# Ramsey Numbers and an Approximation Algorithm for the Vertex Cover Problem.

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# Ramsey Numbers and an Approximation Algorithm for the Vertex Cover Problem

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**Summary.** We show two results. First we derive an upper bound for the special Ramsey numbers $r_k(q)$, where $r_k(q)$ is the largest number of nodes a graph without odd cycles of length bounded by $2k+1$ and without an independent set of size $q+1$ can have. We prove $r_k(q) \leqq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k+2}{k+1} q$. The proof is constructive and yields an algorithm for computing an independent set of that size. Using this algorithm we secondly describe an $O(|V| \cdot |E|)$ time bounded approximation algorithm for the vertex cover problem, whose worst case ratio is $\Delta \leqq 2 - \frac{1}{k+1}$, for all graphs with at most $(2k+3)^k(2k+2)$ nodes (e.g. $\Delta \leqq 1.8$, if $|V| \leqq 146000$).

## 1. Introduction

Let $G=(V,E)$ be an undirected graph. A subset $I \subseteq V$ is called independent set of $G$ iff $E \cap \{\{u,v\} \mid u,v \in I, u \neq v\} = \emptyset$ and a subset $M \subseteq V$ is called vertex cover of $G$ iff $\forall \{u,v\} \in E$: $u \in M$ or $v \in M$. By $\alpha(G)$ and $\lambda(G)$ we denote the cardinalities of a maximum independent set and a minimum vertex cover of $G$.

In the first part of this paper we consider the following function of Ramsey type $r_k(q) := \max\{n \mid \exists \text{ graph } G: |V(G)| = n \text{ and } t_{odd}(G) > 2k+1 \text{ and } \alpha(G) \leqq q\}$, where $t_{odd}(G)$ is the length of a shortest odd cycle in $G$. This function gives a lower bound for the cardinality $\alpha(G)$ of a maximum independent set for graphs without odd cycles of length at most $2k+1$.

We will show that $r_k(q) \leqq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k+2}{k+1} q$ holds for $k \geqq 1$ and $q \geqq 0$. The proof of this result is constructive and it yields an algorithm, which always computes an independent set of size at least $n^{\frac{k}{k+1}}$ for all graphs $G$ with $t_{odd}(G) > 2k+1$ and with $n$ nodes, $n \geqq n_0$ for some sufficiently large $n_0$.

Erdös considered a function similar to $r_k(q)$, which differs from ours in the stronger requirement that the graphs under consideration are not allowed to contain any cycle of length $k$ or less. Let us denote his function by $r'_k(q)$. Then it is shown in [4] $q^{1+\frac{1}{2k}} \leqq r'_k(q) \leqq c \cdot q^{1+1/\left[\frac{k}{2}\right]}$, for some $c>0$. There is no estimation given for the constant $c$.

In order to achieve the upper bound for our function $r_k(q)$ and to obtain a "good" constant (for $q=0$ and 1 our solution of $r_k(q)$ is exact) we have to estimate a breadth first search approach very carefully. Note that our result yields the same upper bound for $r_k$ as the one for $r'_k$ obtained in [4]. This is remarkable since we prohibit only short cycles of odd length and allow short cycles of even length to occur. For a survey of other lower bounds for $\alpha(G)$ see the paper of Griggs [9].

In the second part of this paper we use the above result in order to obtain an $O(|V| \cdot |E|)$ time bounded approximation algorithm for the vertex cover problem. The underlying idea of this algorithm consists of destroying first short cycles of odd length and using afterwards the lower bound for $\alpha(G)$ for graphs without short odd cycles. The algorithm also uses a technique described by Nemhauser and Trotter in [15]. We will show that our algorithm always computes a vertex cover $M$ such that $\frac{|M|}{\lambda(G)} \leqq 2 - \frac{1}{k+1}$, for every graph $G$ with at most $(2k+3)^k(2k+2)$ nodes. That means that the worst case ratio $\Delta$ of our algorithm is bounded by

$$\begin{aligned} &\Delta \leqq 1.67, && \text{if } |V| \leqq 294 \\ &\Delta \leqq 1.75, && \text{if } |V| \leqq 5832 \\ &\Delta \leqq 1{,}8 && \text{if } |V| \leqq 146410 \\ &\Delta \leqq 1.9, && \text{if } |V| \leqq 1{,}588 \cdot 10^{13}. \end{aligned}$$

The proof in Sect. 3 will show that this estimation directly follows from our upper bound for the function $r_k$. We will discuss also the border lines of our approach.

