Review of[1]
Ramsey Theory over the Integers (Second Edition)
by Bruce M. Landman and Aaron Robertson
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1 Introduction

Three classic theorems of Ramsey Theory are:

1. *Ramsey's Theorem:* For all r, m there exists n such that for all f-colorings of the edges of K_n there is a monochromatic K_m (m vertices such that all of the edges between them are the same. This extends to many colors and hypergraphs.

2. *Van Der Warden's Theorem (henceforth VDW's theorem:* For all r, k there exists w such that for all r-colorings of $[w] = \{1, \ldots, w\}$ there is a monochromatic arithmetic sequence of length k (henceforth a k-AP). The Gallai-Witt theorem is a generalization to more dimensions. The Hales-Jewitt theorem is a further generalization.

3. *Schur's Theorem* For all r there exists n such that for all r-colorings of $[n]$ there exists x, y, z the same colors such that $x + y = z$.

Most books on Ramsey Theory focus on both Ramsey's Theorem and VDW's theorem and variants of them (Schur's theorem when generalized can be seen as a variant of VDW's theorem). This book focuses *just* on VDW's theorem and variants of it that *only* involve colorings of initial segments of N. Hence the book goes wide rather than deep. As such the material is accessible to an undergraduate. There are many problems and ideas for research problems in this book that could be tackled by an undergraduate. The authors state accessibility for undergraduates and ideas for projects as explicit goals.

2 Summary of Contents

The first chapters introduce Ramsey Theory and give statements of the classic theorems above. The second chapter states and proof VDW's theorem and gives upper and lower bounds on some VDW numbers. The proof is complete and rigorous. This is a welcome and not common. The proof of VDW's theorem is such that its easier to go part way and leave the rest as an exercise since its messy to write down. By using the color-focusing way to present the theorem they obtain a complete rigorous proof. One downside: they do not present the proof of VDW's theorem for 2 colors and 3-AP's. that uses 2-colorings of blocks-of-5 integers, which is nice to see for intuition. This proof (any version of it, any presentation of it) yields rather large upper bounds for $W(r, k)$ (not primitive recursive). Shelah had an elementary proof that yielded smaller numbers (primitive recursive) though still quite large. Gowers had a proof using advanced mathematics that yielded a bound that was *only* a finite stack of exponentials. Neither of these proof is given. Note that the search for matching upper and lower bounds, and for an elementary proof of more reasonable upper bounds, is ongoing.

Chapters three, four, five, and six have variants of VDW's theorem by replacing AP by other types of sets of numbers:
Chapter three:

1. A *k-term quasi progression of diameter* n is a set of the form $\{x_1 < \cdots < x_k\}$ such that there exists d, for all i, $d \le x_{i+1} - x_i \le n + d$. This can be generalized by letting d be a function of i.

2. Let $k \ge 3$. A *k-term descending wave* is a set of the form $\{x_1 < \cdots < x_k\}$ such that, for all i, $x_i - x_{i-1} \le x_{i+1} - x_i$.

3. A *k-term semi-progression of scope* m is a set of the form $\{x_1 < \cdots < x_k\}$ such that, for all i, $x_i - x_{i-1} \in \{d, 2d, 3d, \ldots, md\}$.

4. A *k-term* p_n*-sequence* is a set of the form $\{x_1, \ldots, x_k\}$ such that there exists a polynomial $p(x) \in \mathsf{Z}[x]$ of degree n such that for all i, $x_{i+1} = p(x_i)$.

By VDW's theorem there for all r, k there is a w such that there is a k-term set of any of the above types to exist (for the p_n sequence you would take $n = 1$). However, this would yield rather large upper bounds. In Chapter three they study all of the above types of sequences and get much more reasonable upper and lower bounds than are known for VDW numbers. We give one example.

Let $DW(k)$ be the least w such that for all 2-colorings of $[w]$ there is a k-term descending wave. They prove $k^2 - k + 1 \le DW(k) \le \frac{k^3}{2} - \frac{k^2}{2} + 1$. They state without proof that more is known: there is a c such that $ck^2 \le DW(k) \le \frac{k^3}{3} - \frac{4k}{3} + 3$.
Chapter four deals with restricting the differences of the AP:

1. A set D is *r-large* if, for all k there exists w such that for all r-colorings of $[w]$ there is a mono set with difference in D. They consider this for both finite and infinite D.

2. A special case of interest: $w'(c, k; r)$ is the least w such that for all r-colorings of $[w]$ there is a mono k-AP with difference $\ge c$. Such a w always exists by VDW's theorem, but can we get a different proof with a better bound?

3. Let f be an increasing function from N to N. $w(f(x), k; r)$ is the least w such that there is a mono set $\{x_1 < x_2 < \cdots < x_k\}$ such that, for all, i $x_{i+1} - x_i = f(i)$.

