

Problems with a Point: Exploring Math and Computer Science

**William Gasarch
Clyde Kruskal**

Book's Origin

- ▶ In 2003 Lance Fortnow started **Complexity Blog**
- ▶ In 2007 Bill Gasarch joined and it was a co-blog.
- ▶ In 2015 various book publishers asked us

Can you make a book out of your blog?

- ▶ Lance declined but Bill said **YES**.

Book's Point

Bill took the posts that had the following format:

- ▶ make a point **about** mathematics
- ▶ do some math to underscore those points

and made those into chapters.

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- ▶ make a point **about** mathematics
- ▶ do some math to underscore those points

and made those into chapters.

Caveat: Not every chapter is quite like that.

To quote Ralph Waldo Emerson

A foolish consistency is the hobgoblin of small minds.

Possible Subtitles

Problems with a Point needed a subtitle.

I proposed

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**Problems with a Point: Mathematical Musing and Math to
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The publisher wisely decided to be less cute and more informative:

Problems with a Point: Exploring Math and Computer Science

Clyde Joins the Project!

After some samples of Bill's writing the publisher said

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After some samples of Bill's writing the publisher said

Please Procure People to Polish Prose and Proofs of Problems with a Point

so

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Now onto some samples of the book!

Point: Students Can Give Odd Answers: Easy

From the Year 2000 Maryland Math Competition:

There are 2000 cans of paint. Show that at least one of the following two statements is true:

- ▶ There are at least 45 cans of the same color.
- ▶ There are at least 45 cans that are different colors.

Work on it.

Point: Students Can Give Odd Answers: Easy

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Work on it.

Answer:

If there are 45 different colors of paint then we are done. Assume there are ≤ 44 different colors. If all colors appear ≤ 44 times then there are $44 \times 44 = 1936 < 2000$ cans of paint, a contradiction.

Note: this was Problem 1, which is supposed to be easy and indeed 95% got it right. What about the other 5%? Next slide.

One of the Wrong Answers. Or is it?

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ANSWER:

Paint cans are grey. Hence there are all the same color. Therefore there are 2000 cans that are the same color.

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- ▶ There are at least 45 cans of the same color.
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ANSWER:

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What do you think:

- ▶ That's just stupid. 0 points.
- ▶ Question says *cans of the same color*. ... The full 30 pts.
- ▶ Not only does he get 30 points, but everyone else should get 0.

Another Wrong Answers. Or is it?

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ANSWER:

If you look at a paint color really really carefully there will be differences. Hence, even if two cans seem to both be (say) RED, they are really different. Therefore there are 2000 cans of different colors.

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What do you think:

- ▶ That's just stupid. 0 points.
- ▶ Well... he's got a point. 30 points in fact.
- ▶ Not only does he get 30 points, but everyone else should get 0.

Point: Students Can Give Odd Answers: Hard

From the year 2007 Maryland Math Competition.

QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

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Note I think I was assigned to grade it since it looks like the kind of problem I would make up, even though I didn't. It was problem 5 (out of 5) and was hard. About 100 students tried it, 8 got full credit, 10 got partial credit

Funny Answers One

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All the vertices are red because I can make them whatever color I want. I can also write at a 30 degree angle to the bottom of this paper (The students answer was written at a 30 degree angle to the bottom of the paper.) if thats what I feel like doing at the moment. Just like $2 + 2 = 5$ if thats what my math teacher says. Math is pretty subjective anyway.

Was Student One Serious?

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Theorem The students is not serious.

Proof Assume, by contradiction, that they are serious. Then they really think math is subjective. Hence they don't really understand math. Hence they would not have done well enough on Part I to qualify for Part II. But they took Part II. Contradiction.

Funny Answers Two

QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Funny Answers Two

QUESTION: Let ABC be a fixed triangle. Let COL be any 2-coloring of the plane where each point is colored with red or green. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

I like to think that we live in a world where points are not judged by their color, but by the content of their character. Color should be irrelevant in the the plane. To prove that there exists a group of points where only one color is acceptable is a reprehensible act of bigotry and discrimination.

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Was Student Two Serious. Yes.

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Was Student Two Serious. Yes. about **Justice!**

The Real Answer to Points in the Plane Problem

Each point in the plane is colored either red or green. Let ABC be a fixed triangle. Prove that there is a triangle DEF in the plane such that DEF is similar to ABC and the vertices of DEF all have the same color.