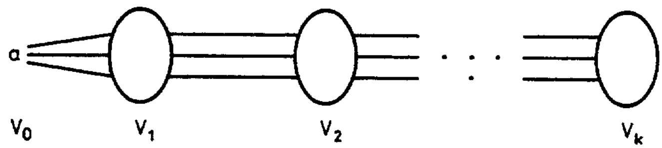
Independently Bar-Yehuda and Even [3] found essentially the same method and showed that it can be applied in the weighted case too. Our introduction of the function $r_k$ leads to an estimation achieving the same worst case ratio for nearly twice as large graphs than their estimation does.

Our algorithm gives a partial answer to the challenging open problem whether there is an approximation algorithm for the vertex cover problem of worst case ratio $\Delta < 2$. It is curious that up to now this problem has resisted all efforts to solve it, because any maximal matching defines a vertex cover, which is at most twice as large as an optimal one (see e.g. [5]). More results of the attempt to solve the above problem can be found in [2, 10, 14]. In [14] we discussed also the worst case ratios which are achievable for graphs of bounded degree.

## 2. An Estimation for the Functions $r_k$

**Theorem 1.** *$r_k(q) \leqq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k+2}{k+1} q$ holds for all $k \geqq 1$, $q \geqq 0$.*

*Proof.* First we will deduce a recurrence relation bounding the size of $r_k$. Let $G = (V, E)$ be a graph without any odd cycle of length not greater than $2k+1$ whose maximum independent set has cardinality $q$. Let $a \in V$ be an arbitrary node

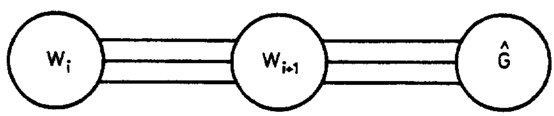



Set $V_i := \{u \in V; \operatorname{dist}(a,u) = i\}$, $0 \leqq i \leqq k$, where $\operatorname{dist}(u,v)$ is the length of the shortest path from $u$ to $v$. Set $W_i := \bigcup_{\substack{j \leqq i \\ j \equiv i \bmod 2}} V_j$. Since any odd cycle has length at least $2k+3$ all the sets $V_i$, $0 \leqq i \leqq k$, are independent sets. The sets $W_i$, $0 \leqq i \leqq k$, are also independent sets since for every $j$ there is no edge connecting nodes from $V_j$ and $V_p$ with $p \leqq j-2$ or $p \geqq j+2$.

Set $y_i := |W_i|$ for $i = 0, \ldots, k$.

Then there exists $i$, $0 \leqq i \leqq k-1$, such that $y_{i+1} \leqq q^{1/k} \cdot y_i$. Assume on the contrary that $y_{i+1} > q^{1/k} \cdot y_i$ holds for all $i = 0, \ldots, k-1$. Then $y_i > q^{i/k}$ for all $i = 0, \ldots, k$ and especially $y_k > q$. This is a contradiction since $W_k$ forms an independent set and we have assumed that the maximum independent set of $G$ has cardinality $q$.

Now let $i$ fulfill $y_{i+1} \leqq q^{1/k} \cdot y_i$. Consider the following partition of $G$ where $\hat{G}$ is the graph induced by the following construction,




$\hat{V} := V - (W_i \cup W_{i+1})$. $W_i$ is an independent set and there exist no edges connecting nodes from $W_i$ and $\hat{G}$, therefore every independent set of $\hat{G}$ has size at most $q - y_i$. Furthermore $\hat{G}$ is a subgraph of $G$ and has no odd cycles of length not greater than $2k+1$. This implies $|\hat{V}| \leqq r_k(q - y_i)$.

$$\Rightarrow \; |V| = y_i + y_{i+1} + |\hat{V}|$$

$$\leqq y_i + \min\{q^{1/k} \cdot y_i, q\} + r_k(q - y_i).$$

$$\Rightarrow \; r_k(q) \leqq \max_{1 \leqq y \leqq q} \{y + \min\{q^{1/k} \cdot y, q\} + r_k(q - y)\}.$$

