Review of[1]
Weakened Ramsey Theory
by Mark R. Budden
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1 Ramsey Theory and Weak Ramsey Theory

The following is well known:

For all 2-colorings of the edges of K_6 there is a monochromatic triangle. That is, there are three vertices so that the edges between them are all the same color.

Ramsey's Theorem generalizes this:

Theorem 1.1 *For all k, c there exists n such that, for all c-colorings of the edges of K_n, there is a monochromatic K_k. We denote the least such n by $R^c(k)$.*

The following is known:

$$128 \leq R^3(4) \leq 229.$$

This means that:

- There is a 3-coloring of K_{128} with no mono K_4. [2]
- For all 3-coloring s of K_{229} there is a mono K_4. [1]

What is you *weaken* your goals? Instead of wanting a mono K_4, you want a K_4 that only uses 2 of the 3 colors.

Def 1.2 Let $n, s, t \in \mathsf{N}$ such that $2 \leq s \leq t$. Let G be a graph.

1. $[n] = \{1, \ldots, n\}$.

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2. $\binom{[n]}{2}$ is the set of all unordered pairs of elements from $[n]$. We use this for the set of edges of K_n.

3. Let $s, t \in \mathsf{N}$ such that $2 \leq s \leq t$.

4. Let COL: $\binom{[n]}{2} \to [t]$. An *s-mono* G is a copy of G where the edges use $\leq s$ colors.

5. Let G be a graph. $r_s^t(G)$ is the least n so that for all t-coloring of $\binom{[n]}{2}$ there is a an s-mono G.

For the problem at hand we are looking for $r_2^3(K_4)$ which is, the least n such that for all 3-colorings of $\binom{[n]}{2}$ there is a 2-mono K_4. Clearly this will be $\leq R^3(4)$. Of more important, there may be tighter bounds on it. There are!

Theorem 1.3 $r_2^3(K_4) = 10$.

This is an example of *weakened Ramsey Theory*. The general philosphy is:

Take a theorem in Ramsey Theory that demands a mono K_k (or some other structures) and weaken the conclusion to obtaining a s-mono K_k. While the conclusion is weaker, note that the number needed to achieve it will be smaller, which is a plus.

Does this lead to interesting math? It does! The next section showcases a such a theorem.

2 Weakened Ramsey Numbers

We want a lower bound on r_2^3. To state their theorem we need some more concepts and notation.

Def 2.1 Let $\mathsf{G} = (G, +, e)$ be a group. Let $S \subseteq G$ such that (1) $e \notin S$ and (2) for all $a \in S$, $a^{-1} \in S$. The *Cayley graph* $\mathrm{Cay}(\mathsf{G}; S)$ has vertex set G and edge set

$E = \{(x, y) \colon (\exists s \in S)[sx = y]\}$.

Convention 2.2 The notation $\mathrm{Cay}(\mathsf{F}_q, \mathsf{F}_q^{\times 3})$ means the following:

- The F_q is the *additive group* of F_q.
- The $\mathsf{F}_q^{\times 3}$ is the nonzero cubes in F_q.

Notation 2.3 Let $G = (V, E)$ be a graph.

1. $\omega(G)$ is the size of the largest clique in G.
2. $\overline{G}$ is the complement of the graph. Formally this is $(V, \binom{V}{2} - E)$.

We can now state an interesting theorem

Theorem 2.4 *Let p be a prime such that $p \equiv 1 \pmod 6$. Let $n = \omega(\text{Cay}(\mathsf{F}_q, \mathsf{F}_q^{\times 3})$. Then $r_2^3(K_{n+1}) \geq p + 1$.*

Note how marvelous this is: the question in weak ramsey theory about $r_2^3(K_{n+1})$ uses group theory!

The book has many results about $r_s^t(G)$ for (1) $G = K_k$, (2) $G = K_{1,n}$, (3) $G = W_n$, and others. There are also results about only allowing Gallai colorings (those with no rainbow triangles).

3 Schur's Theorem and Weak Schur's Theorem

The following is a variant of a theorem of Schur.

Theorem 3.1 *For all m, t there exists n such that, for all t-colorings of $[n]$, there is a set $x_1 < \cdots < x_m$ such that*

- $x_1 + \cdots + x_{m-1} = x_m$,
- $x_1 < \cdots < x_m$, *and*
- *all of the $x_1, \ldots, x_m$ are the same color.*

The least such n is denoted $\hat{S}^t(m)$ (the version without the hat does not require that the elements all be distinct).

Very few values of $\hat{S}^t(m)$ are known. What is we *weaken* our goal? Instead of wanting a mono solution you want a solution that only uses some fixed $s \leq t$ colors.

Def 3.2 Let $m, s, t \in \mathsf{N}$ such that $2 \leq s \leq t$.

1. Let $\text{COL}[n] \to [t]$. A solution to $x_1 + \cdots + x_{m-1} = x_m$ is s-mono if it uses $\leq s$ of the colors.
2. $\hat{S}^t_s(m)$ is the least n so that for all t-coloring of $[n]$ there is a s-mono solution to $x_1 + \cdots + x_{m-1} = x_m$.

The book has some theorems about $\hat{S}^t_s(m)$. We state one of them.

Theorem 3.3 *For all $n \geq 3$:*

For all $1 \leq i < j \leq n$, $x_{ij} = x_{ji}$.

$$\hat{S^t_2}(3) = \begin{cases} \frac{6t+5}{5} & \text{if } t \equiv 0p \pmod 5; \\ \frac{6t+9}{5} & \text{if } t \equiv 1p \pmod 5; \\ \frac{6t+3}{5} & \text{if } t \equiv 2p \pmod 5; \\ \frac{6t+7}{5} & \text{if } t \equiv 3p \pmod 5; \\ \frac{6t+6}{5} & \text{if } t \equiv 4p \pmod 5; \end{cases} \tag{1}$$

4 Other Weak Ramsey Theorems

The book also discusses weak version of Ramsey theory on Hypergraphs, and weak versions of Rado's Theorem (a generalization of Schur's theorem to other linear homogenous equations).

5 Opinion

Asking for the weak version of other theorems in Ramsey Theory is a –great idea for undergraduate projects.

References

[1] L. Boza. New upper bounds for the classical Ramey numbers r(4,4,4), r(3,4,5), r(3,3,6), 2026.
`https://arxiv.org/abs/2603.10851`.

[2] R. Hill and R. W. Irwing. On group partitions associated with lower bounds for symmetric Ramsey numbers. *European Journal of Combinatorics*, 3:35–50, 1982.