An Application of Lovász' Local Lemma—A New Lower Bound for the van der Waerden Number

Zoltán Szabó
Eötvös Lóránd University Budapest, Balzac U.18, 1136 Hungary

ABSTRACT

The van der Waerden number $W(n)$ is the smallest integer so that if we divide the integers $\{1, 2, \ldots, W(n)\}$ into two classes, then at least one of them contains an arithmetic progression of length n. We prove in this paper that $W(n) \geq 2^n/n^\varepsilon$ for all sufficiently large n.

INTRODUCTION

Denote by $W(n)$ smallest integer so that if we divide the integers $\{1, 2, \ldots, W(n)\}$ into two classes, then at least one of them contains an arithmetic progression of length n.

It was proved by L. Lovász [3] that the van der Waerden number $W(n) \geq 2^n/8n$. Berlekamp [2] proved that if p is a prime, then $W(p+1) \geq 2^p \cdot p$. We improve on Lovász's lower bound by showing that $W(n) \geq 2^n/n^\varepsilon$ for all sufficiently large n. We also deal with the case of n-uniform almost disjoint hypergraphs. Let X be a finite set and $A_1, A_2 \cdots A_t$ be subsets of X satisfying $|A_i| = n$ for every $1 \leq i \leq t$ and $|A_i \cap A_j| \leq 1$ (almost disjointness) for every pair (A_i, A_j) will $i \neq j$. It follows from the Lovász' Local Lemma that if $|\langle i\colon P \in A_i\}| \leq 2^{n-3}/n$ for every $P \in X$, then there exists a good two-coloring of X, which means that we can color the points of X with two colors—red and blue—so that none of the A_i will be monochromatic.

Theorem 1 below improves on this result and the proof will be a good model for understanding the case of arithmetic progressions.

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Theorem 1. *Let $(X, \{A_1, \ldots, A_t\})$ be an n-uniform almost disjoint hypergraph. Let $\varepsilon > 0$ be arbitrary but fixed. Suppose that $|\{i: P \in A_i\}| \leq 2^n/n^\varepsilon$ for every $P \in X$. Then there exists a good two coloring of X provided $n > n_0(\varepsilon)$.*

Proof of Theorem 1. Let $L = \{A_1, A_2, \ldots, A_t\}$ and call the elements of L edges. We need the following technical Lemma:

Lemma 1. *If n is large enough and k is a constant depending only on ε, then we can choose t sets $R_1, R_2, \ldots, R_t$ with the following properties:*

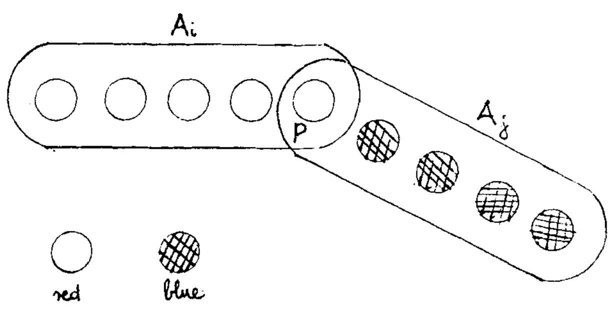
(i) $R_i \subset A_i$ for all $1 \leq i \leq t$
(ii) $]R_i| = k$ for all $1 \leq i \leq t$

(iii) $|\{i: P \in R_i\}| \leq \frac{2^n}{n^\varepsilon} \cdot \frac{2k}{n}$ for all $P \in X$.

Remark. Before proving the existence of such sets R_i we explain the role of the sets R_i:

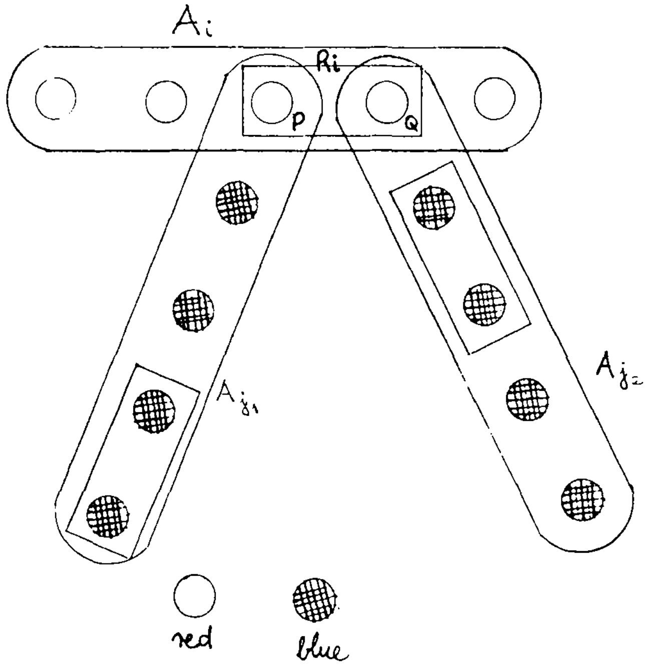
Suppose that we have already colored the points of X, but we have been unlucky and there are some monochromatic A_i's. It is a natural idea to select one point from each monochromatic A_i and then change the color of these points. But we have to be careful about the selections. It may happen that after this modification we get some new monochromatic sets A_i. Let us examine the following situation:

Example 1.



Assume that $\{P\} = A_i \cap A_j$ where A_i is red and $A_j \backslash A_i$ is blue. If we change the color of P from red to blue then we kill A_i but A_j will become monochromatic. Of course we should select another point from A_i. All we have to do is to select one "nice" point from each monochromatic edge A_i. In order to simplify our job we restrict ourselves to R_i, which means that we know from the beginning that our nice point from A_i will be an element of R_i.

Example 2.



Assume that A_i is monochromatically red, and for every $P \in R_i$, there exists an A_j satisfying $A_j \cap A_i = \{P\}$, and $A_j \setminus \{P\}$ is blue. It would be impossible to find a nice point in that situation. In order to avoid situations like Example 2, we will use a version of the Lovász' Local Lemma, viz., Lemma 2.

Proof of Lemma 1. Let us denote d the maximum degree of the hypergraph, i.e.,

$$d = \max_{P \in X} |\{i: P \in A_i\}| .$$

We know from the hypothesis of Theorem 1 that $d \leq 2^n/n^\varepsilon$.

