Suppose the time to failure of each of the components is exponentially distributed with mean $1/\lambda$ and suppose the system operates for exactly T hours each day. Now, a component whose lifetime follows the exponential distribution shows no aging, i.e., the probability that the component survives day $n+1$ given that it survived day n is the same as the probability that a new component will survive day one. Hence the stochastic analysis of this (active redundant) system is equivalent to the analysis in Section 2 providing we identify

(i) the number of rings r with the number of components, and

(ii) the (constant) probability of failure q on a single toss of a ring with the (constant) probability $e^{-\lambda T}$ that a component will survive day one ($= T$ hours)

The play in the ring tossing game can then be identified with the comparison between competing systems built from two different types of components.

ON APPLICATIONS OF VAN DER WAERDEN'S THEOREM

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1. Equivalent versions of the theorem. In [1] B. L. van der Waerden relates how Artin, Schreier, and he were able to find the proof of the following conjecture of Baudet:

(A) *If the set of positive integers is partitioned in any way into two classes, then for any positive integer l at least one class contains a set of l consecutive members of an arithmetic progression. (Henceforth we shall use the phrase "arithmetic progression of length l" to mean a set of l consecutive members of an arithmetic progression.)*

Aside from the ingenuity of the proof which finally arose, one of the most intriguing aspects of the paper is the manner in which these men were able to manipulate Baudet's conjecture into more manageable, yet equivalent forms. The first such manipulative step was to consider the following statement suggested by Schreier:

(B) *For any positive integer l there exists a positive integer $N(l)$ such that if the set $\{1,2,\cdots,N(l)\}$ is partitioned into two classes, then at least one class contains an arithmetic progression of length l.*

This so-called "finite version" of (A) clearly implies (A) and the converse implication is shown in [1] using a Cantor diagonal approach.

From here it was an easy step for Artin to show that (B) is equivalent to:

(C) *For any positive integers k and l there exists a positive integer $N(k,l)$ such that if the set $\{1,2,\cdots,N(k,l)\}$ is partitioned into k classes, some class contains an arithmetic progression of length l.*

This is the statement which van der Waerden proved. (See [1] or [2].)

Since the appearance of the proof several applications of the theorem have been published. (We shall discuss A. Brauer's application to power residues in another section of this paper.) However, it has become apparent that the real potential for application of (C) may lie in the size of the numbers $N(k,l)$. In his proof of (C) van der Waerden constructs numbers $N(k,l)$ which suffice, but it is thought that these constructed numbers are terrifically loose. For example, van der Waerden constructs $N(2,3) = 67$ whereas any $N(2,3) \geqq 9$ will work. And just a glance at his general construction of these numbers suggests that their growth rate is much greater than it need be. As Erdös points out in [3], tightening of the numbers $N(k,l)$ may lead to settling the question of the existence of arbitrarily long strings of prime numbers which are consecutive members of some arithmetic progression.

But the refinement of the numbers $N(k,l)$ will not be achieved without an essentially new proof of (C). Because none have yet been found, one is led back to the reasoning of Artin, Schreier, and van der Waerden that perhaps another version of the statement would be more manageable. Several equivalent forms of (C) have appeared since the proof was published. In this section we present some other equivalent versions of (C). These versions seem more explicitly related to the problem of primes in arithmetic progression than does the statement of (C). We begin with:

(D) *Let* $S = \{a_i\}_{i=1}^{\infty}$ *be any strictly increasing sequence of positive integers. If there exists a positive integer* M *such that* $a_{i+1} - a_i \leqq M$, *for* $i = 1,2,\cdots$, *then there exist among the members of* S *arithmetic progressions of arbitrary length.*

This is quickly seen to be a consequence of (C), for consider the following partition of the set of positive integers into M classes:

$$\begin{aligned}
K_0 &= \{a_i\colon a_i \in S\} = S \\
K_1 &= \{a_i + 1\colon a_i \in S\} \cap K_0{}' \\
K_2 &= \{a_i + 2\colon a_i \in S\} \cap (K_0 \cup K_1)' \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
K_j &= \{a_i + j\colon a_i \in S\} \cap (\textstyle\bigcup_{h=1}^{j-1} K_h)' \\
&\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
K_{M-1} &= \{a_i + (M-1)\colon a_i \in S\} \cap (\textstyle\bigcup_{h=1}^{M-1} K_h)'.
\end{aligned}$$

