Review of[1]

Structure and Randomness:
Pages from Year One of a Mathematical Blog
by Terence Tao
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1 Introduction

In the movie *Oh God! Book II* God (played by George Burns) says:

Mathematics, that was a mistake. I should have made the whole thing a little easier.

While I am not one to argue with God, George Burns, or George Burns playing God, I do not think Mathematics was a mistake. But I do wish it was easier. Or perhaps its easy enough to make so much progress in that it becomes hard.

Terence Tao has a math blog that I try to read but find difficult. Often the mathematics itself is beyond me, but other times I have a sense that I really could understand it if I just gave it a bit more time. How to get that time? I can't really explain it, but having the blog in book form really makes a difference for me. I made the same comment when reviewing both books based on the Blog *Godel's Last Letter* and will likely make the same comment if I review Scott Aaronson's upcoming blog-book. And I don't think its my inner-Luddite talking, as many non-Luddites I've spoken to agree with me.

Making the blog entries into a book removes one obstacle. Now the question arises, is the book worth reading? The short answer is yes. The long answer is that there are several types of chapters– some I could read, some I really couldn't, and some are inbetween. I review the book by giving some examples of each type.

2 I Understood The Entire Chapter!

The chapter *Soft Analysis, Hard Analysis, and the Finite Convergence Principle* discusses the difference between Soft Analysis, which seeks theorems about infinite objects, and Hard Analysis which seeks theorems about finite objects and (the hard part) concrete bounds. His point is that these two are not that far apart and can help each other. He then gives a great example: The infinite cvg theorem from soft analysis. The theorem is:

Every bounded monotone sequence of reals converges.

He then discusses what the *finite cvg theorem* is, going through several candidates. The final theorem deserves to be the analog of the infinite convergence theorem since *the infinite cvg theorem and the finite cvg theorem are equivalent!* and he proves it. I understood this chapter so well that I gave a talk about it in seminar.

The chapter *The Crossing Number Inequality* presents the crossing number inequality and two applications, all with proofs.

Let G be a graph, v be the number of vertices, e be the number of edges, and c be the crossing number (the least number c such that the graph can be drawn in the plane with c crossings). The Crossing Number Inequality states that if $e \geq 4v$ then $c = \Omega(e^3/v^2)$. (The constant obtained by Tao is $\frac{1}{64}$ though better is known with a stronger condition on e and v.)

Let P be a finite set of points in the plane. Let L be a finite set of lines in the plane. What is the maximum number on incidences of points in P on lines in L? The Szemerdi-Trotter theorem proved that it is bounded by $O(L^{2/3}P^{2/3} + L + P)$. Tao shows how this can be derived from the Crossing Number Inequality, which he credits to Szekely.

Given A, a set of reals, one can look at $A+A = \{x+y \mid x, y \in A\}$ and $AA = \{xy \mid x \in A, y \in A\}$. Must it be the case that one of these sets is large? Using the Szemerdi-Trotter theorem one can show that either $|A + A|$ or $|AA|$ is of size $\geq |A|^{1+0.25}$.

This is an excellent exposition of some math that is fairly easy and interesting. It also shows Tao's range: he can talk about hard math, easy math, pure math, applied math, and everything in between.

My favorite chapter was *Hilbert's Nullstellensatz* (henceforth HN). He begins with what is a common dilemma: the proof he has seen of HN is a bit too abstract (too abstract for Terence Tao-Yikes!), and not computational. So he came up with a proof that is more computational and less abstract. Is it new? This is the great thing about blogs— **I don't care!**. Tao thinks it might be an old proof presented differently, but none of this is important. Whats important is that there is a proof of HN which I can and will present in seminar!

What is the HN? I state what he calls the weak HN:

Let F be a fixed algebraically closed field. Let $d \geq 1$. Let $P_1, \ldots, P_m \in F[x_1, \ldots, x_d]$. Then either

1. There exists $\vec{a} \in F^d$ such that $P_1(\vec{a}) = \cdots = P_m(\vec{d}) = 0$, or
2. There exists polynomials $Q_1, \ldots, Q_m \in F[x_1, \ldots, x_d]$ such that $P_1Q_1 + \cdots P_mQ_m = 1$.

3 I Understood Something Interesting from the Chapter!

Most chapters in the book are in this category: I got something out of it but it really was a shade (or several) over my head. Many of them are in the category of *Now I know that that piece of hard math relates to that other piece of hard math.*

The chapter *Ultrafilters, non-standard analysis, and ϵ-management* interested me since I've heard about *applications of logic to "real math"* and heard debates about the issue: are there any (clearly yes), are there many (not clear), and will the algebraic geometer of the future need to know model theory (most algebraic geometer hope the answer is no). Tao claims that using non-standard analysis and ultrafilters can clean up some proofs and he gives some examples.

The chapter *Ratner's Theorem* uses Ratner's theorem, which is about topological spaces and closures, to prove a result in number theory that any undergraduate can understand. I learned the following:

1. If Q is a positive definite quadratic form (which may have irrational coefficients) then $Q(Z^d)$ is a discrete set of positive reals.
2. If Q is a positive definite quadratic form with integer coefficients represents all positive naturals ≤ 290 then it represents all positive naturals.

3. If Q is not positive definite then can $Q(Z^d)$ be dense?

   (a) If there are just two variables then no: Take $Q(x,y) = x^2 - \phi^2 y^2 = (x - \phi y)(x + \phi y)$ where ϕ is the golden ratio. There is an interval around 0 where there is no element of $\{Q(x,y) \mid x,y \in Z\}$.

   (b) There is a difficult proof by Margulis that for all $Q(x,y,z)$ that are not positive definite and have irrational coefficients, $Q(Z^3)$ is dense.

   (c) This result can also be obtained from Ratner's theorem easily.

4 What Else is in the Book?

There are chapters on physics, applied math, open problems. There are expositions of fields (Structure and Randomness, Arithmetic Combinatorics). Lets just say there is a lot in this book.

5 Opinion

Who should read this book? You have to already like mathematics and know some. An excellent undergraduate math major could get something out of some of the chapters. She may also be inspired to learn more. I can imagine any chapter becoming a reading-course or even a research project. The more math you know the more of it you can understand. But be warned- the sheer breadth of knowledge in this book will render some fraction of it not really understandable.

Terence Tao is an excellent writer. Its impressive that he can run a blog that is both well written and has rather hard math in it. The more you put into reading his blog (perhaps in book form), the more you'll get out of it.