First Step. We want to choose one point P_i from each A_i, in such a way that the maximum multiplicity is less or equal to $[d/n] + 1$:

$$\max_{P \in X} |\{i: P_i = P\}| \leq \left[\frac{d}{n}\right] + 1 . \tag{1.1}$$

If we have an arbitrary set of representatives $Q = \{P_1, P_2, \ldots, P_t\}$ where $P_i \in A_i$ for every $1 \leq i \leq t$, we define the multiplicity of a point $P \in X$ by:

$$\deg_Q(P) = |\{i: 1 \leq i \leq t,\ P_i = P,\ P_i \in Q\}|$$

and the maximum multiplicity by:

$$M(Q) = \max_{P \in X} \deg_Q(P) .$$

Let us denote $Y_0 = \{P \in X\colon \deg_Q P = M(Q)\}$ and for $j \geq 1$ $Y_j = Y_{j-1} \cup \{P \in X\colon$ there is an $A_i \in L$ with the property that $P \in A_i$ and $P_i \in Y_{j-1}\}$. It is trivial that $Y_0 \subset Y_1 \subset Y_2 \cdots Y_k \subset \cdots$. Let $Y = \cup_{k=0}^{\infty} Y_k$. Since X has a finite number of elements, there is a number k such that $Y_k = Y$. It follows from the construction of the sets Y_j that

$$\{P \in X\colon \text{there is an } A_i \in L \text{ with the property that } P \in A_i \text{ and } P_i \in Y\} \subset Y$$

Let $\min_Q M(Q) = M_0$, where Q runs over all sets of representatives. Let Q_0 be a set with $M(Q_0) = M_0$ such that the number of points of multiplicity M_0 is minimal. We claim:

$$M_0 - 1 \leq \deg_{Q_0} P \leq M_0 \qquad \text{for every} \qquad P \in Y. \tag{1.2}$$

The proof of (1.2) is indirect: Suppose that there exists a point $P \in Y$ with $\deg_{Q_0} P \leq M_0 - 2$. Let us denote

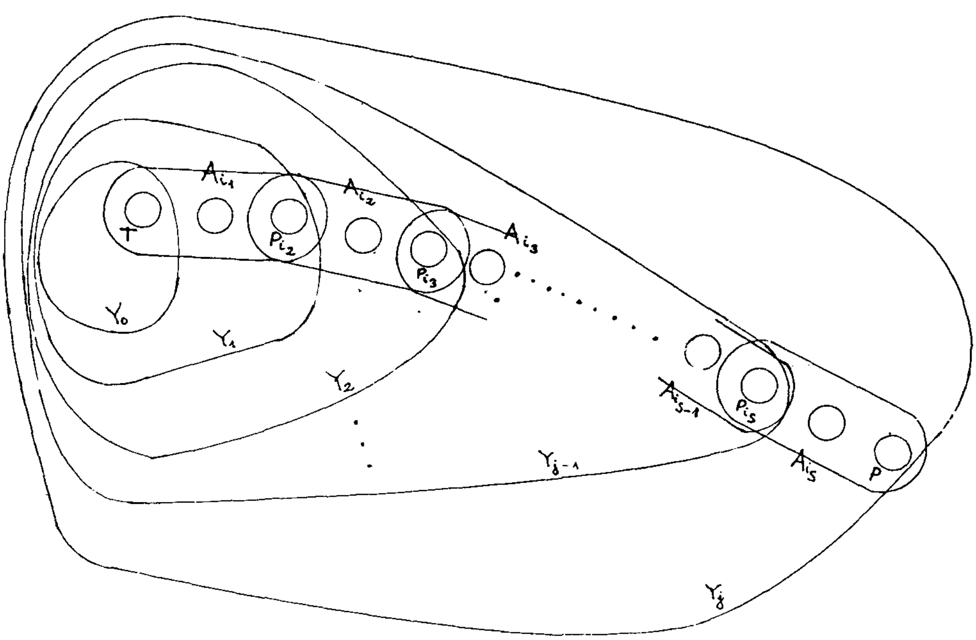
$$S = \min_{P \in Y_j} j.$$

It follows from the construction of the sets Y_j that we can find a point $T \in Y$, edges $A_{i_1}, A_{i_2} \cdots A_{i_S} \in L$ which satisfy the following properties:

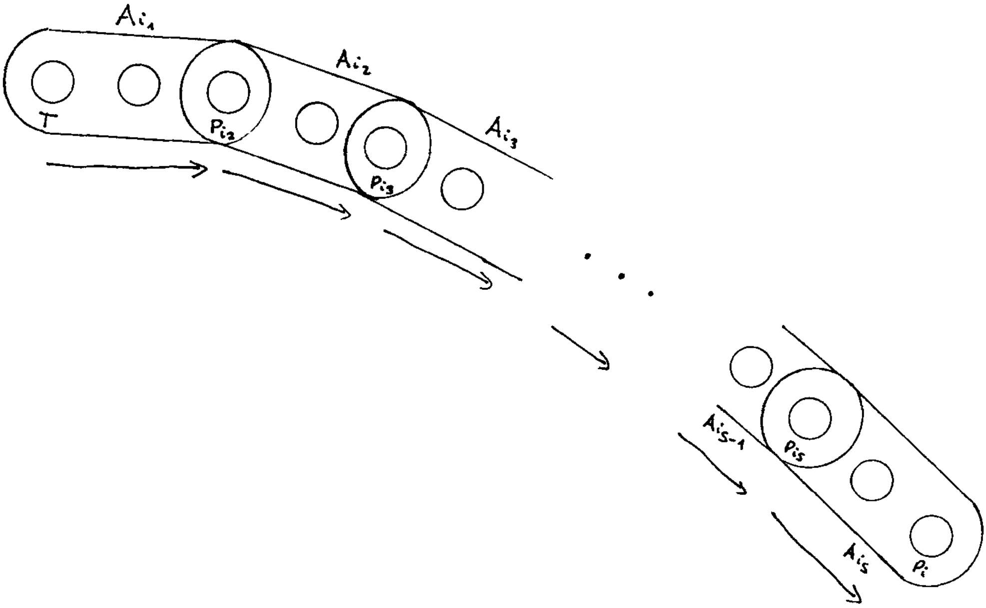
$$T = P_{i_1} \quad \text{and} \quad T \in Y_0$$

$$A_{i_j} \cap A_{i_j+1} = P_{i_j+1} \quad \text{and} \quad P_{i_j+1} \in Y_j \qquad \text{for every} \qquad i \leq j \leq S-1$$

$$\text{and} \quad P \in A_{i_S}$$



We can modify Q_0: Instead of the point T we rather choose the point $A_{i_1} \cap A_{i_2}$ from the edge A_{i_1}, instead of the point $A_{i_j} \cap A_{i_{j+1}}$ we rather choose the point $A_{i_{j+1}} \cap A_{i_{j+2}}$ for every $1 \leq j \leq S-2$, and instead of the point $A_{i_{S-1}} \cap A_{i_S}$ we rather choose the point P from the edge A_{i_S}.



