Y}{2m}$ the number $\prod M = n$ has $\binom{2m}{m}$ solutions $n = a \cdot b$ in A. □

From this group of 'structural' applications two more (recent) examples should be mentioned: one is related to the complexity of Boolean functions and the other to the 'natural orderings' of power sets and cubes.

An n-dimensional Boolean function is a mapping $f: \{0, 1\}^n \to \{0, 1\}$. f is called *symmetric* if $f(a_1, \ldots, a_n)$ depends on $\sum_{i=1}^{n} a_i$ only.

Every Boolean function may be viewed as a partition of the set $\{0, 1\}^n$ into two parts or, alternatively, as a partition of the power set $\mathscr{P}(n)$. This suggests to apply the Ramsey theorem. One can prove, e.g., the following theorem.

Theorem 2.2. *For every n there exists an N such that for every N-dimensional Boolean* *function f there exists an interval $I = [\vec{0}, \vec{a}]$ of length n such that f restricted to I is a symmetric Boolean function.*

Also several other Ramsey type results such as the Hales–Jewett theorem on the Finite Union Theorem (see, e.g., [3]) may be applied in this way to Boolean functions. However, it is intereasting that the Ramsey theorem was applied to get results which are useful from the point of view of complexity of Boolean functions. This was done by several researchers and this article covers the part done by P. Pudlák on the formula size of Boolean functions. He proceeds as follows:

Let Ω be any complete base of connectives (e.g., $\vee$, $\wedge$, $\neg$). Denote by $L_\Omega(f)$ the formula size of f; i.e., the smallest size of a formula which realizes f. (Here size means the total number of occurrences of variables.)

Theorem 2.3 ([11]). *For every complete base Ω there exists an ε_Ω such that if f is an n-dimensional Boolean function and if*

$$L_\Omega(f) \leq \varepsilon_\Omega \cdot n(\log\log n - \log r),$$

then there exist indexes $1 \leq i_1 < i_2 < \cdots < i_r \leq n$ such that

$$f|x_1, \ldots, x_r = b(x_{i_1} \oplus \cdots \oplus x_{i_r}, x_{i_1} \vee \cdots \vee x_{i_r})$$

for a Boolean function b. ($\oplus$ denotes mod 2 addition.)

Explicitly, this means that f restricted to the interval $[\vec{0}, \vec{a}]$ (where $\vec{a}$ is given by indexes $i_1, \ldots, i_r$) is symmetric and, moreover, on all odd levels has the same value and on all even levels with the possible exclusion of 0 has the same value. Schematically, this is shown in Fig. 3.

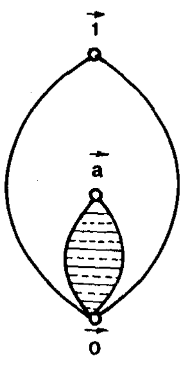


Fig. 3.

Theorem 2.3 is a sharpening of the Hodes–Specker theorem that gives a much slower growing bound (instead of the factor log log n a function slower than $\log^* n$).

Moreover, Theorem 2.3 gives asymptotically the best possible growth rate so the use of the Ramsey theorem is fitting the pattern (see [11]).

The idea of the proof of Theorem 2.3 [11] is simple but the details are more technical. Therefore, we give a sketch here only.

Sketch of Proof of Theorem 2.3. Suppose $L(f)=N$, f is n-dimensional. Then there exists a formula α equivalent to f such that the total number of occurrences of variables in α is N. It follows that at least $\frac{1}{2}n$ variables occur at most $2N/n$ times. Put $k=2N/n$ and let β be the formula α restricted to those $m\geq\frac{1}{2}n$ variables which occur at most k times. Now define subformulas $\beta\{x_i, x_j\}$ in a suitable way—$\beta\{x_i, x_j\}$ is the subformula of β induced by x_i and x_j. As the number of occurrences of x_i, x_j is k, the number l of all possible non-isomorphic subformulas of β induced by 2 variables is small and bounded from above by 2^{ck}, with c a constant. In this situation color a pair $\{x_i, x_j\}$ by the shape of the induced formula $\beta\{x_i, x_j\}$. Now if $m\geq l^{R\cdot l}\geq r(2, l, R)$, i.e., if