It is clear from the nature of S and the construction of the classes, K_j, that this is indeed a partition of the set of positive integers into M classes. Now by (C) there must be an arithmetic progression of length l (arbitrary) in some class, say K_n. Suppose this arithmetic progression is $\{b, b+d, b+2d, \cdots, b+(l-1)d\}$. Since $b \in K_n$, we have $b = a + n$ for some $a \in S$. Similarly each member of this progression is n greater than some member of S. Thus, $\{a, a+d, a+2d, \cdots, a+(l-1)d\} \subset S$, and the result is obtained.

Now (D) immediately yields:

(E) *If the set of positive integers is partitioned into two classes, then at least one of the following holds*:

(1) *One class contains arbitrarily long strings of consecutive integers.*

(2) *Both classes contain arithmetic progressions of arbitrary length.*

This is clear since if (1) does not occur, then (D) applies in both classes. And since (E) readily implies the Baudet conjecture (A), we see that (D) and (E) are each equivalent to van der Waerden's theorem (C).

Now (D) and (E) both yield "finite versions":

(D′) *For any M and l there is a positive integer $N_d(M,l)$ such that any strictly increasing finite sequence $\{a_i\}_{i=1}^m$ of positive integers with differences bounded by M (i.e., $a_{i+1}-a_i \leqq M$) and with $a_m - a_1 \geqq N_d(M,l)$ will contain an arithmetic progression of length l.*

(E′) *For any M and l there is a positive integer $N_e(M,l)$ such that whenever the set $\{1,2,\cdots,N_e(M,l)\}$ is partitioned into two classes at least one of the following holds:*

(1) *One class contains M consecutive numbers.*

(2) *Both classes contain arithmetic progressions of length l.*

From either of the statements (D′) or (E′) one sees the connection between van der Waerden's theorem and the problem of primes forming consecutive members of an arithmetic progression. If one can sharpen the number $N_d(M,l)$ enough and observe a relationship between this number and the rate of growth of gaps between consecutive prime numbers, one may be able to settle the question.

We have not been able to generally sharpen an estimate for $N_d(M,l)$, but we have found best possible values of $N_d(M,l)$ for some values of M and l. These are presented in section 3 of this paper.

2. An application of Witt's generalization of van der Waerden's theorem. Very shortly after van der Waerden published his proof in [**2**], A. Brauer [**4**] showed, among other things, that for any positive integers k and l there is a number $Z(k,l)$ such that for any prime $p > Z$ with $p \equiv 1 \pmod{k}$ the reduced residue system $\{1,2,\cdots,p-1\}$ modulo p contains l consecutive numbers, each of which is a kth power residue modulo p. Several authors since have found uniform upper bounds on this string of consecutive kth power residues for fixed k and l. (See for example, [**5**]–[**10**].)

J. H. Jordan [**11**], without the assurance of a general theorem like Brauer's, stepped into the domain of Gaussian integers $\mathbb{Z}[i]$ and proceeded to find several uniform upper bounds for what he called "consecutive" Gaussian integers which are all kth power residues of a prime of sufficiently large norm. The question arises, then, as to whether there is a theorem like Brauer's for the Gaussian domain. In this section we show that there is such a theorem.

It is Ernst Witt's generalization [**12**] of the van der Waerden theorem which allows one to use an approach analogous to Brauer's in not only the Gaussian integers, but in some other domains as well. Witt's theorem may be stated as follows:

Let $\Gamma = \{\gamma_1,\gamma_2,\cdots,\gamma_l\}$ be a fixed set of Gaussian integers. For any positive integer k there is a positive integer $N(k,l)$ such that if the set

$$\Delta = \left\{\sum_{j=1}^{l} a_j\gamma_j : a_j \in Z,\ a_j \geqq 0,\ \sum_{j=1}^{l} a_j = N(k,l)\right\}$$

is partitioned into k classes, some class will contain a homothetic image, Γ', of Γ.