Then we get a modified set of representatives where the new degree of T is $d_0 - 1$, the new degree of P is increasing by 1 and the degree of P_{i_j} and of all the other points remained the same. So it is easy to see that we get a "more optimal set," which is a contradiction. This proves (1.2).

Let us consider the following cardinality: $V = |\{i\colon A_i \subset Y\}|$. It follows from the definition of degree d of the hypergraph that $V \leq d \cdot |Y|/n$. On the other hand, we know that $U\{A_i\colon \text{where } i \text{ satisfy } P_i \in Y\} \subset Y$. So it follows from (1.2) that $V \geq |Y|(M_0 - 1)$. So we have proved that $M(Q_0) \leq [d/n] + 1$ which means that Q_0 satisfies (1.1).

Let $B_i = A_i \backslash P_i$ for every $1 \leq i \leq t$. We know that $|B_i| = n-1$, $|B_i \cap B_j| \leq 1$ and $|\{i\colon P \in B_i\}| \leq d$ for every $P \in X$. It completes the first step.

General Step. Use the procedure described in the first step to the sets $B_1, B_2, \ldots, B_t$. We take k steps. Because of the almost disjointness, if $n > k+1$ then after j step we get t different almost disjoint sets with cardinality $n-j$. At the end, we get the sets $D_1, D_2 \cdots D_t$, $|D_i| = n-k$ for every $1 \leq i \leq t$. Let $R_i = A_i \backslash D_i$.

By the repeated use of inequalities (1.1) and by the assumption $d \leq 2^n/n^\varepsilon$ we have

$$|\{i\colon P \in R_i\}| \leq \sum_{j=0}^{k-1} \left[\frac{2^n}{n^\varepsilon} \cdot \frac{1}{n-j}\right] + 1 \qquad \text{for every} \qquad P \in X\,.$$

If n is large enough,

$$|\{i: P \in R_i\}| \leq \frac{2^n}{n^\varepsilon} \cdot \frac{2k}{n}$$

and Lemma 1 follows. ■

The Lovász' Local Lemma (see Spencer [4]). *Let $\mathcal{G}$ be a simple graph on the vertex set $V(\mathcal{G}) = \{1, 2, \ldots, n\}$ and let an event T_i be associated with each vertex i. Suppose that there are real numbers $x_1, x_2, \ldots, x_n$, $0 < x_i < 1$ such that*

(a) every T_i is independent of the set of all T_j's for which j is not adjacent to i.

(b) $P(T_i) \leq (1 - x_i) \prod_{\{i,j\}} x_j$ $\quad i = 1, 2, \ldots, n$. Then $P(\bar{T}_1 \cdot \bar{T}_2 \cdot \ldots \cdot \bar{T}_n) > 0$.

Lemma 2 (See Beck [1]). *Let X be a finite set and $B_1, B_2 \cdots B_s$ be not necessarily distinct subsets of X. Assume that $|B_i| \geq n$ for all i. For every B_i let there be given a 2-coloring: f_i: $B_i \to \{red, blue\}$. If $\Sigma_{i:p \in B_i} [1 - (1/n)]^{-|B_i|} \cdot 2^{-|B_i|} \leq 1/n$ for every $p \in X$ then there exists a 2-coloring f: $X \to \{red, blue\}$ such that for every i*

$$f|_{B_i} \neq f_i .$$

Proof of Lemma 2. Following Beck [1], color the points of X with red and blue at random, independently of each other with probability $\frac{1}{2}$. Let T_i denote the event that $f|_{B_i} = f_i$. Then $P(T_i) = 2^{-|B_i|}$ as there are $2^{|B_i|}$ ways to color B_i and one of these comes into consideration. Observe that if $B_{i_1}, B_{i_2} \cdots B_{i_k}$ are disjoint from B_i, then T_i is independent of $T_{i_1} \cdots T_{i_k}$. So if we form $\mathcal{G}$ such that $\{i, j\}$ iff $B_i \cap B_j \neq \emptyset$, then this graph and the associated events satisfy condition (a) of Lovász' Local Lemma.

Moreover, we shall prove that condition (b) is satisfied as well. Indeed, let $1 - x_i = [1 - (1/n)]^{-|B_i|} \cdot 2^{-|B_i|}$ then

$$(1 - x_i) \prod_{\{i, j\} \in \mathcal{G}} x_j \geq (1 - x_i) \prod_{p \in B_i} \prod_{j:\, p \in B_j} x_j \geq (1 - x_i) \prod_{p \in B_i} \Big(1 - \sum_{j:\, p \in B_j} (1 - x_j)\Big)$$
$$\geq (1 - x_i)\left(1 - \frac{1}{n}\right)^{|B_i|}$$

since

$$\sum_{j:p \in B_j} (1 - x_j) = \sum_{i:p \in B_i} \left(1 - \frac{1}{n}\right)^{-|B_i|} \cdot 2^{-|B_i|} \leq \frac{1}{n} .$$

Thus we have

$$(1 - x_i) \prod_{\{i, j\} \in \mathcal{G}} x_j \geq (1 - x_i)\left(1 - \frac{1}{n}\right)^{|B_i|} = 2^{-|B_i|} = P(T_i)$$

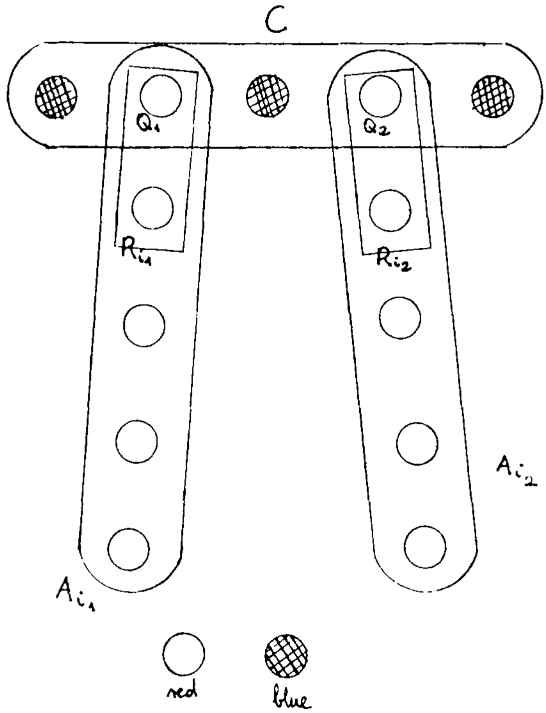
so that (b) is satisfied.