$$\tfrac{1}{2}n\geq 2^{2^{c2N/n}\cdot R},$$

then using the Ramsey theorem there exists a set X of R variables such that all the subformulas of β which are induced by pairs of variables from X are isomorphic (i.e., the set X is homogeneous). The most technical part of the proof consists of proving that this homogeneous set of variables gives a subformula of the desired type. $\square$

Theorem 2.3 has several corollaries. Particularly, the following holds.

Corollary 2.4. $L_\Omega(f)\geq\eta_\Omega n\log\log n$ *for every symmetric n-dimensional Boolean function with the exception of* 16 *functions.*

Let us briefly mention the last application of the Ramsey theorem which is most freely related, yet it is somehow typical. It is motivated by the following problems.

(P1) Consider a totally ordered set X with n elements, $p\ll n$. What are the natural orderings of $\binom{X}{p}$?

(P2) Consider the set $\{0, 1, \ldots, p-1\}^n$ of all words of length n with entries from the set $\{0, 1, \ldots, p-1\}$. What are the natural orderings of this set?

Below we shall define what we mean by a natural ordering and we shall characterize them in both problems.

Roughly speaking, what makes the lexicographic ordering natural is the fact that it is computed locally which means that it is invariant on sub-objects. Specifying the notions of a sub-object and the invariance we shall formulate the above problems in an exact way. This is more easy for subsets and let,me treat this case first.

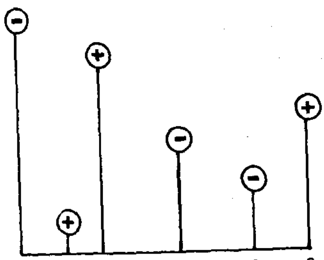
Let $(X, \leq)$ be a totally ordered set. An ordering $\preccurlyeq$ of $\binom{X}{p}$ is said to be *canonical* if for every pair A, B of subsets of X, $A=\{a_1, \ldots, a_k\}_<$, $B=\{b_1, \ldots, b_k\}_<$ the following holds:

$$\{a_{i_1}, \ldots, a_{i_p}\}_< \preccurlyeq \{a_{j_1}, \ldots, a_{j_p}\}_< \text{ iff } \{b_{i_1}, \ldots, b_{i_p}\}_< \preccurlyeq \{b_{j_1}, \ldots, b_{j_p}\}_<.$$

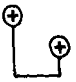
All canonical orderings of $\binom{X}{p}$ may be characterized as follows.

Theorem 2.5. *Let $(X, \leq)$ be a totally ordered set. All canonical orderings of $\binom{X}{p}$ are described in the following way: First we fix a permutation $\pi : \{1, \ldots, p\} \to \{1, \ldots, p\}$ and a mapping $s : \{1, \ldots, p\} \to \{+, -\}$. Then we define $\{x_1, \ldots, x_p\}_< \preccurlyeq \{y_1, \ldots, y_p\}_<$ iff there exists an i_0 such that $x_{\pi(i)} = y_{\pi(i)}$ for $i < i_0$, $x_{\pi(i_0)} \leq y_{\pi(i_0)}$ if $s(i_0) = +$, $x_{\pi(i_0)} \geq y_{\pi(i_0)}$ if $s(i_0) = -$.*

The canonical ordering $\preccurlyeq$ determined by the pair (π, s) may be visualized by means of the following diagram:



(which should be read from the top). For example, the lexicographic ordering of pairs corresponds to the following diagram:



The situation is not so easy for words over a (finite) alphabet, i.e., for set-valued cubes. Only the main ideas will be indicated here.

