The Muffin Problem

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How it Began

A Recreational Math Conference
(Gathering for Gardner)
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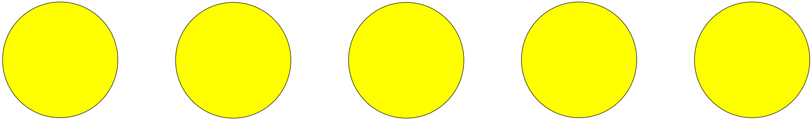
I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

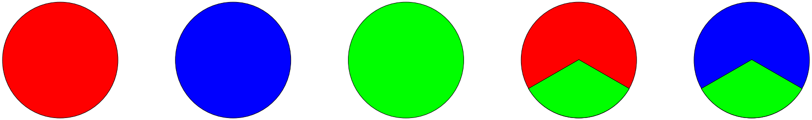
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ *where nobody gets a tiny sliver?*



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	**BLUE**	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

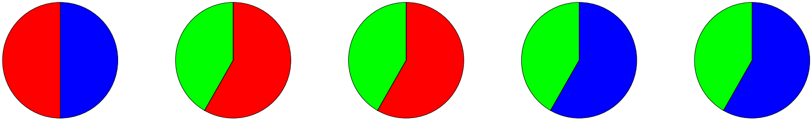
Is there a procedure with a larger smallest piece?

YES WE CAN!

Five Muffins, Three People–Proc by Picture

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN'T!

Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

General Problem

$f(m,s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5,3) = \frac{5}{12}$ here.

We have shown $f(m,s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).

Amazing Results!/Amazing Theorems!

1. $f(43,33) = \frac{91}{264}$.
2. $f(52,11) = \frac{83}{176}$.
3. $f(35,13) = \frac{64}{143}$.

All done by hand, no use of a computer
by Co-author Erik Metz is a muffin savant !

Have **General Theorems** from which **upper bounds** follow.
Have **General Procedures** from which **lower bounds** follow.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?
NO

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
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$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \ge \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \ge \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
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4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.
Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

Floor-Ceiling Thm (FC Thm) Generalizes $f(5,3) \le \frac{5}{12}$

$$f(m,s) \le FC(m,s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1 - \frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s} \rceil}$.

Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s\lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\le 1 - \frac{m}{s\lfloor 2m/s \rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k},\ k \ge 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

Note: A Mod 3 Pattern.
Theorem: For all $m \ge 3$, $f(m,3) = FC(m,3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Note: A Mod 4 Pattern.
Theorem: For all $m \geq 4$, $f(m, 4) = FC(m, 4)$.
FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = FC(m, s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$
Note: A Mod 5 Pattern.
Theorem: For all $m \geq 5$ **except m=11**, $f(m, 5) = FC(m, 5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11,5) \geq \frac{13}{30}$.
2. $f(11,5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

If $f(5,11) < \frac{11}{25}$ then FC-conjecture is false!

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So

$$\frac{13}{30} \leq f(11,5) \leq \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

If $f(5,11) < \frac{11}{25}$ then FC-conjecture is false!

WE SHOW: $\mathbf{f(11,5) = \frac{13}{30}}$

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 shares.
$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.
We call a share that goes to a person who gets 5 shares a **5-share**.

$f(11,5)=\frac{13}{30}$, Fun Cases

Case 4.1: is $\le \frac{1}{2}$. Then there is a piece

$$\ge \frac{(11/5)-(1/2)}{3} = \frac{17}{30}.$$

The other piece from the muffin is

$$\le 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.$$

Case 4.2: All 4-shares are $> \frac{1}{2}$. So there are $4s_4 = 12$ 4-shares. There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

Essence of the Interval Method

1. Every muffin cut into two pieces.
2. Find L such that some students get either L or $L+1$ pieces.
3. Find how many students get L $(L+1)$ pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval
6. Get a contradiction out of this.

Note: Can turn Interval Theorem into a function INT such that $f(m,s) \leq INT(m,s)$.