By the application of Lovász' Local Lemma we obtain $P(\bar{T}_1 \cdot \ldots \cdot \bar{T}_S) > 0$, i.e., there exists a good 2-coloration. This proves Lemma 2. ■

Remark. In other words, if we have a "few" prohibited color configurations, then there exists a 2-coloring which contains no prohibited configurations. It is important that the sets $B_1 \dots B_s$ are not necessarily different. This means that we may prohibit more then just one coloring on the same subset B. In the following, we will describe the prohibited color-configurations.

We recall that $L = \{A_1, \dots, A_t\}$ and k is a constant depending only on ε. Let $k = [5/\varepsilon]$.

Definition. Configuration of Type 1.



Let B be the following subset: $B = (\cup_{S=1}^{m} A_{i_S}) \cup C$ where the edges C, $A_{i_1}, \dots A_{i_S} \dots A_{i_m}$ are elements of L satisfying

$$A_{i_S} \cap C = Q_S \quad \text{and} \quad Q_S \in R_{i_S} \qquad \text{for every} \qquad 1 \leq S \leq m.$$

Here $Q_1, Q_2 \dots Q_m$ are mutually distinct points. Assume that the set $\cup_{S=1}^{m} A_{i_S}$ is red, and the set $C \setminus \cup_{S=1}^{m} A_{i_S}$ is blue. Then we call the subset B with this coloring configuration of type 1.

For example, if $P \in R_i$ in Example 1, then this is a configuration of type 1 with $m = 1$. We cannot avoid these configurations even with the help of Lemma 2 since the configuration has $2n - 1$ points and there are too many configurations which contain a fixed point P.

We want to prohibit the configurations of type 1 at least in the case when m is large. We distinguish two cases.

Definition. We call a configuration of type 1 as a *prohibited configuration of type* 1.*a* if

$$\frac{n}{2} \geq m \geq m_0 = \left[\frac{5}{\varepsilon}\right].$$

Let (f_1, B_1), $(f_2, B_2) \cdots (f_l, B_l)$ be all the prohibited configurations of type 1.a. We call

$$\sum_{j: P \in B_j} \left(1 - \frac{1}{n}\right)^{-|B_j|} \cdot 2^{-|B_j|}$$

the contribution with respect to $P \in X$ of the configurations of type 1.a in Lemma 2.

Lemma 3. *The contribution of the prohibited configurations of type* 1.*a is* $0(1/n^2)$ *for every* $P \in X$.

Proof of Lemma 3. Fix a permutation of X. Fix m.

Suppose that $P \in C$. There are $\leq 2^n/n^\varepsilon$ choices to C since $d \leq 2^n/n^\varepsilon$. Then we prescribe the m points of C: $\binom{n}{m}$ possibilities. We have a permutation of the points of X, so we have an order to the m points—$Q_1, Q_2 \cdots Q_m$ also. At first we will prescribe the edge A_{i_1} adjacent to Q_1, then A_{i_2} adjacent to Q_2, and so on. We call the edge A_{i_1} and also the indices S "almost determined" if there exists $j < S$ with $A_{i_j} \cap A_{i_S} \neq \emptyset$. Suppose that there are exactly z different "almost determined" indices. There are $\binom{m}{z}$ possibilities to fix this z element index set. We select the sets A_{i_j} one-by-one.

If the indices S is not "almost determined," then there are only $2^n/n^\varepsilon \cdot 2k/n$ choices to A_{i_s} because of the property $Q_S \in R_{i_S}$ and the property (*iii*) of Lemma 1.

If S is "almost determined" then there are $\leq n^2$ possibilities to A_{i_S}: Before we determine A_{i_S} we already know which point is Q_i and we know that one of the elements of A_{i_S} is in the set $\cup_{j<S} A_{i_j}$. Using the fact that two points determine A_{i_S} we get the previous inequality.

If A_i is "almost determined" then $|A_{i_S} \setminus \cup_{j=1}^{S-1} A_{i_j}| \geq n - S + 1$ because of the almost disjoint property. It follows:

$$|B| \geq n - m + (m - z)n + z\left[\frac{n}{2}\right].$$

Summarizing the previous calculations, we have at most

$$\frac{2^n}{n^\varepsilon} \cdot \binom{n}{m} \cdot \binom{m}{z} \cdot (n^2)^z \cdot \left(\frac{2^n}{n^\varepsilon} \cdot \frac{2k}{n}\right)^{m-z}$$

configurations, each contain at least $n - m + (m - z)n + z \cdot [n/2]$ points. So it follows that the contribution of that case

$$\leq \left(\frac{8k}{n^{\varepsilon}}\right)^{m-z+1} \cdot \left(\frac{8n^3}{2^{[n/2]}}\right)^{z}.$$

We know that $m \geq m_0$ and $0 \leq z \leq m$. It is easy to see that in each case the contribution $= 0(1/n^4)$. Recall that m and z were fixed. Summing up all these cases we get that the contribution of the configurations of type $1.\mathrm{a} = 0(1/n^2)$ under the assumption $P \in C$.

Next *suppose that* $P \not\in C$. In this case, first fix the edge adjacent to P: $\leq 2^n/n^{\varepsilon}$ possibilities and call this edge A_0. Then prescribe C: $\leq k \cdot 2^n/n^{\varepsilon}$ possibilities. Then from the same calculation and the same argument as before we get that the contribution of this case (with fixed z and m):

$$\leq 8k \cdot \left(\frac{8k}{n^{\varepsilon}}\right)^{m-z} \cdot \left(\frac{8n^3}{2^{[n/2]}}\right)^{t} = 0\left(\frac{1}{n^4}\right).$$

If z and m are not fixed, we get $0(1/n^2)$ as before. Thus we proved Lemma 3. ∎

Definition. Let B be a configuration of type 1. with $m = [n/2]$. Assume that the color of the set $\cup_{S=1}^{[n/2]} A_{i_S}$ is red, and the coloring of $C \setminus \cup_{S=1}^{[n/2]} A_{i_S}$ is arbitrary. Then we call B with this coloring, *prohibited configuration of type* $1.b$.

Lemma 4. *The contribution of the prohibited configurations of type* $1.b$ *are* $0(1/n^2)$.

Proof. Repeating the argument of Lemma 3 we obtain that the contribution

$$\leq \frac{(8k)^{[n/2]-z+1}}{(n^{\varepsilon})^{[n/2]-z}} \cdot \left(\frac{8n^3}{2^{[n/2]}}\right)^{z} \cdot 2^{[n/2]+1}$$

if we fix z.