We will show the theorem for every fixed $k$ by induction on $q$. $r_k(0) = 0$ and therefore the theorem holds for $q = 0$. In order to do the induction step, we set

$$f(q,y)=y+\min\{q^{1/k}\cdot y,q\}+\frac{k}{k+1}(q-y)^{\frac{k+1}{k}}+\frac{k+2}{k+1}(q-y)$$
$$=\frac{k+2}{k+1}q-\frac{1}{k+1}\cdot y+\min\{q^{1/k}\cdot y,q\}+\frac{k}{k+1}(q-y)^{\frac{k+1}{k}}$$

and we have to determine the maximum of this function for $1\leqq y\leqq q$. For $q^{\frac{k-1}{k}}\leqq y\leqq q$ the function is decreasing monotonically and therefore we have to consider only the range $1\leqq y\leqq q^{\frac{k-1}{k}}$. In this range $f$ is differentiable and

$$\frac{\partial f}{\partial y}=-(q-y)^{1/k}+q^{1/k}-\frac{1}{k+1},\qquad \frac{\partial^2 f}{(\partial y)^2}=\frac{1}{k}(q-y)^{\frac{1-k}{k}}\geqq 0.$$

Therefore any extremum in the range $1\leqq y\leqq q^{\frac{k-1}{k}}$ is a minimum and the maximum is taken at $y=1$ or at $y=q^{\frac{k-1}{k}}$.

In order to prove our theorem we have to show

$$f(q,1)=\frac{k}{k+1}(q-1)^{\frac{k+1}{k}}+\frac{k+2}{k+1}q+q^{1/k}-\frac{1}{k+1}$$
$$\leqq\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q \tag{1}$$

and

$$f(q,q^{\frac{k-1}{k}})=\frac{k}{k+1}(q-q^{\frac{k-1}{k}})^{\frac{k+1}{k}}+\left(1+\frac{k+2}{k+1}\right)q-\frac{1}{k+1}q^{\frac{k-1}{k}}$$
$$\leqq\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q. \tag{2}$$

First we prove (1). We set

$$g(q)=\frac{k}{k+1}(q-1)^{\frac{k+1}{k}}-\frac{k}{k+1}q^{\frac{k+1}{k}}+q^{1/k}-\frac{1}{k+1}$$

and we have to show that $g(q)\leqq 0$ holds for all $q\geqq 1$. $g(1)=0$ and it is easy to see that $g(q)<0$ holds if $q$ is large enough. Now we have to consider extremal points.

$$g'(q)=(q-1)^{1/k}-q^{1/k}+\frac{1}{k}q^{1/k-1}$$

i.e. $g'(\hat{q})=0\Rightarrow(\hat{q}-1)^{1/k}=\hat{q}^{1/k}-\frac{1}{k}\hat{q}^{1/k-1}$

$$\Rightarrow g(\hat{q})=\frac{k}{k+1}(\hat{q}-1)\left(\hat{q}^{1/k}-\frac{1}{k}\hat{q}^{1/k-1}\right)-\frac{k}{k+1}\hat{q}^{\frac{k+1}{k}}+\hat{q}^{1/k}-\frac{1}{k+1}$$
$$=\frac{1}{k+1}(\hat{q}^{1/k-1}-1)<0,\qquad \text{since } \hat{q}>1.$$

Now we have to prove (2).

$$\frac{k}{k+1}\left(q-q^{\frac{k-1}{k}}\right)^{\frac{k+1}{k}}+q-\frac{1}{k+1}q^{\frac{k-1}{k}}\leqq\frac{k}{k+1}q^{\frac{k+1}{k}}$$

$$\Leftrightarrow\ k(1-q^{-1/k})^{\frac{k+1}{k}}+(k+1)q^{-1/k}-q^{-2/k}\leqq k.$$

This time we define the function

$$h(x)=k\cdot(1-x)^{\frac{k+1}{k}}+(k+1)\cdot x-x^2-k$$

and we have to show that $h(x)\leqq 0$ holds for $0\leqq x\leqq 1$. $h(0)=h(1)=0$ hold and we have to consider still the extremal points.

$$h'(x)=-(k+1)(1-x)^{1/k}+(k+1)-2x$$

i.e. $h'(\hat{x})=0\Rightarrow(1-\hat{x})^{1/k}=1-\frac{2}{k+1}\hat{x}$

$$\Rightarrow\ h(\hat{x})=k(1-\hat{x})\left(1-\frac{2}{k+1}\hat{x}\right)+(k+1)\hat{x}-\hat{x}^2-k$$

$$=-\hat{x}\cdot\left(\frac{2k}{k+1}-1\right)+\hat{x}^2\cdot\left(\frac{2k}{k+1}-1\right)$$

$$=\hat{x}\cdot\left(\frac{2k}{k+1}-1\right)(\hat{x}-1)<0.\quad\square$$

In the proof of Theorem 1 we even have shown the following stronger result. If an independent set of size $q$ is determined for a graph $G$ without odd cycles of length $2k+1$ or less by the breadth first search approach described in the first part of the proof, then $G$ has at most $\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q$ nodes, our estimation of $r_k(q)$.