FC CONJECTURE STILL SORT OF TRUE

FC Conj: For all $m \geq s$, $f(m,s) = FC(m,s)$. FALSE

Theorem: For fixed s, for $m \geq \frac{s^3+2s^2+s}{2}$ $f(m,s) = FC(m,s)$.

Statistics: For $3 \leq s \leq 50$, $s+1 \leq m \leq 59$:

$f(m,s) = FC(m,s)$ in 683 cases
$f(m,s) = INT(m,s)$ in 194 cases

Still 108 cases left. Need new technique!

The Buddy-Match Method! (BM)

Can FC and INT do everything?
No.
They are very good when $\frac{2m}{s} > 3$ but NOT so good otherwise.
We do a concrete example of **The Buddy-Match Method**

$$f(43, 39) \le \frac{53}{156}$$

(We have matching lower bound also)

Definition: Assume we have a protocol where all students get 2 or 3 shares. If x is a 2-share then the other share that student has is the shares **match**. Note that $M(x) = \frac{m}{s} - x$.
Warning: We will apply M to intervals. These intervals have to have only 2-shares in them! But they will!

$f(43,39) \le \frac{53}{156}$

Theorem $f(43,39) \le \frac{53}{156}$ ($\ge$ also known).
Assume there is an $(43,39)$-procedure with smallest piece $> \frac{53}{156}$.
Can assume all muffins cut in 2 pieces, all students get ≥ 2 shares.

Case 1: A student gets ≥ 4 shares. Some share $\le \frac{43}{39\times 4} < \frac{53}{156}$.

Case 2: A student gets ≤ 1 shares. Can't occur.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.

How Many Students get Two Shares? Three Shares?

Let s_2 (s_3) be the number of 2-students (3-students).

$$\begin{aligned} 2s_2 + 3s_3 &= 86 \\ s_2 + s_3 &= 39 \text{ Get } s_2 = 31 \text{ and } s_3 = 8 \end{aligned}$$

Case 3.1, 3.2, 3.3, 3.4:
($\exists$) 3-share $\ge \frac{66}{156}$. Rm. Now 2-shares $\ge \frac{43}{39} - \frac{66}{156} = \frac{53}{78}$.

So some share $\le \frac{53}{156}$.
By similar reasoning (Case 3.2, 3.3, 3.4) we have:

$$\underset{\frac{53}{156}}{(} \quad 24 \text{ 3-shs} \quad \underset{\frac{66}{156}}{)[} \quad 0 \text{ shs} \quad \underset{\frac{69}{156}}{](} \quad 62 \text{ 2-shs} \quad \underset{\frac{103}{156}}{)}$$

The Buddy-Match Method

$$\underset{\frac{53}{156}}{(} \quad \text{24 3-shs} \quad \underset{\frac{66}{156}}{)[} \quad \text{0 shs} \quad \underset{\frac{69}{156}}{](} \quad \text{62 2-shs} \quad \underset{\frac{103}{156}}{)}$$

$$|(\tfrac{53}{156}, \tfrac{69}{156})| = 24$$

$$|B(\tfrac{53}{156}, \tfrac{69}{156})| = |\tfrac{87}{156}, \tfrac{103}{156}| = 24$$

$$|M(\tfrac{87}{156}, \tfrac{103}{156})| = |\tfrac{69}{156}, \tfrac{85}{156}| = 24$$

$$|(\tfrac{53}{156}, \tfrac{69}{156}) \cup (\tfrac{69}{156}, \tfrac{85}{156}) \cup (\tfrac{87}{156}, \tfrac{103}{156})| = 24 \times 3 = 72$$

$$|(\tfrac{85}{156}, \tfrac{87}{156})| = 86 - 72 = 14.$$

More Buddy-Match Method

$|(\frac{85}{156}, \frac{87}{156})| = 14$. Buddy-Match yields $|(\frac{53}{156}, \frac{55}{156})| = 14$

$|[\frac{66}{156}, \frac{69}{156}]| = 0$. Buddy-Match yields $|[\frac{55}{156}, \frac{58}{156}]| = 0$.

