1 Introduction

Notation 1.1 Let $p_1, \ldots, p_n$ be labelled points in

$$U = [0,1]^2;$$

coincidences are allowed. Define

$$r_i = \min_{j \neq i} |p_i - p_j|,$$

Our goal is to find an upper bound on $\sum_{i=1}^n r_i^2$ We would like to also show that it is tight, meaning that some set of points achieves $\sum_{i=1}^n r_i^2$.

2 The Case of $n = 4$

Theorem 2.1 *There is a set of four points p_1, p_2, p_3, p_4 such that $\sum_{i=1}^n r_i^2 = 4$.*

Proof: Take the p_i's to be the four corners of the unit square. ∎

We show that, for all p_1, p_2, p_3, p_4, $\sum_{i=1}^4 r_i^2 \leq 4$.
We need some lemmas first.

Lemma 2.2 *The graph K_4 has exactly three Hamiltonian cycles.*

Proof: Let the vertices be $1, 2, 3, 4$. We assume that 1 is both the starting and ending point. We list all sequences of length four that begin with 1. We note which sequences are cycle-equivalent.

1. $(1, 2, 3, 4)$
2. $(1, 2, 4, 3)$
3. $(1, 3, 2, 4)$
4. $(1, 3, 4, 2) \equiv (1, 2, 4, 3)$.
5. $(1, 4, 2, 3) \equiv (1, 3, 2, 4)$.
6. $(1, 4, 3, 2) \equiv (1, 2, 3, 4)$.

∎

Lemma 2.3 *Let p_1, p_2, p_3, p_4 be four distinct points in the unit square. Let G be the weighted graph with vertices $\{p_1, p_2, p_3, p_4\}$, all of the edges, and the weight of (p_i, p_j) is $d(p_i, p_j)^2$. Then there is a Hamiltonian cycle $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$ such that*

$$d(p_{i_1}, p_{i_2})^2 + d(p_{i_2}, p_{i_3})^2 + d(p_{i_3}, p_{i_4})^2 + d(p_{i_4}, p_{i_1})^2 + \leq 4.$$

Proof: Let $p_i = (x_i, y_i)$. Relabel the points so that $x_1 \leq x_2 \leq x_3 \leq x_4$.

Let

$d_{12} = d(x_1, x_2)$

$d_{23} = d(x_2, x_3)$

$d_{34} = d(x_3, x_4)$

$d_{41} = d(x_4, x_1)$.

Note that

- $dx_{12} + dx_{23} + dx_{34} = dx_{14} = dx_{41}$.
- $dx_{12} + dx_{24} = dx_{14} = dx_{41}$.
- $dx_{43} + dx_{13} = dx_{13} + dx_{34} = dx_{14} = dx_{41}$.

We look at all three Hamiltonian cycles from Lemma 2.2 and compute the sum of square differences of the x-values.

1. $(p_1, p_2, p_3 3, p_4)$

$$dx_{12}^2 + dx_{23}^2 + dx_{34}^2 + dx_{41}^2 \leq (dx_{12} + dx_{23} + dx_{34})^2 + dx_{41}^2 \leq 2dx_{41}^2 \leq 2$$

2. (p_1, p_2, p_4, p_3)

$$(dx_{12}^2 + dx_{24}^2) + (dx_{43}^2 + dx_{13}^2) \leq (dx_{12} + dx_{24})^2 + (dx_{43})^2 + dx_{13}^2) \leq 2dx_{41}^2 \leq 2$$

3. (p_1, p_3, p_2, p_4)

$$dx_{13}^2 + dx_{32}^2) + (dx_{24}^2 + dx_{41}^2)$$

This one we are not going to bound.

Of the three Hamiltonian cycles, two of them have sums of squares of x-differences ≤ 2.

By the exact same reasoning:

of the three Hamiltonian cycles, two of them have sums of squares of y-differences ≤ 2.

There must be at least one Hamiltonian cycle that has both properties. By renumbering assume its (p_1, p_2, p_3, p_4).

$d(p_1, p_2)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$d(p_2, p_3)^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2$

$d(p_3, p_4)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2$

$d(p_4, p_1)^2 = (x_1 - x_4)^2 + (y_1 - y_4)^2$

Hence

$d(p_1, p_2)^2 + d(p_2, p_3)^2 + d(p_3, p_4)^2 + d(p_4, p_1)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 + (x_4 - x_3)^2 + (y_4 - y_3)^2 (x_1 - x_4)^2 + (y_1 - y_4)^2 = dx_{12}^2 + dx_{23}^2 + dx_{34}^2 + dx_{41}^2 + dy_{12}^2 + dy_{23}^2 + dy_{34}^2 + dy_{41}^2 + \leq 2 + 2 = 4$

∎

Theorem 2.4 *Let p_1, p_2, p_3, p_4 be points in the unit square. Then $\sum_{i=1}^{n} r_i^2 \leq 4$.*

Proof: If the points are not distinct then we can move them so that they are. This movement can only add to the sum. Hence we assume p_1, p_2, p_3, p_4 are all distinct.

Let G be the weighted graph with vertices $\{p_1, p_2, p_3, p_4\}$, all of the edges, and the weight of (p_i, p_j) is $d(p_i, p_j)^2$. By Lemma 2.3 there is a Hamiltonian cycle sum of the weights ≤ 4. By renumbering assume it is (p_1, p_2, p_3, p_4).

Note the following:

$r_1 \leq d(p_1, p_2)$

$r_2 \leq d(p_2, p_3)$

$r_3 \leq d(p_3, p_4)$

$r_4 \leq d(p_4, p_1)$

Square each equation, add them, and use Lemma 2.3 to get

$$\sum_{i=1}^{4} r_i^2 \leq d(p_1, p_2)^2 + d(p_2, p_3)^2 + d(p_3, p_4)^2 + d(p_4, p_1)^2 \leq 4.$$

∎