The algorithm *AI* based upon this approach for determining an independent set can be formulated as follows.

**Input**: Graph $G=(V,E)$ with $t_{\text{odd}}(G)>2k+1$;

1. determine the smallest number $q\in\mathbb{N}$ satisfying

$$|V|\leqq\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q;$$

2. $I:=\emptyset$;
3. delete all isolated nodes from $G$ and add them to $I$;
4. **while** $G\neq\emptyset$ **do**
   **begin**
4.1 choose $a\in V(G)$ and determine the sets $W_0, W_1, \ldots, W_i, W_{i+1}$ until $|W_{i+1}|\leqq q^{1/k}\cdot|W_i|$ or $i=k-1$;
4.2 $I:=I\cup W_i$; $G:=G-(W_i\cup W_{i+1})$;
4.3 delete all isolated nodes from $G$ and add them to $I$;
   **end**;

**Output**: Independent set $I$;

By breadth first search statement 4.1 can be performed within time $O(|E|)$ and so the whole while-loop is time bounded by $O(|V|\cdot|E|)$.

**Lemma 1.** *Algorithm AI, which works for every graph $G=(V,E)$ with $t_{odd}(G)>2k+1$ within time $O(|V|\cdot|E|)$ computes an independent set $I\subseteq V$ of size at least $q$, where $q$ is the smallest number satisfying $|V|\leqq\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q$.*

## 3. Approximating the Vertex Cover Problem

Our algorithm for approximating the vertex cover problem is based especially upon the algorithm $AI$ from Corollary 1 and a result of Nemhauser and Trotter shown in [15].

If $C$ is an odd cycle in $G$ with $n=2l+1$ nodes then every vertex cover of $G$ contains at least $l+1$ nodes from $C$, i.e. taking all nodes from $C$ into a vertex cover of $G$ leads to a local ratio of at most $\frac{2l+1}{l+1}=2-\frac{1}{l+1}$.

Nemhauser and Trotter describe an algorithm for computing an independent set for node weighted graphs. We will shortly reformulate their algorithm into an algorithm for computing a vertex cover for unweighted graphs.

**Input**: Graph $G=(V,E)$;

1. compute the bipartite graph $H$ from $G$ by setting

$$V(H):=V\cup\hat{V},\quad \text{where } \hat{V}:=\{\hat{v}\mid v\in V\} \quad \text{and} \quad E(H):=\{\{u,\hat{v}\}\mid\{u,v\}\in E\};$$

2. determine a minimum vertex cover $M\subseteq V(H)$ of $H$;
3. set $V_1:=\{v\in V\mid v\notin M\wedge\hat{v}\notin M\}$,
   $V_2:=\{v\in V\mid v\in M\wedge\hat{v}\in M\}$, and
   $V_3:=\{v\in V\mid \text{either } v\in M \text{ or } \hat{v}\notin M\}$;

**Output**: $V_1$, $V_2$, $V_3$;

**Theorem 2.** [15]. *The Nemhauser-Trotter algorithm computes within time $O(\sqrt{|V|}\cdot|E|)$ a partition of $V$ into $V_1, V_2, V_3$ such that*

a) *$V_2$ is a minimum vertex cover for the graph induced by $V_1\cup V_2$,*
b) *there are no edges $\{u,v\}$ with $u\in V_1$ and $v\in V_3$, and*
c) *$\lambda(G')\geqq\frac{1}{2}|V_3|$, where $G'$ is the graph induced by $V_3$.*

Computing a minimum vertex cover for a bipartite graph mainly relies on computing a maximum matching, see [12]. This can be done within time $O(\sqrt{|V|}\cdot|E|)$, see e.g. [6].

We can now formulate the following algorithm $VC$ for computing a vertex cover.