(Here we shall say that Γ' is homothetic to Γ if $\Gamma' = \lambda\Gamma + \alpha = \{\lambda\gamma + \alpha : \gamma \in \Gamma\}$ where λ is a positive integer and α is an arbitrary Gaussian integer.) Now if in the above statement $\gamma_J \in \Gamma$ is such that $|\gamma_J| \geqq |\gamma_i|$, $1 \leqq i \leqq l$, then we see

$$\left| \sum_{j=1}^{l} a_j\gamma_j \right| \leqq \sum_{j=1}^{l} (|a_j|\,|\gamma_j|) \leqq |\gamma_J| \sum_{j=1}^{l} |a_j| = |\gamma_J|\,N(k,l)$$

since $a_j \geqq 0$. Thus, we may also say that if the set of all Gaussian integers with norm not greater than $(|\gamma_J|\,N(k,l))^2$ is partitioned into k classes, then some class will contain a homothetic image of Γ. Let $N(k,\Gamma) = (\max\{|\gamma_j| : \gamma_j \in \Gamma\})N(k,l)$. Now we can prove:

THEOREM 1. *Given any finite set of Gaussian integers, say $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_l\}$, and any sufficiently large Gaussian prime, π, there is a set $P = \{\rho_1, \rho_2, \cdots, \rho_l\}$ of quadratic residues modulo π such that P is a translation of Γ.*

Proof. Let $D(\Gamma) = \max\{|\gamma_j - \gamma_i| : \gamma_i, \gamma_j \in \Gamma\}$ be called the diameter of Γ. By choosing our prime π with large enough norm we can imbed any set of Gaussian integers having finite diameter in a reduced residue system modulo π. (See Hardman and Jordan [13].) Hence, we see that for π with sufficiently large norm there is either a translated image of Γ close to the origin consisting entirely of quadratic residues or there is a quadratic nonresidue, ν, modulo π such that $1 \leqq |\nu| \leqq D(\Gamma) + 1$. Let R represent any finite array of Gaussian integers which is large enough to contain some translation of Γ and $\nu\Gamma = \{\nu\gamma_j : \gamma_j \in \Gamma\}$.

Now consider π to be a Gaussian prime such that $|\pi| > 2N(2,R) + 1$, thus assuring that a reduced residue system (of the Hardman-Jordan type) modulo π, when broken into two classes will contain a homothetic image of R in one of the classes. In particular, either the class of quadratic residues or the class of quadratic nonresidues will contain such an image, say $R' = \lambda R + \alpha$ for some nonnegative integer λ and α a Gaussian integer. Now if λ and the elements of R' have the same quadratic nature modulo π, then multiplication by λ^{-1} modulo π yields $R'' = R + \lambda^{-1}\alpha$ consisting entirely of quadratic residues modulo π.

If λ and the elements of R' are of differing quadratic nature modulo π, then $R'' = R + \lambda^{-1}\alpha$ consists of quadratic nonresidues. But now $R + \lambda^{-1}\alpha$ contains as a subset some translation of $\nu\Gamma$, say $\nu\Gamma + \alpha'$. Since ν is a quadratic nonresidue, $\Gamma + \nu^{-1}\alpha'$ is made up of quadratic residues modulo π, and we are done.

Following Brauer's argument in similar fashion, one can establish

THEOREM 2. *Given any finite set Γ of Gaussian integers and any positive integer k, there exists a translation of Γ consisting entirely of kth power residues modulo any sufficiently large Gaussian prime π with $N(\pi) \equiv 1 \pmod{k}$.*

In the interest of moving toward applications of Witt's Theorem similar to those mentioned in the first section of this paper, we note that this theorem has the following equivalent versions analogous to (D) and (E) of the preceding section:

(A) *If $\Sigma = \{\sigma_i\}_{i=1}^{\infty}$ is any sequence of Gaussian integers for which there exists a finite set $H = \{\eta_1, \eta_2, \cdots, \eta_M\}$ of Gaussian integers such that*

$$\mathbf{Z}[i] \subset \Sigma \cup (\Sigma + \eta_1) \cup (\Sigma + \eta_2) \cup \cdots \cup (\Sigma + \eta_M)$$

then for any $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_l\} \subset \mathbf{Z}[i]$, Σ *contains a homothetic image of* Γ.