The following picture captures what we know so far about 3-shares.

$$
\begin{array}{ccccccc}
(& 14 &)[& 0 &](& 10 &) \\
\frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{66}{156}
\end{array}
$$

Big Shares and Small Shares

$$\underset{\frac{53}{156}}{(} \quad 14 \quad \underset{\frac{55}{156}}{)[} \quad 0 \quad]\underset{\frac{58}{156}}{(} \quad 10 \quad \underset{\frac{66}{156}}{)}$$

- Shares in $(\frac{53}{156}, \frac{55}{156})$ are *small shares*;
- Shares in $(\frac{58}{156}, \frac{66}{156})$ are *large shares*;

Notation d_i is numb of students who have i small shares ($3 - i$ large shares).

$d_0 = 0$ since $3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}$.

$d_3 = 0$ since $3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}$.

SO there are NO d_0-students or d_3-students.

d_1 and d_2 Students Cause a Gap!

$$\underset{\frac{53}{156}}{(} \quad 14 \quad \underset{\frac{55}{156}}{)[} \quad 0 \quad \underset{\frac{58}{156}}{](} \quad 10 \quad \underset{\frac{66}{156}}{)}$$

d_1: If a d_1-student has a large shares $\ge \frac{61}{156}$ then he will have

$$> \frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of d_1-student are in $(\frac{58}{156}, \frac{61}{156})$.

d_2: If a d_2-student has a large shares $\le \frac{62}{156}$ then he will have

$$< \frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of a d_2-student are in $(\frac{62}{156}, \frac{66}{156})$.

Upshot Upshot: There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$

Even More Buddy Match

The following picture captures what we know so far about 3-shares.

$$\begin{array}{ccccccccccc} (& 14 &)[& 0 &](& x &)[& 0 &](& y &) \\ \frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{61}{156} & & \frac{62}{156} & & \frac{66}{156} \end{array}$$

Use Buddy-Match to show that $|(\frac{61}{156}, \frac{62}{156})| = |(\frac{62}{156}, \frac{63}{156})|$. So:

$$\begin{array}{ccccccccccc} (& 14 &)[& 0 &](& x &)[& 0 &](& y &) \\ \frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{61}{156} & & \frac{63}{156} & & \frac{66}{156} \end{array}$$

$$x + y = 10.$$

Use Buddy-Match to show that $|(\frac{58}{156}, \frac{61}{156})| = |(\frac{63}{156}, \frac{66}{156})|$ so they are are both 5.

$$\begin{array}{ccccccccccc} (& 14 &)[& 0 &](& 5 &)[& 0 &](& 5 &) \\ \frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{61}{156} & & \frac{63}{156} & & \frac{66}{156} \end{array}$$

Equations

$$
\begin{array}{ccccccccccc}
(& 14 &)[& 0 &](& 5 &)[& 0 &](& 5 &) \\
\frac{53}{156} & & \frac{55}{156} & & \frac{58}{156} & & \frac{61}{156} & & \frac{63}{156} & & \frac{66}{156}
\end{array}
$$

Only the d_2-students use $(\frac{63}{156}, \frac{66}{156})$. Every d_2 student uses one share from that interval:

$$d_2 = 5.$$

Each d_i student uses i shares from $(\frac{53}{156}, \frac{55}{156})$:

$$1 \times d_1 + 2 \times d_2 = 14 : \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8 : \text{ So } 5 + 4 = 8. \text{CONTRADICTION!}$$

The Essence of The Buddy-Match Method

1. Works when $\lceil \frac{2m}{s} \rceil = 3$: Just 2-shares and 3-shares.
2. $2m$ pieces, s_2 students get 2 shares, s_3 students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3-shares-interval find intervals that cover **almost** the entire interval. Missing an interval (a, b).
5. Use BM on (a, b) to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPs within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in $\mathbf{N}$.