**Input:** Graph $G=(V,E)$;

1. determine the smallest number $k\in\mathbb{N}$ such that $|V|\leqq(2k+3)^k\cdot(2k+2)$; $M:=\emptyset$;
2. **while** there is an odd cycle $C$ of length $\leqq 2k+1$ in $G$ **do**
   **begin**
   $M:=M\cup V(C)$; $G:=G-V(C)$;
   delete all isolated vertices from $G$
   **end;**
3. apply the Nemhauser-Trotter-algorithm to $G$ yielding a partition $V_1, V_2, V_3$ of $V(G)$ as described in Theorem 2;
   $M:=M\cup V_2$; $\quad G':=G-(V_1\cup V_2)$;
4. apply the algorithm $AI$ to $G'$, which computes an independent set $I\subseteq V(G')$ for $G'$ as described in Lemma 1;
   $M':=V(G')-I$; $\quad M:=M\cup M'$;

**Output:** Vertex Cover $M$;

**Theorem 3.** *The algorithm VC works for every graph $G=(V,E)$ within time $O(|V|\cdot|E|)$ and computes a vertex cover $M\subseteq V$ such that $\frac{|M|}{\lambda(G)}\leqq 2-\frac{1}{k+1}$, where $k$ is the smallest integer satisfying $|V|\leqq(2k+3)^k\cdot(2k+2)$.*

*Proof.* We first consider the running time of the algorithm $VC$. The execution times of the statements 3. and 4. are bounded by $O(\sqrt{|V|}\cdot|E|)$ and $O(|V|\cdot|E|)$ by Theorem 2 and Lemma 1. The deletion of all odd cycles of length at most $2k+1$ can be performed within time $O(|V|\cdot|E|)$. For a breadth first search starting from an arbitrarily chosen node $a$ finds in time $O(|E|)$ a cycle $C$ of odd length at most $2k+1$ or it finds that there is no odd cycle $C$ through $a$ of that length (see [13]). This search has to be performed at most $|V|$ times.

It should be clear that the set $M$ computed by the algorithm $VC$ is a vertex cover of $G$. In order to prove the worst case ratio of the algorithm $VC$ we use the following simple fact: If $U_1, U_2$ is a partition of the vertex set of a graph $G=(V,E)$ and $M_i\subseteq U_i$ is a vertex cover of the graph $G_i$ induced by $U_i$ with $\frac{|M_i|}{\lambda(G_i)}\leqq\beta$ for $i=1,2$ such that $M_1$ contains all nodes from $U_1$, which are connected by an edge with some node from $U_2$, then $M_1\cup M_2$ is a vertex cover of $G$ and $\frac{|M_1\cup M_2|}{\lambda(G)}\leqq\beta$.

During every while loop of statement 2 an odd cycle $C$ of length at most $2k+1$ is deleted from $G$ and all nodes from $C$ are put into the vertex cover $M$. I.e. we have chosen $M_c:=V(G_c)$ as a vertex cover for the graph $G_c$ induced by the vertex set of $C$ and we know that $\lambda(G_c)\geqq\frac{|V(G_c)|+1}{2}$. Thus

$$\frac{|M_C|}{\lambda(G_c)}=\frac{|V(G_c)|}{\frac{|V(G_c)|+1}{2}}=2-\frac{2}{|V(G_c)|+1}\leqq 2-\frac{2}{2k+1+1}=2-\frac{1}{k+1}.$$

By Theorem 2 we know that $V_2$ is a minimum vertex cover for the graph $G_1$ induced by $V_1 \cup V_2$. Then $\frac{|V_2|}{\lambda(G_1)}=1$. Furthermore there are no edges between vertices of $V_1$ and $V_3$. It remains to show that the vertex cover $M'$ computed by statement 4 of $VC$ for the graph $G'$, induced by the vertex set $V_3$, satisfies $\frac{|M'|}{\lambda(G')} \leqq 2-\frac{1}{k+1}$. By the consideration above we have proved our theorem.