Fix a 2-coloring of the plane.

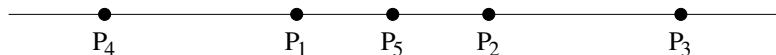
There are 3 equally-spaced mono points on x -axis

Proof Clearly there are two points on the x -axis of the same color: p_1, p_2 are RED. If p_3 , the midpoint of p_1, p_2 , is RED then p_1, p_3, p_2 are all RED. DONE. Hence we assume p_3 is GREEN.

Let p_4 be such that $|p_1 - p_4| = |p_2 - p_1|$. If p_4 is RED then p_4, p_1, p_2 are all RED. DONE. Hence we assume p_4 is GREEN.

Let p_5 be such that $|p_5 - p_2| = |p_2 - p_1|$. If p_5 is RED then p_1, p_2, p_5 are all RED. DONE. Hence we assume p_5 is GREEN.

Only case left p_3, p_4, p_5 are all GREEN. DONE.



Finish Proof By Picture

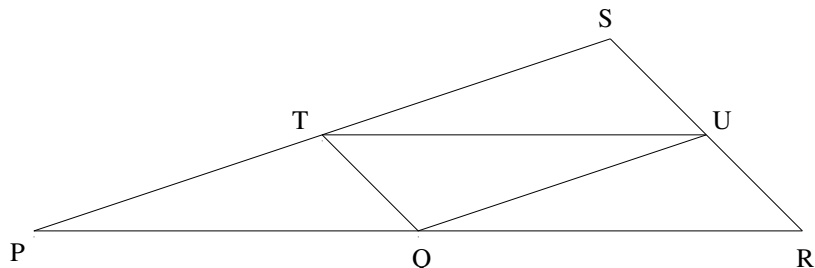


Figure: Triangle Similar to ABC with Monochromatic Vertices

P, Q, R are RED.

If T or U or S are RED then get RED Triangle similar to ABC .

If not then ALL of T, U, S are GREEN, so get GREEN triangle similar to ABC .

Point: What is a Simple Function?

Bill assigned the following in Discrete Math: For each of the following sequences find a **simple function** $A(n)$ such that the sequence is $A(1), A(2), A(3), \dots$

1. 10, -17, 24, -31, 38, -45, 52, \dots
2. -1, 1, 5, 13, 29, 61, 125, \dots
3. 6, 9, 14, 21, 30, 41, 54, \dots

Caveat: These are NOT trick questions.
Work on it.

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1. 10, -17, 24, -31, 38, -45, 52, \dots $A(n) = (-1)^{n+1}(7n + 3)$.
2. -1, 1, 5, 13, 29, 61, 125, \dots $A(n) = 2^n - 3$.
3. 6, 9, 14, 21, 30, 41, 54, \dots $A(n) = n^2 + 5$.

A Student asks — What is a Simple Function?

One student, in earnest, emailed Bill the following:

*We never defined **Simple Function** in class so I went to Wikipedia. It said that **A Simple Function is a linear combination of indicator functions of measurable sets.** Is that what you want us to use?*

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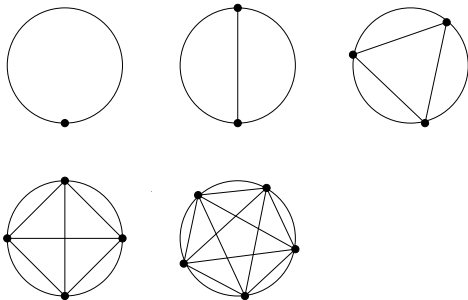
The student got the first one right, but left the other two blank.

Point: When Do Patterns Hold?

The last question brings up the question of when patterns do and don't hold. We looked for cases where a pattern *did not* hold.

First Non-Pattern: n Points on a circle

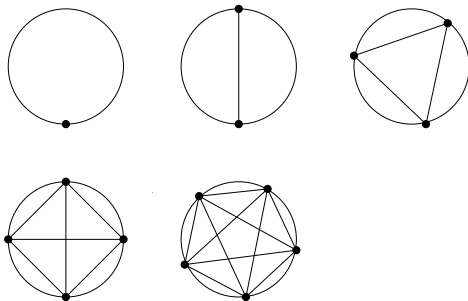
What is the max number of regions formed by connecting every pair of n points on a circle. For $n = 1, 2, 3, 4, 5$:



Tempted to guess 2^{n-1} .