In cases $z \geq 2$ we have that the contributions $\leq (8n^3)^2/2^{[n/2]} = 0(1/n^3)$.

In cases $z \leq 2$ we have that the contributions $\leq (2/n^{\varepsilon/2})^{n/2} = 0(1/n^3)$ which proves Lemma 4. ∎

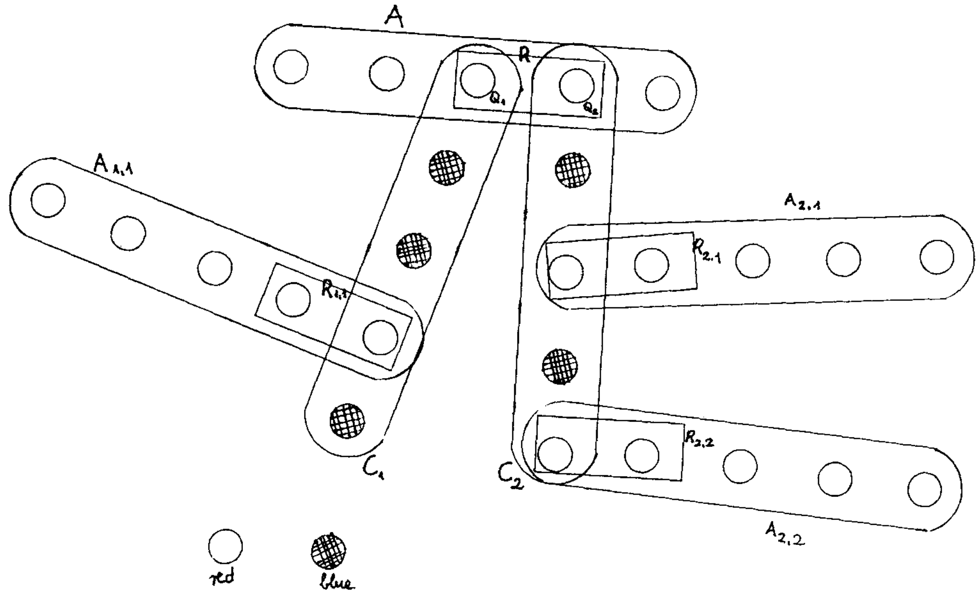
Remark. It is evident that if there is no prohibited configuration of type 1.b, then there is no configurations of type 1. with $m \geq n/2$ also.

Now we change the role of the blue and red colors and prohibit the reversed cases where $\cup_{S=1}^{m} A_{i_S}$ is blue and $C \setminus \cup_{S=1}^{m} A_{i_S}$ is red.

We call a point P "*bad*" if there is a configuration of type 1. in which P is one of points Q_i. We call the point P nice otherwise. We call an edge "*almost monochromatic*" if there exists a configuration of type 1. in which the edge is in the position of C.

It is easy to see that in the Example 2, the set A_j is almost blue and every points in R_i are bad. We want to avoid the situation in which there exist a monochromatic edge A_i and every points $P \in R_i$ are bad.

Definition. *Configuration of Type* 2.



A is a monochromatic red edge which has only bad points in its subset R. More formal: Let the sets A, $C_1 \cdots C_k$ and $A_{i,j}$ $1 \le i \le k$, $i \le j \le m_i$ be elements of L. A is red and the edges $A_{i,j}$ are not necessarily different red edges. Suppose further that the edges $C_1, C_2, \ldots, C_k$ are almost disjoint blue edges, and the edges $A, A_{i,1}, \ldots, A_i, m_i, C_i$ form a configuration of type 1 with $m = m_i + 1$ for every $1 \le i \le k$.

We know that $R = \{Q_1 \cdots Q_k\}$. Suppose that $A \cap C_i = Q_i$ for every $1 \le i \le k$. Then the set $(\cup_{i=1}^{k}(C_i \cup_{j=1}^{m_i} A_{i,j})) \cup A$ with this 2 coloring is called a configuration of type 2.

Definition. A configuration of type 2 is called a *prohibited configuration of type* 2 if $0 \le m_i \le m_0 - 2$ for every $1 \le i \le k$.

Remark. Since we have already prohibited the configurations of type 1 with $m \ge m_0$, there is no need to deal with configuration of type 2 with $m_i \ge m_0 - 1$ for some i.

Lemma 5. *The contribution of the prohibited configurations of type* 2 *are* $0(1/n^2)$.

The calculation is similar to the case of configuration of type 1. *Now we do not have to waste our time with the "almost determined" edges, but we have a little trouble, because the edges* $A_{i,j}$ *are not necessarily different. For this reason we introduce the following notation*:

$$\text{Let } m_i' = |\{j : 1 \le j \le m_i, \nexists(i', j') \quad \text{such that} \quad i' < i \quad \text{and} \quad A_{i,j} = A_{i',j'}\}|$$

Calculation. Because of the almost disjoint property, every configuration has

$$\geq n\left(1+k+\sum_{j=1}^{k} m_j'\right)-\binom{\sum_{j=1}^{k} m_j'+k+1}{2} \quad \text{points}. \tag{1.3}$$

Fix the numbers $m_1', m_2' \cdots m_j'$. All we have to do is to count all the possibilities, to count the number of all such configurations which contain a fix point P. We will calculate only the case $P \in A$ because the other cases are very similar and lead to the same result apart from constant factor.

By using the same counting approach as in Lemma 3 we get the following upper bound:

$$\frac{2^n}{n^\varepsilon}\cdot\left(\frac{2^n}{n^\varepsilon}\right)^k\cdot\prod_{S=1}^{m}\binom{n}{m_S'}\cdot\left(\frac{2^n}{n^\varepsilon}\cdot\frac{2k}{n}\right)^{m_S'}.$$

Using (1.3) we get that the contribution of this case

$$\leq\left(\frac{1}{n^\varepsilon}\right)^{k+1}\left(\frac{8k}{n^\varepsilon}\right)^{(\Sigma_{S=1}^{k} m_S')}\cdot 2^{\binom{1+k+\sum_1^k m_S'}{2}}\leq\left(\frac{1}{n^\varepsilon}\right)^{k+1}\cdot 2^{([5/\varepsilon]^4)}.$$

Now let the numbers $m_1', m_2' \cdots m_S'$ be arbitrary. We get the upper bound:

$$\left(\frac{1}{n^\varepsilon}\right)^{k+1}\cdot 2^{[5/\varepsilon]^4}\cdot m_0^k = C\cdot\left(\frac{1}{n^\varepsilon}\right)^{k+1}=0\left(\frac{1}{n^2}\right).$$