Set $V':=V_3$. Then $M'=V'-I$, where $I$ is the independent set of $G'$ computed by the algorithm $AI$ and $|I| \geqq q$, where $q$ is the smallest integer such that

$$|V'| \leqq \frac{k}{k+1} q^{\frac{k+1}{k}} + \frac{k+2}{k+1} q =: \varphi(q)$$

by Lemma 1, i.e. $|M'| \leqq |V'|-q$. We have assumed $|V'| \leqq |V| \leqq (2k+3)^k(2k+2)$ and $\varphi(\hat{q})=(2k+3)^k(2k+2)$ holds for $\hat{q}=(2k+3)^k$. Therefore $q \leqq (2k+3)^k$ holds and this implies:

$$\begin{aligned}
\frac{|M'|}{\lambda(G')} &\leqq \frac{|V'|-q}{\frac{1}{2}|V'|} = 2-\frac{2q}{|V'|} \\
&\leqq 2-\frac{2q}{\frac{k}{k+1}q^{\frac{k+1}{k}}+\frac{k+2}{k+1}q} = 2-\frac{2(k+1)}{k \cdot q^{1/k}+k+2} \\
&\leqq 2-\frac{2(k+1)}{k(2k+3)+k+2} = 2-\frac{1}{k+1}. \quad \square
\end{aligned}$$

We want to discuss the border lines of this algorithmic method. The proof of the last theorem shows that we can guarantee a worst case ratio of $\Delta \leqq 2 - \frac{1}{k+1}$ for all graphs $G=(V,E)$ with $|V| \leqq n(k)$, where $n(k)$ fulfills the condition that every graph without odd cycles of length $2k+1$ or less has an independent set of cardinality at least $\frac{|V|}{2k+2}$. Our estimation $n(k)=(2k+3)^k(2k+2)$ is based on the bound for $r_k$ in Theorem 1. It is very unlikely that our bounds are optimal. We can compare our results with the optimal ones only for the case $k=1$. Our function $r_1(q)$ is equal to the well studied Ramsey-function $R(3,q)$ and it is known that $R(3,q) \leqq c \cdot \frac{q^2}{\log q}$ holds, [1]. Moreover our estimation of $n(1)$ yields that every triangle-free graph with at most 20 nodes always has an independent set containing one quarter of its nodes, whereas we know by results shown in [8], that 32 is the correct number with this property. For the case $k=2$ we got in [14] a result which is a little bit better than the one presented in this paper.

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## References

1. Ajtai, M., Komlós, J., Szemeredi, E.: A note on Ramsey numbers. J. Comb. Theory, Ser. A **29**, 354–360 (1980)
2. Bar-Yehuda, R., Even, S.: On approximating a vertex cover for planar graphs. Proc. 14th Ann. ACM Symp. Th. Computing, pp. 303–309. San Francisco 1982
3. Bar-Yehuda, R., Even, S.: A $2-\frac{\log\log n}{2\log n}$ performance ratio for the weighted vertex cover problem. Technion Haifa, Technical Report #260, January 1983
4. Erdös, P.: Graph theory and probability, Can. J. Math. **11**, 34–38 (1959)
5. Erdös, P., Gallai, T.: On the minimal number of vertices representing the edges of a graph, pp. 181–203. Magyar Tudományos Akadémia Matematikai Kutato Intézetének Közleményei VI. Evfolyam 1961
6. Even, S.: Graph algorithms. Rockville, MD: Computer Science Press 1979
7. Fajtlowicz, S.: On the size of independent sets in graphs. In: Proc. 9th S.E. Conf. Combinatorics, Graph Theory and Computing, Boca Raton, Congressum Numerantium No. XXI, pp. 269–274. Utilitas Mathematical-Publ. Inc. 1978
8. Graver, J.E., Yackel, J.: Some graph theoretic results with Ramsey's theorem. J. Comb. Theory, Ser. A **4**, 125–175 (1968)
9. Griggs, J.R.: Lower bounds on the independence number in terms of the degrees. J. Comb. Theory, Ser. B **34**, 22–39 (1983)
10. Hochbaum, D.S.: Efficient bounds for the stable set, vertex cover and set packing problems. Discrete Appl. Math. **6**, 243–254 (1983)
11. Hopkins, G., Staton, W.: Girth and independence ratio. Can. Math. Bull. **25**, 179–186 (1982)
12. König, D.: Theorie der endlichen und unendlichen Graphen. Leipzig: Akademische Verlagsgesellschaft 1936
13. Monien, B.: The complexity of determining a shortest cycle of even length. Computing **31**, 355–369 (1983)
14. Monien, B., Speckenmeyer, E.: Some further approximation algorithms for the vertex cover problem. In: Proc. 8th Coll. on Trees in Algebra and Programming (CAAP 83), L'Aquila. Lecture Notes in Computer Science **159**, 341–349 (1983)
15. Nemhauser, G.L., Trotter, L.E.: Vertex packing: structural properties and algorithms. Math. Program. **8**, 232–248 (1975)

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