A set D need not be r-large for any r. There are functions f such that $w(f(x), k; r)$ does not exist. Chapter 4 has theorems about both when these things do and do not happen. We give two examples. (1) If D is finite then for D is not 2-large. (2) The Fibonacci numbers are not 4-large. (3) If $p(x) \in \mathsf{Z}[x]$ and $p(0) = 0$ then the image of p is large (this is stated by not proven).

Chapters five has 15 pages on sequences of the form $\{x, ax + d, bx + 2d\}$. They then look at homothetic copies of sequences. Chapter six deals with the differences not being equal but being congruent mod m for some m. Chapter seven offers yet more variants on the notion of an AP.

Chapters eight and nine are on a different kind of variant than those of chapters three through seven. Recall VDW's theorem for $r = 2$ and $k = 4$:

There exists w such that for all 2-colorings COL of $[w]$ there exists a, d such that

$$COL(a) = COL(a+d) = COL(a+2d) = COL(a+3d).$$

We rewrite this in terms of equations.

There exists w such that for all 2-colorings COL of $[w]$ there exists distinct e_1, e_2, e_3, e_4 such that

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4)$$

and

$$\begin{aligned} e_2 - e_1 &= e_3 - e_2 \\ e_2 - e_1 &= e_4 - e_3. \end{aligned}$$

We rewrite these equations:

$$\begin{aligned} e_1 - 2e_2 + e_3 + 0e_4 &= 0 \\ e_1 - e_2 - e_3 + e_4 &= 0. \end{aligned}$$

Hence VDW's theorem for $r = 2$ and $k = 4$ can be rewritten as:

There exits w such that for all 2-colorings COL of $[w]$ there exist distinct e_1, e_2, e_3, e_4 such that

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4),$$

$$A\vec{e} = \vec{0},$$

where A is

$$\begin{pmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and $\vec{e} = (e_1, e_2, e_3, e_4)$.

What other matrices have this property? Schur's theorem, which was proven before VDW's theorem, answers this question for the single equation $x + y - z = 0$. Chapter eight looks at variants of Schur's theorem including counting how many monochromatic solutions there are and generalizing to equations in more than one variable. The full generalization: Let $(a_1, \ldots, a_n)$ be a tuple of integers. Let $L(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_k$. The following are equivalent

- For all r, k there exists w such that for all r-colorings of $[w]$ there exists a monochromatic solution of $L(\vec{x}) = 0$.
- Some subset of $\{a_1, \ldots, a_n\}$ sums to 0.

Rado's theorem gives a condition C on matrices A (which we omit her but is in the book) such that the following are equivalent.

- For all r, k there exists w such that for all r-colorings of $[w]$ there exists a monochromatic solution of $A(\vec{x}) = 0$.
- A satisfies condition C.

Chapter nine has Rado's theorem and many variants. Chapter ten has still more variants on VDW type theorems.

3 Opinion

This book is excellent for what is set out to do: there are many variants of VDW's theorem and they can form the basis for many student projects. This is not just my speculation: the first edition of this book *did* inspire many projects and many of the open problems from it have been solved. The book is well written.

From the review one might think *Gee, there are A LOT of variants. Are they all interesting?* This is a matter of taste. But one cautionary note to not get overwhelmed by them: Pick A FEW that you find interesting and read about those. You may read others later but don't (as I did) read the book in a week.

The Gallai-Witt theorem is omitted, which makes sense since this is about Ramsey Theory over *the integers.* However, just the theorem *for all 2-colorings of the lattice points in the plane there is a mono square* is on the level of undergraduates, and would have been nice. They omit the Hales-Jewitt theorem which is a very wise decision since it is somewhat abstract and seems to be just the right cut-off point.

They do not have the polynomial VDW theorem though they do state a corollary of it (if $p(x) \in \mathsf{Z}[x]$ and $p(0) = 0$ then the image of p is large). I (1) state the full Poly VDW, (2) say why I think it should have been included, and (3) say why I might be wrong.

Polynomial VDW theorem: For all r, for all $p_1, \ldots, p_k \in \mathsf{Z}[x]$ such that $(\forall i)[p_i(0) = 0]$, there exists W such that, for all r-colorings of $[W]$ there exists a, d such that

$$a, a + p_1(d), a + p_2(d), \ldots, a + p_k(d)$$

are all the same color.

This theorem has an elementary proof, due to Walters, that the authors know about (Walters paper is in the bibliography). The proof uses the color-focusing method which is the same method used in the books proof of VDW theorem. This theorem clearly fits into the theme of VDW on the integers. Undergraduate projects can be made out of (I speak from experience). There is a very nice contrast between the proof of VDW's theorem, which is an ω^2-induction, and the proof of Poly VDW which is an ω^ω-induction.

So why might I be wrong? First off, the book is already 380 pages. By contrast *Joy of Factoring* and *Asymptopia* which are in the same series, are 293 and 189 pages respectively. Second, and I admit this freely, my view may be prejudiced since the elementary proof of Poly VDW is what got me into Ramsey theory in the first place. Having said that, if the writers ever do a third edition, they should at least consider the idea.