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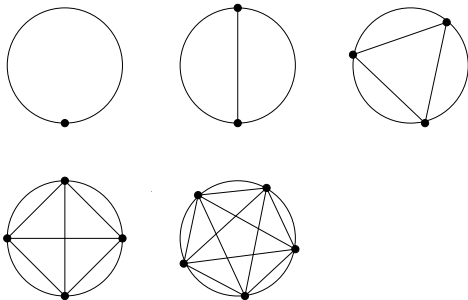


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But for $n = 6$, the number of regions is only 31.

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But for $n = 6$, the number of regions is only 31.

The actual number of regions for n points is $\binom{n}{4} + \binom{n}{2} + 1$.

Second Non-Pattern: Borwein Integrals

$$\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} = \frac{\pi}{2}$$

⋮

$$\int_0^{\infty} \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} = \frac{\pi}{2}$$

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But

$$\int_0^{\infty} \frac{\sin x}{x} \frac{\sin \frac{x}{3}}{\frac{x}{3}} \frac{\sin \frac{x}{5}}{\frac{x}{5}} \frac{\sin \frac{x}{7}}{\frac{x}{7}} \frac{\sin \frac{x}{9}}{\frac{x}{9}} \frac{\sin \frac{x}{11}}{\frac{x}{11}} \frac{\sin \frac{x}{13}}{\frac{x}{13}} \frac{\sin \frac{x}{15}}{\frac{x}{15}} =$$

$$\frac{467807924713440738696537864469\pi}{935615849440640907310521750000} \sim 0.9999999999852937186 \times \frac{\pi}{2}$$

Why the breakdown at 15?

Because

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{13} < 1$$

but

$$\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{15} > 1.$$

For more Google

Borwein Integral

Point: Finding Problems to Work on

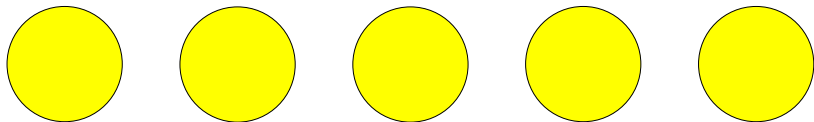
A Recreational Math Conference (Gathering for Gardner) May 2016

Bill found a pamphlet:

The Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

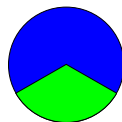
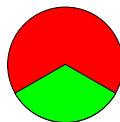
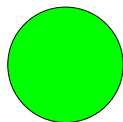
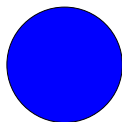
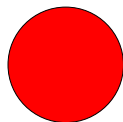
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	RED	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	BLUE	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

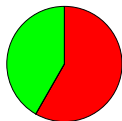
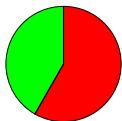
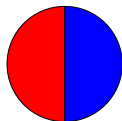
Is there a procedure with a larger smallest piece?

Work no it

Five Muffins, Three People—Proc by Picture

Person	Color	What they Get
Alice	RED	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	BLUE	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN'T!

Five Muffins, Three People—Can't Do Better Than

$$\frac{5}{12}$$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \quad \text{Great to see } \frac{5}{12}$$

General Problem

The Muffin Problem:

How can you divide and distribute m muffins to s students so that each student gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

This Problem went from recreational Mathematics to Serious Math when we replaced $(5,3)$ with (m,s) .

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2. Book with Four Authors (To appear in 2020)

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6. A future talk in this forum?

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All from

ONE PAMPHLET!

Our Book on Amazon

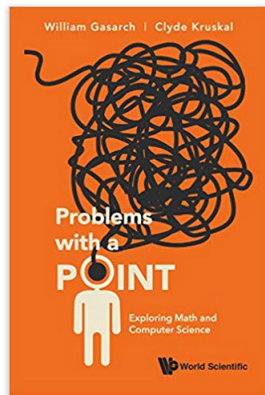
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by [William Gasarch](#) (Author), [Clyde Kruskal](#) (Author)

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ISBN-13: 978-9813279971

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