This statement is proved in the same way as statement (D) of the preceding section was proved. And from (A) we get:

(B) *For any partition of the set of Gaussian integers into two classes one of the following must occur*:

(1) *One class contains a translation of any finite set of Gaussian integers.*

(2) *Both classes contain a homothetic image of any finite set of Gaussian integers.*

To establish (B) one needs only to observe that in such a partition if we view each class as a sequence of Gaussian integers, either the condition of (A) will apply to both sequences (and, hence, (2) holds) or that condition will fail to hold for one of the sequences. In the latter case the class K where the condition fails will have arbitrarily large "holes" in it in the sense that for any positive integer M we could find $\xi \in \mathbf{Z}[i]$ such that

$$\{\zeta \in \mathbf{Z}[i]: |\zeta - \xi| \leqq M\} \cap K = \varnothing.$$

(For, if not, we could use $H = \{\zeta \in \mathbf{Z}[i]: |\zeta| \leqq M\}$ in the statement of (A).)

So (B), like (E) of the preceding section, brings forth the question of "patterns" of prime Gaussian integers. Of course, just as there are arbitrarily large gaps in the set of rational primes, so there are arbitrarily large holes in the set of Gaussian primes. But still one is led to study a "finite version" of (B) as weighed against the growth rate of holes in the set of Gaussian primes. Here, as in the rational case, such studies have so far yielded little fruit because of the unwieldy size of the constants involved in finite versions of Witt's Theorem. This is not unexpected since Witt's proof is essentially the same as that of van der Waerden in the rational case.

3. Some numerical results. Let $\{a_i\}_{i=1}^m$ be a strictly increasing sequence of positive integers such that for some fixed positive integer M we have $a_{i+1} - a_i \leqq M$ for $i = 1, 2, \cdots, m-1$. From statement (D′) of the first section of this paper we know of the existence of a number $N(M,l)$ such that if $a_m - a_1 \geqq N(M,l)$, then among the members of the given sequence there is an arithmetic progression of length l. We direct our attention to the number $N(M,l)$. Clearly, once such a number is found, any larger number would serve the same purpose. Let $N^*(M,l) = \min\{N(M,l)\}$. Under this definition, displaying a value of $N^*(M,l)$ for some M and l implies the existence of a sequence $\{a_i\}_{i=1}^m$ with differences between successive members bounded by M such that $a_m - a_1 = N^*(M,l) - 1$, and such that the sequence contains no arithmetic progression of length l. In presenting our numerical results we shall also present such sequences which show our constants to be correct. Actually we shall give the sequence of differences associated with the original sequence; that is, if $\{a_i\}_{i=1}^m$ is a sequence to be displayed, we shall instead display the sequence $\{d_i\}_{i=1}^{m-1}$ where $d_i = a_{i+1} - a_i$ for $i = 1, 2, \cdots, m-1$. We shall also impose on our sequences the condition that $d_i + d_{i+1} > M$ for $i = 1, 2, \cdots, m-2$. One easily sees that this condition in no way alters the generality in computations of $N^*(M,l)$.

To give an easy example of how the computations were made, let us consider the calculation of $N^*(2,3)$. That is, we consider all sequences of differences d_i with each $d_i = 1$ or 2 and with $d_i + d_{i+1} > 2$. Suppose $d_1 = 1$. Then $d_2 = 2$ and $d_3 = 1$ or 2. If, however, $d_3 = 2 = d_2$, then two consecutive differences are alike which, of course, means three members of the original sequence are in arithmetic progression. So we consider the case when $d_3 = 1$. This means $d_4 = 2$, and again since $d_4 + d_3 = d_2 + d_1$, we have an arithmetic progression of length $l = 3$ in the original sequence. This exhausts all cases with $d_1 = 1$. Similar argument shows that with $d_1 = 2$ one gets the sequence of differences $\{2,1,2\}$ before exhausting all possibilities. A corresponding original sequence might be $\{1,3,4,6\}$. This is, in one sense, the longest such sequence with no three terms in arithmetic progression. Since here, in the notation of the preceding paragraph, $a_m - a_1 = 5$, we get $N^*(2,3) = 6$.