Proof of Lemma 5 is complete. ■

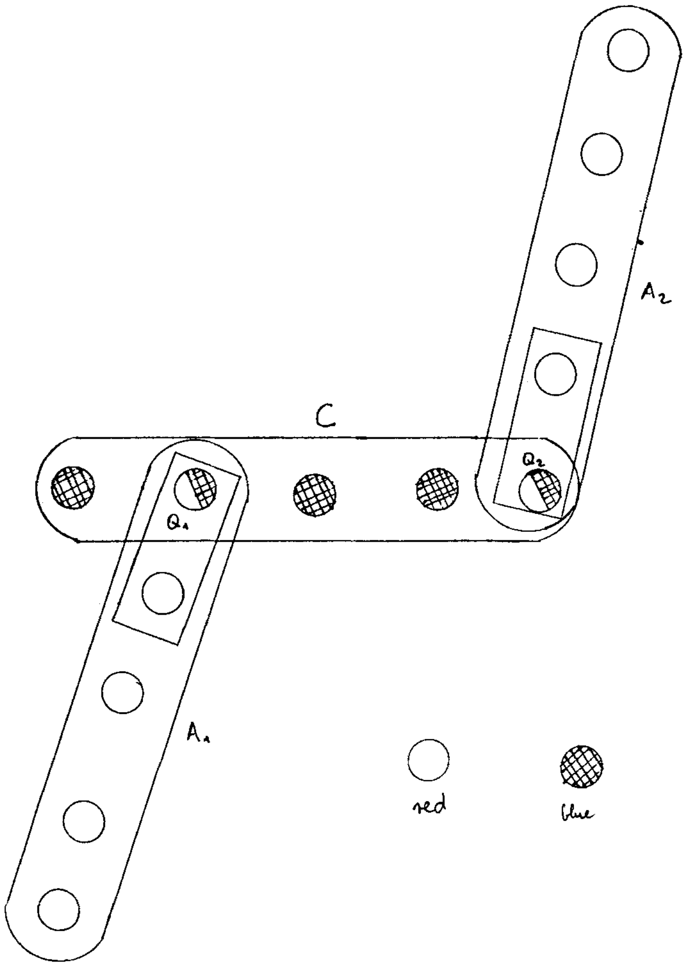
Now we change the role of the blue and red colors and prohibit the reversed prohibited configurations of type 2.

First Coloring. We are using Lemma 2 to the prohibited configurations. It follows from Lemma 3, Lemma 4, Lemma 5 that we can use the Lemma 2. We get a two coloring. It follows from the previous remarks that there is no configuration of type 2 at all. This means that in every monochromatic edge A_i there is a nice point, which is an element of R_i.

Second Coloring. If A_i is monochromatic then select one nice point $P_i \in R_i$. Then change the color of the set

$$Z=\bigcup_{\substack{i:A_i\\ \text{monochromatic}}} P_i .$$

We claim that after this second coloring we get a good 2-coloring of X. The proof is indirect: Suppose that C is monochromatically blue, then we killed it: it cannot remain blue after the second coloring. If C was not blue and turned to blue, then it follows from the construction of the second coloring that C was almost blue after the first coloring:



We know that the subset $Z \cap C = \{Q_1, \ldots, Q_s\}$ were turned from red to blue. But in that case $Q_1, Q_2 \cdots Q_s$ are bad points, and we get a contradiction with the construction of the second coloring. Of course the same argument shows the lack of the monochromatically red edges. Q.E.D.

Remark. You may be suspicious that we kill the monochromatic edges and take care of the properly 2-colored edges. But is it possible that in the second coloring we turn a red edge of blue? Notice that in that case the monochromatically red edges would be described as an almost blue also; and this is an ordinary case of which we have already taken care in the proof.

Theorem 2. *The van der Waerden number $W(n) \geq 2^n/n^\varepsilon$ for arbitrary $n \geq n_0(\varepsilon)$.*

In the case of the arithmetic progressions X will be the set of the natural numbers in the interval $[1, 2^n/n^\varepsilon]$. The edges will be arithmetic progressions of length n. It is easy to see that we have $\leq 2^n/n^\varepsilon$ edges which contain a fixed point P. There are difficulties with the property of being almost disjoint since the arithmetic progres-

sions do not satisfy it. At any rate it is a rare situation that two arithmetic progressions have more than one common point, and we will see later that it is possible to imitate the previous proof. The almost disjoint hypergraph is a natural model for us and this is the reason why we deal with it before the arithmetic progressions. In order to overcome the difficulties we have to define some new "prohibited configurations."

Definition. Let B be a red arithmetic progression of length $[n+\log^2 n]$. Then we call B *prohibited configuration of type 3.*

Lemma 6. *The contributions of this configuration are* $0(1/n^2)$.

Calculation. The number of the configuration which contains a fixed point P, $\leq 2^n/n^\varepsilon$. The distribution:

$$\leq \frac{2^n}{n^\varepsilon}\cdot\left(1-\frac{1}{n}\right)^{-[n+\log^2 n]}\cdot\left(\frac{1}{2}\right)^{[n+\log^2 n]}\leq\left(\frac{1}{n}\right)^{n\cdot\log 2}=0\left(\frac{1}{n^2}\right).$$

Definition. We call an arithmetic progression of length t a *generalized edge* if

$$n\leq t<[n+\log^2 n].$$

We say that two generalized edges *interfere with each other* if they have a common point and they have the same difference. If two generalized edges do not interfere with each other, we call those "*substantially different.*"

Definition. Suppose that A_1 and A_2 are two substantially different generalized edges and they have more than one common point. Let the color of $A_1\cup A_2$ be red. We call the subset $A_1\cup A_2$ with this coloring *prohibited configuration of type* 4.

Lemma 7. *The contribution of this configuration is* $0(1/n^2)$.

Calculation. It is easy to see that the inequality $|A_1\cup A_2|\geq[3/2n]$ follows from the assumption of being substantially different. On the other hand, the number of the configurations which contain a fixed point P

$$\leq\log^2 n\cdot(n+\log^2 n)^4\cdot\frac{2^n}{n^\varepsilon}.$$

So it follows that the contribution

$$\leq\left(\frac{1}{2}\right)^{3/2n}\cdot 2^n\cdot n^4=0\left(\frac{1}{n^2}\right).$$

Now we define the subsets R_i. Suppose that A_i an arbitrary generalized edge. This means that $A_i=\{a_1,\ldots,a_t\}$, where $a_1<a_2<\cdots<a_t$ and $n\leq t<[n+$

$\log^2 n]$. Let

$$R_i = \{a_{[n/2]+j} | 1 \le j \le k\}, \text{ where } k \text{ constant depending only on } \varepsilon .$$

It is easy to see that the sets R_i satisfy the following condition:

$$|\{i: P \in R_i\}| \le \frac{2^n}{n^\varepsilon} \cdot \frac{\log^2 n \cdot k}{n} \qquad \text{for every} \qquad P \in X . \tag{2.1}$$

Definition. We call two generalized edges A_1, A_2 *almost disjoint* if $|A_1 \cap A_2| \le 1$.