Let A be a finite set, n a positive integer. The set A^n will be called n-dimensional cube over A. We can identify the elements of A^n either with words of length n (over A) or with functions $f : \{0, 1, \ldots, n-1\} \to A$.

An m-dimensional subcube S of A^n is determined by an $f_0 \in A^n$ and nonempty disjoint sets $\omega_0, \omega_1, \ldots, \omega_{m-1}$ of $\{0, 1, \ldots, n-1\}$. S is then the set of all functions $f \in A^n$ which are constant on every set ω_i, $i = 0, \ldots, m-1$ and which coincide with f_0 outside of $\bigcup_{i=0}^{m-1} \omega_i$.

Clearly $|S| = |A^m|$. Moreover, assuming $\min \omega_0 < \min \omega_1 < \cdots < \min \omega_{m-1}$, the mapping $\Phi : A^m \to A^n$ defined by

$$\Phi(f)(j) = \begin{cases} f_0(j) & \text{for } j \notin \bigcup_{l=0}^{m-1} \omega_l, \\ f(i) & \text{for } j \in \omega_i \end{cases}$$

is an isomorphism of A^m and S. This mapping is called the standard isomorphism of S and A^m.

Similarly, we can define the standard isomorphism of m-dimensional subcubes S and S' of A^n.

Using these concepts we define canonical ordering of a cube as follows.

Definition 2.6. An ordering $\preccurlyeq$ of A^n is said to be *canonical* if for every $m \leqslant n$ and for each m-dimensional subcubes S and S' the following holds:

$$x \preccurlyeq y \text{ iff } \Phi(x) \preccurlyeq \Phi(y) \quad \text{for each } x, y \in S$$

(here $\Phi : S \to S'$ is the standard isomorphism).

The description of canonical orderings of A^n is slightly involved but it has an interesting structure:

Every canonical ordering of A^n is determined by three conditions:

(i) an ordering $\leqslant$ of A;

(ii) an interval tree T on $(A, \leqslant)$;

(iii) a quasi-order $\leqslant_q$ which extends T.

Here the undefined notions have the following meaning:

- An interval on $A = \{a_1, \ldots, a_k\}_<$ is a set of the form $\{a_i, a_{i+1}, \ldots, a_j\}$, $i \leqslant j$.
- An interval tree T on $(A, \leqslant)$ is a (directed) tree whose vertices are subintervals on $(A, \leqslant)$, the immediate successors of each vertex (i.e., an interval) I form a partition of I and all leaves are singletons.
- A quasi-order $\leqslant_q$ is a reflexive, transitive relation. We assume that T (considered as a relation) is a subset of $\leqslant_q$.

Given a triple $(\leqslant, T, \leqslant_q)$ we may define the order $\preccurlyeq = \preccurlyeq(\leqslant, T, \leqslant_q)$ as follows: Let $T(0), T(1), \ldots, T(r)$ be the enumeration of the equivalence classes given by $\leqslant_q$, $T(0) \leqslant_q T(1) \leqslant_q \cdots \leqslant_q T(r)$. Given two words $x, y \in A^n$ we put

$$x \preccurlyeq y \text{ iff there exists an } i_0 \in \{0, 1, \ldots, r\} \text{ such that}$$

$$x \restriction T(i)|_{\mathrm{ISucc}\, T(i)} = y \restriction T(i)|_{\mathrm{ISucc}\, T(i)} \quad \text{for all } i < i_0,$$

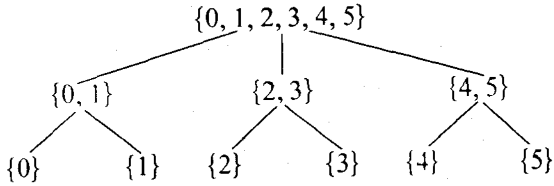
while

$$x \restriction T(i_0)|_{\mathrm{ISucc}\, T(i_0)} \leqslant' y \restriction T(i_0)|_{\mathrm{ISucc}\, T(i_0)}.$$

Here $x \restriction B$ is the subword of x formed by all entries from $B \subseteq A$; the factorized word $x|_{(B_1, \ldots, B_l)} = (x'_1, \ldots, x'_m)$ is defined by $x'_i = \min B_j$, where $x_i \in B_j$; $\leqslant'$ is the lexicographic ordering induced by A^m (for each $m \leqslant n$); and ISucc $T(i)$ is the set of all immediate successors of $T(i)$.

