Review of[1]
An Introduction to Ramsey Theory:
Fast Functions, Infinity, and Metamathematics
by Matthew Katz and Jan Reimann
AMS, 2018
207 pages, Softcover, $52.

Review by
William Gasarch (`gasarch@cs.umd.edu`)
Department of Computer Science
University of Maryland, College Park, MD

1 Introduction

The first theorem in Ramsey Theory is:

> For all 2-colorings of the edges of K_6 there is a monochromatic K_3.

This generalizes to the first real theorem:

> For all m there exists $R = R(m)$ such that, for all 2-colorings of the edges of K_R there is a monochromatic K_m.

More generally, Ramsey theory is a branch of combinatorics that deals with statements of the form

> If you color a large enough BLAH you will have a nice monochromatic sub-BLAH.

The book under review is at first a standard book on Ramsey Theory, but which then takes a turn into connections to logic. Logic? Where does logic come in?

Recall Gödel's incompleteness theorem which we summarize as:

> There are statements S such that S is true of the natural numbers but cannot be proven in Peano Arithmetic.

This is a very important theorem since it shows that Peano Arithmetic cannot do everything in Number Theory. However, the statement S is not natural. Paris and Harrington came up with a natural statement in Ramsey theory that is not provable in Peano Arithmetic. I have always wanted a clean self-contained treatment of the Paris-Harrington result and why it is not provable in Peano Arithmetic. Is this book that treatment? Yes!

We give some definitions that we use throughout the review:

Definition

1. PA is Peano Arithmetic. It is a system of axioms and rules of inference. Most theorems in number theory can be proven in it. Almost all (all but a finite number :-)) interesting theorems in number theory can be proven in it.

2. If ϕ is a statement that cannot be proven in PA then we write $PA \vdash \phi$. Note that ϕ must be written in the language of PA.

3. If ϕ is a statement and it cannot be proven in PA then we write $PA \not\vdash \phi$. Note that ϕ must be written in the language of PA. If ϕ was a theorem in analysis (e.g., the intermediate value theorem) then technically $PA \not\vdash \phi$ is true but one would never write that.

4. PH is the Paris-Harrington statement which I will define later in this review.

Definition: Let $k, n \in \mathbb{N}$. Then $[n]$ denotes the set $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ is the set of all k-sets of $[n]$. $\binom{\mathbb{N}}{k}$ is the set of all k-sets of $\mathbb{N}$.
Definition: Let $c \in \mathbb{N}$ and let COL be a c-coloring of $\binom{[n]}{k}$ (respectively $\binom{\mathbb{N}}{k}$). We say $H \subseteq [n]$ (respectively $H \subseteq \mathbb{N}$) is *homogenous with respect to COL* if all elements of $\binom{H}{k}$ are the same color.

2 Summary of Contents

Chapter 1 is mostly standard material on Finite Ramsey Theory. We state Ramsey's Theorem for hypergraphs, which is one of the results they prove.

> **Ramsey's Theorem for Hypergraphs:** For all c, k, m, there exists n such that for all c-colorings of $\binom{[n]}{k}$ there is a homogenous set of size m.

The book gives proofs, upper bounds on n, and lower bounds on n. The lower bounds were shown by the probabilistic method. One could say this is an application of the probabilistic method but that's a bit odd since the probabilistic method was originally invented by Paul Erdős for this purpose. The authors also give a proof of Ramsey's Theorem (for hypergraphs) by Nesetril which is more abstract and new (at least to me).

Chapter 2 is about the infinite Ramsey Theorem:

> **Ramsey's Theorem for Infinite Hypergraphs:** For all c, k, for all c-colorings of $\binom{\mathbb{N}}{k}$, there is an infinite homogenous set.

This can be proved directly, or from the finite Ramsey Theorem. Alternatively one can prove the finite Ramsey Theorem from the infinite. This chapter looks at variants of this on cardinals and large cardinals.

Chapter 3 is about the Growth of Ramsey Numbers
We state van der Waerden's (VDW) Theorem:

> **Theorem:** For all c, k there exists $W = W(k, c)$ such that for all c-colorings of $[W]$ there exists a, d such that a, $a+d$, ..., $a+(k-1)d$ are all the same color. (A monochromatic arithmetic sequence.)

They present the original proof that gives Ginormous bounds on $W(k, c)$. They then give Shelah's' proof of the Hales-Jewitt Theorem which yields much smaller bounds on $W(k, c)$. They don't seem to mention (unless I missed it) that Gowers gave a proof with even smaller bounds. If this is their example of a large growing Ramsey function it's not quite right. The proof of VDW's Theorem does give a bound that grows very fast, but what it is bounding ends up not being that fast growing. The distinction should have been made more clear. Having said all that, this is a fine presentation of the proofs given.

They also prove the Paris-Harrington (PH) Ramsey Theorem:

Definition: A *large set* is a finite set of $\mathbb{N}$ where the size of the set is bigger than the smallest element.

One might wonder if you can get a homogenous set that's large. That's not quite right since if 1 is in the set then it's already homogenous. But what if you start the numbering of the vertices with a larger number? With that in mind:

> **PH Ramsey's Theorem for Hypergraphs:** For all c, k, there exists n such that for all c-colorings of $\binom{\{k,k+1,\ldots,k+n\}}{k}$ there is a large homogenous set of size m. We denote m by $PH(k, c)$.

One way to prove this is from the Infinite Ramsey Theorem. This proof does not give any bounds on n. Is there a proof that gives bounds on n? See Chapter 4.

Chapter 4 is the proof that $PA \not\vdash PH$. One consequence of this is that there is no proof that gives bounds on $PH(k, c)$.

There are two ways to prove $PA \not\vdash PH$. One way is to show that the PH function grows so quickly that it can't be proven to exist in PA. The other way is to use non-standard models of PA, indiscernibles, and, ironically, Ramsey Theory! That is the way the authors proved it. Their presentation is self-contained and can be followed by a non-logician. It will take some time to get through but is well worth it.

3 Opinion

This is a great book! This finally gives me the self-contained treatment of $PA \not\vdash PH$ that I feared I would have to write myself if someone else didn't write it. It is well written and gives the reader just enough logic to understand the proof.

I consider the $PA \not\vdash PH$ to be the point and highpoint and purpose of the book. However, there is other good material in there as well on both Ramsey Theory and Logic.

Who should read this book? Who shouldn't read this book?

Alice is thinking of reading this book. Alice's knowledge of Ramsey Theory is α, and of logic is β. Alice's mathematical maturity is γ. We take $\alpha, \beta, \gamma \in \{0, 1, \ldots, 100\}$. For what values of α, β, γ should Alice read this book? Clearly for $(\alpha + \beta)\gamma \geq 150$. I am, of course, kidding.

If Alice knows some Ramsey Theory but little logic, she can skim the Ramsey Theory and learn logic. She won't just learn the logic needed for $PA \not\vdash PH$, she will also learn about large cardinals, ordinals, and some non-standard models of PA. She will even learn how to apply Ramsey theory to logic.

If Alice knows some logic but little Ramsey Theory she can skim the logic and learn Ramsey Theory. She won't just learn Ramsey Theory. She will learn about the interactions of Ramsey Theory to her field of Logic.

If Alice knows neither but has lots of math maturity she could read and understand the book, though it will be a tough read.

Only if Alice lacks knowledge and maturity would this book be too hard for her.