Definition. Suppose that the sets $A, A_{i_1} \cdots A_{i_m}$ are substantially different edges and $A \cap A_{i_j} \in R_{i_j}$ for every $1 \le j \le m$ and $m = [5/\varepsilon]$. Let the color of the set $(\cup_{j=1}^m A_{i_j}) \cup A$ be red. We call the configuration with this coloring *configuration of type* 5. We say that a color configuration of type 5 is a *prohibited configuration of type* 5, if the sets $A, A_{i_1}, \ldots, A_{i_m}$ are pairwise almost disjoint.

Lemma 8. *The contribution of this configuration is* $0(1/n^2)$.

Calculation. Using the same counting approach as in Lemma 3, using (2.1) and the almost disjoint assumption we have the following upper bound:

$$\sum_{t=0}^{[5/\varepsilon]} C \cdot \left(\frac{\log^2 n}{n^\varepsilon}\right)^{[5/\varepsilon]-t} \cdot \left(\frac{(n+\log^2 n)^3 \cdot (\log^2 n)}{2^{n/2}}\right)^t = 0\left(\frac{1}{n^2}\right).$$

Remark. Because of the contributed configuration of type 4, we do not have to deal with configurations of type 5 which do not satisfy the almost disjoint property.

Definition. Suppose that the sets $C, A_1 \cdots A_m$ are substantially different generalized edges and $|C| = n$. Suppose that the set $C \cap (\cup_{j=1}^m A_j) = \{P_1, P_2, \ldots, P_m\}$ and the points $P_j \in R_j$ for every $1 \le j \le m$. Let the color of the set $\cup_{j=1}^m A_j$ be red and the color of the set $C \backslash \cup_{j=1}^m A_j$ blue. We call this configuration, *configuration of type* 6.

Lemma 9. *If a two coloration does not contain prohibited configurations of types* 4 *and* 5, *then any configuration of type* 6 *satisfies the following conditions*:

(i) The sets $A_1, \ldots, A_m$ are pairwise almost disjoint.

(ii) $|A_j \cap C| \le \left[\frac{5}{\varepsilon}\right]$ for every $1 \le j \le m$.

Proof. (i) is trivial. (ii) Suppose that $|A_j \cap C| \ge [5/\varepsilon] + 1$. Then $(A_j \backslash P_j) \cap C = \{P_{i_1} \cdots P_{i_t}\}$ where $t \ge [5/\varepsilon]$. Then the sets $A_{i_1}, A_{i_2} \cdots A_{i_{[5/\varepsilon]}}$ and A_j form a prohibited configuration of type 5. The contradiction proves Lemma 9. ∎

Definition. We call a configuration of type 6 *prohibited configuration of type 6.a*, if the sets C, $A_1 \cdots A_m$ satisfy the property (i), (ii) of Lemma 9 and

$$\frac{n}{2} \geq m \geq m_0 = \left[\frac{5}{\varepsilon}\right].$$

Definition. A configuration is *prohibited configuration of type 6.b*, if the subset is a configuration of type 6 which satisfy (i), (ii) and $m = [n/2]$ the set $\cup_{j=1}^{m} A_j$ is red, the coloring of the set $C \backslash \cup_{j=1}^{m} A_j$ is arbitrary.

Lemma 10. *The contribution of type* 6.*a and type* 6.*b are* $0(1/n^2)$. *The proof of the Lemma goes along exactly the same lines as the proof of Lemma* 3, *using the assumption* (*i*), (*ii*) *of the Lemma* 9.

Definition. We call an arithmetic progression of length n *almost blue edge* if there exists a configuration of type 6 in which our arithmetic progression agrees with C. Suppose that A is an arithmetic progression of length t, $A = \{a_1, \ldots, a_t\}$ where $n \leq t$ and the sets: $\{a_{i+j} | 1 \leq j \leq n\}$ are almost blue edges for every $0 \leq i \leq t - n$. In that case A is called *almost blue generalized arithmetic progression*. If $n \leq t < [n + \log^2 n]$ then we call the set A *almost blue generalized edge*.

Remark. Suppose that the sets A, A_1 are not substantially different arithmetic progressions of length $\geq n$ and $A_1 \subset A$. Then if A is almost blue then A_1 is almost blue also. Now suppose that a two coloring does not contain prohibited configurations of types 4, 5, 6.a, 6b. Then we know that every almost blue arithmetic progression of length n has $\leq [5/\varepsilon]$ red points. So if A is a generalized almost blue arithmetic progression of length $[n + \log^2 n]$, then A has $\leq 2 \cdot [5/\varepsilon]$ red points, because $A = A_1 \cup A_2$, where A_1 and A_2 are almost blue arithmetic progressions of length n.

Definition. Suppose that A is an arithmetic progression of length $[n + \log^2 n]$ and A has $\leq 2 \cdot [5/\varepsilon]$ red points. In that case A is called prohibited configuration of type 7.

Lemma 11. *The contribution of type* 7 *is* $0(1/n^2)$.

Calculation. Using Lemma 6, we have the following upper bound:

$$\left(\frac{1}{n}\right)^{(\log 2)\cdot n} \cdot \sum_{k=1}^{[5/\varepsilon]} \binom{[n + \log^2 n]}{k} \leq C \cdot \left(\frac{1}{n}\right)^{n \cdot \log 2 - [5/\varepsilon]} = 0\left(\frac{1}{n^2}\right).$$

Definition. Let the sets A_1 and A_2 be two substantially different generalized edges which have more than one common point. Suppose that A_1 have $\leq 2 \cdot [5/\varepsilon]$ red points and A_2 have $\leq 2 \cdot [5/\varepsilon]$ red points. In that case the set $A_1 \cup A_2$ with that coloring is called *prohibited configuration of type* 8.

Lemma 12. *The contribution of type* 8 *is* $0(1/n^2)$.

Calculation. Using Lemma 7 we have the following upper bound

$$\frac{n^4}{n^{\log 2\cdot n}} \cdot \left(\sum_{k=1}^{[5/\varepsilon]} \binom{n}{k} \right)^2 \leq C \cdot n^{-\log 2\cdot n+4+[5/\varepsilon]^2} = 0\left(\frac{1}{n^2}\right).$$

Remark. We know from the previous remark and Lemmas 10 and 11, that we do not have to deal with such two colorings, which contain almost blue arithmetic progressions of length $\geq [n + \log^2 n]$ or contain two substantially different almost blue generalized edges, which have more than one common points.