Note: Can turn BM technique into a function $BM(m, s)$ such that $f(m, s) \leq BM(m, s)$.

Statistics

For $3 \leq s \leq 50$, $s+1 \leq m \leq 59$:

$f(m,s) = FC(m,s)$ in 683 cases
$f(m,s) = INT(m,s)$ in 193 cases
$f(m,s) = BM(m,s)$ in 85 cases.
$f(m,s) = ERIK(m,s)$ in 5 cases.

$f(m,s)$ is OPEN in 18 cases.

$\sim$ 98% of the cases solved!

A Guess that Works. But Why?

1) We suspected there was a constant X such that:

$$(\forall k \geq 1)\left[f(21k+11, 21k+4) \leq \frac{7k+X}{21k+4}\right]$$

2) We knew that $f(11,4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several k.

4) But the proof for $f(11,4)$ $(k=0)$ **cannot use BM and is totally unrelated to the proof for $k \geq 1$.**
Note: This technique always worked!

Another Guess that Works But we Don't Know Why

Want to know $f(41,19)$. Can't use BM.
$41-19=22$. So try to prove, diff d is always Mod $3d$ pattern.
Need X:

$$(\forall k \ge 1)\left[f(66k+41, 66k+19) \le \frac{22k+X}{66k+19}\right]$$

Find X using BM and linear algebra (have program for that).
Get conj: $f(41,19)=\frac{X}{19}$.
Note: This seems to always work but have not been able to use to get new results yet.

Open Problems: $1 \le s \le 40$, $s+1 \le 59$

M	S	LB	UB	Method for UB
41	19	980/2280	983/2280	*ERIK*
41	23	195/483	196/4839	*FC*
54	25	215/500	216/500/	*FC*
59	26	134/312	135/312	*FC*
47	29	140/348	141/348	*FC*
49	30	340/840	343/840	*FC*
52	31	152/372	153/372	*INT*
55	31	75/186	76/186	*FC*
59	33	159/396	160/396	*FC*
55	34	164/408	165/408	*FC*
57	35	56/140	57/140	*FC*
47	36	74/216	75/216	*FC*
48	37	512/1480	515/1480	*BM*

Open Problems: $41 \le s \le 50$, $s+1 \le 59$

M	S	LB	UB	Method for UB
55	42	$86/252$	$87/252$	*FC*
53	43	$183/516$	$184/516$	*INT*
55	43	$90/258$	$91/258$	*INT*
56	43	$59/172$	$60/172$	*FC*
59	45	$92/270$	$93/270$	*FC*

Programs

We have a program that on input (m, s):

1. We we used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).

Results

1. FC, INT, and BM upper bounds on $f(m,s)$
2. For fixed s, for $m \geq\sim s^3$, $f(m,s) = FC(m,s)$.
3. For all $m \geq s$ $f(m,s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m,s)$. Mod s pattern
5. For $s = 8, \ldots, 100$ conjectures for $f(m,s)$. $f(m,s)$ seems to be a mod s pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s+d,s)$. Mod $3d$ pattern.
7. For all d conjecture that our Theorem Generator gives $f(s+d,s)$.
8. Conjecture that for all a, d there exists X such that

$$(\forall k \geq 0)\left[f(3dk+a+d, 3dk+a) \leq \frac{dk+X}{3dk+a}\right]$$

Open Problems-Complexity

Consider:
Given m, s in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. The problem IS in FPT: $(\forall m \geq s^3)[f(m, s) = FC(m, s)]$.

Conjectures That are Surely True

1. $f(m,s)$ has mod s pattern with a few exceptions (known for large m).
2. $f(s+d,s)$ has mod $3d$ pattern with no exceptions.
3. $f(m,s)$ only depends on m/s.
4. If $f(m,s)=\frac{a}{b}$ then s divides b.

Accomplishment I Am Most Proud of

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Accomplishment I Am Most Proud of:

Convinced

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 3 college student (Erik, Jacob, Daniel)
- 1 professor (John D)

that the most important field of Mathematics is **Muffinry.**