The following table displays some values of $N^*(M,l)$ which we have found using essentially the above technique and the CDC 3800 computer at the Research Computation Center, Naval Research Laboratory, Washington, D.C.

$N^*(M,l)$	Sequences of differences $\{d_i\}_{i=1}^{m-1}$ of maximal length
$N^*(2,3) = 6$	$\{2,1,2\}$
$N^*(3,3) = 18$	$\{3,2,3,1,3,2,3\}$
$N^*(4,3) = 27$	$\{1,4,3,4,2,4,3,4,1\}$
$N^*(5,3) = 64$	$\{5,4,5,3,5,4,5,1,5,4,5,3,5,4,5\}$
$N^*(6,3) = 102$	$\{5,6,4,6,2,6,4,5,6,5,3,5,6,5,4,6,2,6,4,6,5\}$
$N^*(2,4) = 15$	$\{2,2,1,2,2,1,2,2\}$
$N^*(3,4) = 57$	$\{3,3,2,2,3,3,1,3,3,2,3,3,1,3,3,2,3,3,1,3,1,3,2\}$
$N^*(2,5) = 29$	$\{1,2,1,2,2,2,1,2,2,2,1,2,2,2,1,2,1\}$
$N^*(2,6) = 57$	$\{2,2,1,2,1,2,2,2,1,2,2,2,2,1,2,1,2,1,2,1,2,2,2,2,1,2,2,2,1,2,1,2,2\}$

We also found the partial results $N^*(4,4) \geqq 160$ and $N^*(2,7) \geqq 193$.

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ELEMENTARY EVALUATION OF $\zeta(2n)$

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1. Introduction. Let B_j denote the jth Bernoulli number (defined below in Section 2). Euler's formula

$$\zeta(2n) \equiv \sum_{k=1}^{\infty} k^{-2n} = \frac{(-1)^{n-1}(2\pi)^{2n}B_{2n}}{2(2n)!} \qquad (n \geqq 1) \tag{1.1}$$

is one of the most beautiful results of elementary analysis. Perhaps the three most common methods of proving (1.1) are by the use of the Fourier series for the Bernoulli polynomials [**4**, p. 524], by the use of the calculus of residues in conjunction with the Laurent expansion of $\cot x$ (given below) in terms of Bernoulli numbers [**10**, pp. 141–143], and by the method of Euler, described in [**1**], for example. T. M. Apostol [**1**] recently gave a proof of (1.1) that uses knowledge of symmetric functions and one of Newton's formulas. The idea for Apostol's proof can be found in the Yagloms' book [**14**, pp. 131–133], although they only establish (1.1) for $n = 1$ and $n = 2$. I. Skau and E. Selmer [**11**] use a similar method, but they do not explicitly evaluate $\zeta(2n)$ in terms of Bernoulli numbers.

Apostol's paper [**1**] contains a survey of "elementary" methods used to establish (1.1). An even more recent paper of E. L. Stark [**12**] contains a lengthy bibliography of papers on the evaluation of $\zeta(2)$ and $\zeta(2n)$. To the references cited in the two aforementioned papers, one can add the paper of R. Hovstad [**3**] and H. Rademacher's book [**9**, pp. 121–124].

In this paper, two new proofs of (1.1) are given. The proofs use only elementary calculus. The first proof, especially, is suitable for presentation in an ordinary calculus class.

2. Properties of Bernoulli numbers and polynomials. The Bernoulli polynomials $B_n(x)$, $0 \leqq n < \infty$, are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)t^n/n! \qquad (|t| < 2\pi). \tag{2.1}$$