Instead of giving a less formalized definition let us consider an illustrative example.

Example 2.7. Given $A=\{0,1,2,3,4,5\}$; $0<1<2<3<4<5$, we have the following interval tree T:



with quasi-order $\leqslant_q$ on T as follows:

$$\{0,1,2,3,4,5\} \leqslant_q \{2,3\} \leqslant_q \{0,1\} \approx_q \{4,5\}.$$

In this case $T(0)=A$, $T(1)=\{2,3\}$, $T(2)=\{\{0,1\},\{4,5\}\}$.

Consider words $x=(0,1,0,2,3,5,5)$, $y=(1,0,0,3,3,4,5)$. Then

$$x \leqslant y \quad \text{as} \quad x|_{\text{ISucc } T(0)} = y|_{\text{ISucc } T(0)} = (0,0,0,2,2,4,4),$$

$$x' = x \restriction T(1) = (2,3), \quad y' = y \restriction T(1) = (3,3) \quad \text{and}$$

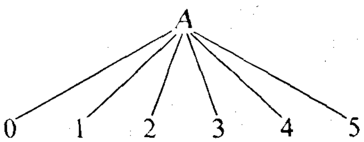
$$x'|_{\text{ISucc } T(1)} = x', \quad y'|_{\text{ISucc } T(1)} = y'.$$

Other examples are the following ones:

$$(0,1,3,5,5,5) < (0,0,4,0,0,0),$$

$$(0,1,2,5,5,5) > (0,0,3,0,0,0).$$

Fig. 4 depicts the covering diagram of the ordering $\leqslant$ for $n=2$. Compare also the interval tree which corresponds to the lexicographic ordering:



The above description of canonical orderings nicely fits in the general framework of Ramsey type statements. We have the following results (which in fact provided the original motivation for this research).

Theorem 2.8. *For every positive integer p, n there exists an N such that for every ordering $\leqslant$ of $\binom{N}{p}$ there exists a subset $X \subseteq N$, $|X| \geqslant n$ such that $\leqslant$ restricted to $\binom{X}{p}$ is a canonical ordering.*

Theorem 2.9. *For every positive integer p, n there exists an N such that for every*

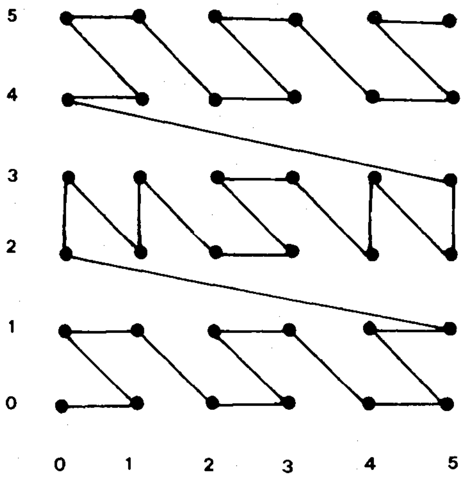


Fig. 4.

ordering $\preccurlyeq$ *of* p^N *there exists an n-dimensional subcube* $Q \subseteq p^N$ *such that* $\preccurlyeq$ *restricted to* Q *is a canonical ordering.*

These last results were obtained by Leeb and Prömmel [5] (for sets) and by Prömmel, Rödl, Voigt and the present author [8] (for cubes).

Acknowledgment

I would like to thank R. Graham and P. Pudlák, and also the guest-editor of this issue, J. Diaz, for their remarks which improved the quality of this paper.

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