example there does not exist a sequence of domains D_1, D_2, D_3, ... closing down on the point O and such that, for each n, the boundary of D_n is compact.

[4] Jones, F. B., "Concerning Certain Topologically Flat Spaces," *Trans. Am. Math. Soc.*, **42**, 53–93 (1937).

[5] See Axiom 1.

[6] See the proof of Theorem 1 of Chapter I.

[7] See the proposition labeled "Theorem 25" in Chapter II. That this proposition is not a consequence of Axioms 0, 1 and 2 may be seen with the aid of Example 1.

ON SETS OF INTEGERS WHICH CONTAIN NO THREE TERMS IN ARITHMETICAL PROGRESSION

BY R. SALEM AND D. C. SPENCER

DEPARTMENTS OF MATHEMATICS, HARVARD UNIVERSITY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY, AND STANFORD UNIVERSITY

Communicated October 26, 1942

Let S be a set of non-negative integers, all different from one another. We say briefly that S is "progression-free" if any three distinct integers of S never form an arithmetical progression, i.e., if $a + a' \neq 2a''$ whenever a, a', a'' are different and belong to S.

If the elements of a progression-free set S do not exceed a given N, then the number of elements of S has clearly a maximum $\nu = \nu(N)$.

It has been widely conjectured that, as $N \to \infty$, $\nu(N) = 0(N)^{\alpha}$ where α is a positive constant inferior to 1. A more precise conjecture has assigned to α the value log 2/log 3 which corresponds to the progression-free sequence of integers whose digits in the ternary scale are 0 and 2 only.[1]

The purpose of the present note is to prove that the conjecture $\nu(N) = 0(N^{\alpha})$ is false for every $\alpha < 1$. We shall prove that, as $N \to \infty$,

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log\log N}}$$

for every $\epsilon > 0$.

Let d be an integer > 2 and n an integer divisible by d. Having fixed d and n, let $S(d, n)$ be the set of all integers given by the expression

$$A = a_1 + a_2(2d - 1) + \ldots + a_n(2d - 1)^{n-1}$$

where the "digits" a_i are integers subjected to the following condition: exactly n/d digits are equal to zero, n/d digits are equal to 1, n/d digits are equal to 2, etc...., and n/d digits are equal to $d - 1$. Thus the number of integers of $S(d, n)$, all different, is

$$\mu(d, n) = \frac{n!}{[(n/d)!]^d} \tag{1}$$

On the other hand, for all numbers A of $S(d, n)$ we have

$$A < (2d - 1)^n. \tag{2}$$

The set $S(d, n)$ is progression-free. Suppose that A, A', A'' belong to the set and that $A + A' = 2A''$. Let a_i, a_i', a_i'' be the digits of rank i in A, A', A'', respectively. Since $a_i + a_i' \leqslant 2d - 2$ and $2a_i'' \leqslant 2d - 2$, the equality $A + A' = 2A''$ implies $a_i + a_i' = 2a_i''$ for all i. Now there are in A'' exactly n/d digits equal to zero, and if $a_h'' = 0$, then necessarily $a_h = a_h' = 0$; i.e., the n/d digits equal to zero occupy the same places in A, A' and A''. Next, there are, in A'', n/d digits equal to 1, and if $a_k'' = 1$, then since $a_k \neq 0$ and $a_k' \neq 0$, the equality $a_k + a_k' = 2a_k''$ implies $a_k = a_k' = 1$; i.e., the n/d digits equal to 1 correspond in A, A', A''. Generally, if $a_l'' = m$, a_l and a_l' being different from $0, 1, 2, \ldots m - 1$, then $a_l = a_l' = m$ and the n/d digits equal to m have the same ranks in A, A', A''. Going up to $m = d - 1$, we prove that $A = A' = A''$, that is to say $S(d, n)$ is progression-free.

Now if n and n/d are large enough we have by (I)

$$\mu(d, n) > \frac{n^n\sqrt{2\pi n}\ e^{-n}}{[(n/d)^{n/d}\sqrt{2\pi(n/d)}\ e^{-n/d}]^d}\frac{1}{C^d}$$

C being a constant (as near to 1 as we please). Thus

$$\mu(d, n) > (d/\gamma n)^{d/2}\, d^n \tag{3}$$

γ being a constant (as near to 2π as we please).

Let us now fix an N and let us choose d such that

$$(2d - 1)^{d\omega(d)} \leqq N < (2d + 1)^{(d + 1)\,\omega(d + 1)} \tag{4}$$

where $\omega(d)$ is an integer increasing infinitely with d and such that $\dfrac{\omega(d)}{\log d} \to \infty$ but $\dfrac{\log \omega(d)}{\log d} \to 0$ as $d \to \infty$. Let us construct the set $S(d, n)$ with $n = d\omega(d)$. We have by (2), (3) and (4)

$$\nu(N) \geqslant \nu[(2d - 1)^{d\omega(d)}] \geqslant \mu(d, d\omega(d)) > \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} d^{d\omega(d)}$$

$$\frac{\nu(N)}{N} > \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} \frac{d^{d\omega(d)}}{(2d + 1)^{(d + 1)\,\omega(d + 1)}}$$

Now, as $N \to \infty$, $d \to \infty$, and

$$\log\left(\frac{N}{\nu(N)}\right) < (d + 1)\omega(d + 1)\log(2d + 1) - d\omega(d)\log d + \frac{d}{2}\log\omega(d) + \frac{d}{2}\log\gamma = d\omega(d)[\log 2 + o(1)], \quad (5)$$

if we suppose, as we may, that $\omega(d)$ increases regularly enough to have $\omega(d + 1) - \omega(d) = o(1)$. By (4)

$$\log N \geqslant d\omega(d)\log(2d - 1)$$

$$\log\log N < \log(d + 1) + \log\omega(d + 1) + \log\log(2d + 1)$$

and so

$$\frac{\log N}{\log\log N} > d\omega(d)[1 + o(1)]. \quad (6)$$

From (5) and (6) it plainly follows that, as $N \to \infty$

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log\log N}}$$

for every $\epsilon > 0$.

Remark.—The sequence constructed above is finite and the construction depends on N. Therefore it should be pointed out that by a slight modification of the argument, we can form an infinite "progression-free" sequence of integers such that the number of terms of the sequence not exceeding N is, for $N \to \infty$, greater than $N^{1 - \frac{a}{\log\log N}}$, a being a constant.

Extension to Sets of Points.—Let E be a set of points in (0, 1) such that if x and y belong to E, then $(x+y)/2$ belongs to E if and only if $x = y$ (property P). It is known that E is of measure zero.[2] An adaptation of the above argument yields a perfect set E having the property P and whose Hausdorff dimensionality is greater than every $\alpha < 1$. The proof, together with other remarks on sets of points having the property P, will appear elsewhere.

[1] See Erdos and Turan, *Jour. London Math. Soc.*, **11**, 261–264 (1936).

[2] See Ruziewicz, S., *Fundamenta Mathematicae* **7**, 141–143 (1925).