Definition of Type 9. Suppose that subset A is a generalized red edge and suppose that the sets $C_1, C_2 \cdots C_{[5/\varepsilon]}$ are substantially different almost blue generalized edges. It follows that for every $1 \leq i \leq [5/\varepsilon]$ there exist m_i generalized red edges $A_{i,1}, A_{i,2} \cdots A_{i,m_i}$ satisfying

$$C_i \cap R_{ij} \neq \phi \qquad \text{for every} \qquad 1 \leq j \leq m_i$$

and

$$C_i \cap R \neq \phi .$$

(It does not interest us whether these m_i sets $A_{i,1} \cdots A_{i,m_i}$ are substantially different or not). We call $\cup_{j=1}^{[5/\varepsilon]} (C_j \cup_{i=1}^{m_i} A_{j,i}) \cup A$ with the 2-coloring configuration of type 9. Since we have already prohibited a great deal of configurations we can suppose that

(i) C_i has $\leq 2 \cdot [5/\varepsilon]$ red points for every $1 \leq i \leq [5/\varepsilon]$,
(ii) $m_i \leq 2 \cdot [5/\varepsilon]$ for every $1 \leq i \leq [5/\varepsilon]$,
(iii) $|C_i \cap C_j| \leq 1$ for every pair $i \neq j$,
(iv) $|A_{i,j} \cup A_{i',j'}| < [n + \log^2 n]$ for every pair $A_{i,j}$, $A_{i',j'}$ which interfere with each other,
(v) $|A_{i,j} \cap A_{i',j'}| \leq 1$ for every pair which does not interfere with each other.

Definition. If a configuration of type 9 satisfies the property i–v then we call it prohibited configuration of type 9.

Lemma 13. *The contribution of type* 9 *is* $0(1/n^2)$. *The calculation is very similar to the case of type* 2. *Now we can use the properties i, ii, iii, iv, and v instead of the almost disjoint property. We get that the contribution of type* 9

$$\leq C \cdot \left(\frac{\log^2 n}{n^{\varepsilon}}\right)^{[5/\varepsilon]} = 0\left(\frac{1}{n^2}\right).$$

Now we prohibit all the configurations in the reversed cases.

First Coloring. Using Lemma 2 we get a coloring which contains no prohibited configurations of types 3, 4, 5, 6a, 6b, 7, 8, 9.

Definition. We call a point P "bad" blue point, if P is blue and there exists an almost red generalized edge which contains it. We call a point bad red point, if P red and there exists an almost blue generalized edge which contains it. We call P nice otherwise.

Suppose that A is monochromatic red edge. We recall that $|R| = k$ and k constant. Let $k = 2 \cdot [5/\varepsilon]^2 + 1$. We know that every almost blue generalized edge has $\leq 2 \cdot [5/\varepsilon]$ red points. So it follows from the lack of configuration of type 9 that at least one of the points of R is nice.

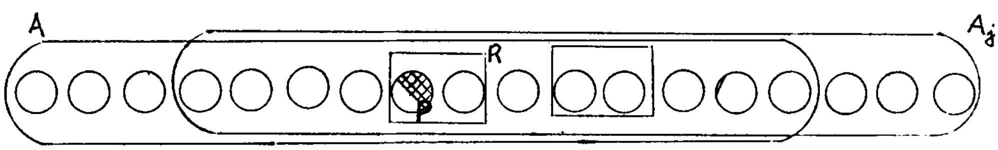
We say that two monochromatic arithmetic progressions of length n are in relation to each other if they interfere with each other. It follows from type 3: then this relation is an equivalence relation.

Second Coloring. We select representatives from every equivalence class. We get arithmetic progressions of length n—$A_{i_1}, \ldots, A_{i_S}$. Then we select nice points $P_{i_1}, \ldots, P_{i_S}$ satisfying $P_{i_j} \in R_{ij}$. Then we change the color of $Z = \cup_{j=1}^{S} P_{i_j}$. We claim that after the second coloring we get a good 2-coloring of X:

Let A be one of the representatives. We can suppose that A is red. We know that after the second coloring, A has at least one blue point P. On the other hand, it follows from type 5: then A has at most $[5/\varepsilon]$ blue points after the second coloring.

Suppose then A_1 is equivalence with A. We know from type 3 that

$$|A \cup A_j| < [n + \log^2 n].$$



We recall that if $A = \{a_1, \ldots, a_t\}$, then $R = \{a_{[n/2]+1}, a_{[n/2]+2}, \ldots, a_{[n/2]+k}\}$. Trivially, $P \in A_j$ and it is easy to see that A_j is properly two colored after the second coloring/

Suppose that A_i is an arbitrary edge which contains P and interferes with A. It is easy to see that A_i has at least one blue point and at least $[n/2] - k - [5/\varepsilon] - 1$ red points which means that A_i is also properly two colored after the second coloring.

Finally, suppose that the edge C satisfy the condition: Either $P \not\subseteq C$ or A does not interfere with C, for every selected representative A. Suppose that we were unlucky and C is monochromatic after the second coloring. We can suppose that C is blue. Let $Z \cap C = \{Q_1, \ldots, Q_l\}$. It follows from the construction of the second coloring that after the first coloring there exist substantially different edges $A_1 \cdots A_l$ which do not interfere with C and satisfy: $Q_i \in R_i \cap C$ for every $1 \leq i \leq l$, $\cup_{i=1}^{l} A_i$ is red, and $C \backslash \cup_{i=1}^{l} A_i$ is blue. It is easy to see that C is almost blue edges and Q_i are "bad" points for every $1 \leq i \leq l$. It is a contradiction to the construction of the second coloring, where we did not change the color of the "bad" points, and this contradiction proves our claim and Theorem 2. ▭

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REFERENCES

[1] J. Beck, "A remark concerning arithmetic progressions," *J. Comb. Theory A.*, **29**(3), 376–379 (1979).

[2] E.R. Berlekamp, 'A construction for partitions which avoid long arithmetic progressions," *Canad. Math. Bull.*, **11**, 409–414 (1968).

[3] P. Erdös and L. Lovász, "Problems and results on 3-chromatic hypergraphs and some related questions," in *Infinite and Finite Sets*, Colloquia Math. Soc., János Bolyai, Vol. 10, 1973.

[4] J. Spencer, "Asymptotic lower bounds for Ramsey functions," *Discrete Math.*, **20**, 69–76 (1976).

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