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example there does not exist a sequence of domains D_1, D_2, D_3, \dots closing down on the point O and such that, for each n , the boundary of D_n is compact.

⁴ Jones, F. B., "Concerning Certain Topologically Flat Spaces," *Trans. Am. Math. Soc.*, **42**, 53-93 (1937).

⁵ See Axiom 1.

⁶ See the proof of Theorem 1 of Chapter I.

⁷ See the proposition labeled "Theorem 25" in Chapter II. That this proposition is not a consequence of Axioms 0, 1 and 2 may be seen with the aid of Example 1.

ON SETS OF INTEGERS WHICH CONTAIN NO THREE TERMS IN ARITHMETICAL PROGRESSION

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Let S be a set of non-negative integers, all different from one another. We say briefly that S is "progression-free" if any three distinct integers of S never form an arithmetical progression, i.e., if $a + a' \neq 2a''$ whenever a, a', a'' are different and belong to S .

If the elements of a progression-free set S do not exceed a given N , then the number of elements of S has clearly a maximum $\nu = \nu(N)$.

It has been widely conjectured that, as $N \rightarrow \infty$, $\nu(N) = O(N)^\alpha$ where α is a positive constant inferior to 1. A more precise conjecture has assigned to α the value $\log 2 / \log 3$ which corresponds to the progression-free sequence of integers whose digits in the ternary scale are 0 and 2 only.¹

The purpose of the present note is to prove that the conjecture $\nu(N) = O(N^\alpha)$ is false for every $\alpha < 1$. We shall prove that, as $N \rightarrow \infty$,

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log \log N}}$$

for every $\epsilon > 0$.

Let d be an integer > 2 and n an integer divisible by d . Having fixed d and n , let $S(d, n)$ be the set of all integers given by the expression

$$A = a_1 + a_2(2d - 1) + \dots + a_n(2d - 1)^{n-1}$$

where the "digits" a_i are integers subjected to the following condition: exactly n/d digits are equal to zero, n/d digits are equal to 1, n/d digits are equal to 2, etc. . . . , and n/d digits are equal to $d - 1$. Thus the number of integers of $S(d, n)$, all different, is

$$\mu(d, n) = \frac{n!}{[(n/d)!]^d} \tag{1}$$

On the other hand, for all numbers A of $S(d, n)$ we have

$$A < (2d - 1)^n. \tag{2}$$

The set $S(d, n)$ is progression-free. Suppose that A, A', A'' belong to the set and that $A + A' = 2A''$. Let a_i, a_i', a_i'' be the digits of rank i in A, A', A'' , respectively. Since $a_i + a_i' \leq 2d - 2$ and $2a_i'' \leq 2d - 2$, the equality $A + A' = 2A''$ implies $a_i + a_i' = 2a_i''$ for all i . Now there are in A'' exactly n/d digits equal to zero, and if $a_n'' = 0$, then necessarily $a_n = a_n' = 0$; i.e., the n/d digits equal to zero occupy the same places in A, A' and A'' . Next, there are, in A'' , n/d digits equal to 1, and if $a_k'' = 1$, then since $a_k \neq 0$ and $a_k' \neq 0$, the equality $a_k + a_k' = 2a_k''$ implies $a_k = a_k' = 1$; i.e., the n/d digits equal to 1 correspond in A, A', A'' . Generally, if $a_i'' = m, a_i$ and a_i' being different from 0, 1, 2, . . . $m - 1$, then $a_i = a_i' = m$ and the n/d digits equal to m have the same ranks in A, A', A'' . Going up to $m = d - 1$, we prove that $A = A' = A''$, that is to say $S(d, n)$ is progression-free.

Now if n and n/d are large enough we have by (I)

$$\mu(d, n) > \frac{n^n \sqrt{2\pi n} e^{-n}}{[(n/d)^{n/d} \sqrt{2\pi(n/d)} e^{-n/d}]^d C^d}$$

C being a constant (as near to 1 as we please). Thus

$$\mu(d, n) > (d/\gamma n)^{d/2} d^n \tag{3}$$

γ being a constant (as near to 2π as we please).

Let us now fix an N and let us choose d such that

$$(2d - 1)^{d\omega(d)} \leq N < (2d + 1)^{(d+1)\omega(d+1)} \tag{4}$$

where $\omega(d)$ is an integer increasing infinitely with d and such that $\frac{\omega(d)}{\log d} \rightarrow \infty$ but $\frac{\log \omega(d)}{\log d} \rightarrow 0$ as $d \rightarrow \infty$. Let us construct the set $S(d, n)$ with $n = d\omega(d)$. We have by (2), (3) and (4)

$$\begin{aligned} \nu(N) \geq \nu[(2d - 1)^{d\omega(d)}] &\geq \mu(d, d\omega(d)) > \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} d^{d\omega(d)} \\ \frac{\nu(N)}{N} &> \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} \frac{d^{d\omega(d)}}{(2d + 1)^{(d+1)\omega(d+1)}} \end{aligned}$$

Now, as $N \rightarrow \infty, d \rightarrow \infty$, and

$$\log \left(\frac{N}{\nu(N)} \right) < (d + 1)\omega(d + 1) \log (2d + 1) - d\omega(d) \log d + \frac{d}{2} \log \omega(d) + \frac{d}{2} \log \gamma = d\omega(d)[\log 2 + o(1)], \quad (5)$$

if we suppose, as we may, that $\omega(d)$ increases regularly enough to have $\omega(d + 1) - \omega(d) = o(1)$. By (4)

$$\log N \geq d\omega(d) \log (2d - 1)$$

$$\log \log N < \log (d + 1) + \log \omega(d + 1) + \log \log (2d + 1)$$

and so

$$\frac{\log N}{\log \log N} > d\omega(d)[1 + o(1)]. \quad (6)$$

From (5) and (6) it plainly follows that, as $N \rightarrow \infty$

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log \log N}}$$

for every $\epsilon > 0$.

Remark.—The sequence constructed above is finite and the construction depends on N . Therefore it should be pointed out that by a slight modification of the argument, we can form an infinite “progression-free” sequence of integers such that the number of terms of the sequence not exceeding N is, for $N \rightarrow \infty$, greater than $N^{1 - \frac{a}{\log \log N}}$, a being a constant.

Extension to Sets of Points.—Let E be a set of points in $(0, 1)$ such that if x and y belong to E , then $(x+y)/2$ belongs to E if and only if $x = y$ (property P). It is known that E is of measure zero.² An adaptation of the above argument yields a perfect set E having the property P and whose Hausdorff dimensionality is greater than every $\alpha < 1$. The proof, together with other remarks on sets of points having the property P , will appear elsewhere.

¹ See Erdos and Turan, *Jour. London Math. Soc.*, **11**, 261–264 (1936).

² See Ruziewicz, S., *Fundamenta Mathematicae*, **7**, 141–143 (1925).