The Muffin Problem

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How it Began

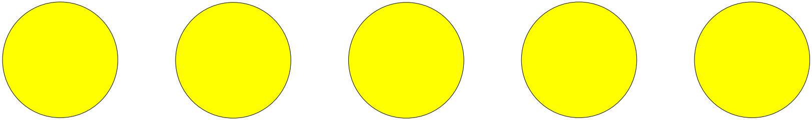
A Recreational Math Conference
(Gathering for Gardner)
May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival:
A Sample of Mathematical Puzzles
Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

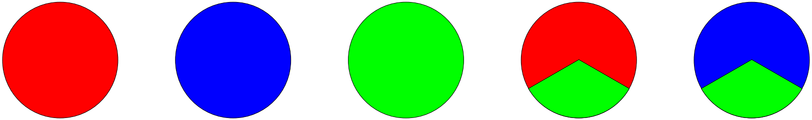
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	**RED**	$1+\frac{2}{3}=\frac{5}{3}$
Bob	**BLUE**	$1+\frac{2}{3}=\frac{5}{3}$
Carol	GREEN	$1+\frac{1}{3}+\frac{1}{3}=\frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- **YES**
- **NO**

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

VOTE

- **YES**
- **NO**

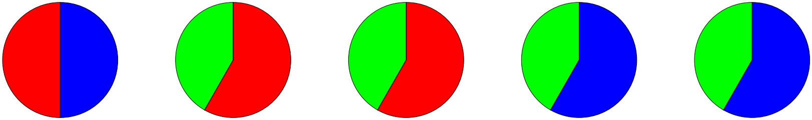
YES WE CAN!

We use **!** since we are excited that we can!

Five Muffins, Three People–Proc by Picture

Person	Color	What they Get
Alice	**RED**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	**BLUE**	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.
Is there a procedure with a larger smallest piece?
VOTE

- **YES**
- **NO**

Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.
Is there a procedure with a larger smallest piece?
VOTE

- **YES**
- **NO**

NO WE CAN'T!

We use **!** since we are excited to prove we can't do better!

Five Muffins, Three People–Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins,3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N. We want $N \le \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$-sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.
(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \le \frac{1}{3} < \frac{5}{12}$.
(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\le \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \qquad \text{Great to see } \frac{5}{12}$$

Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece$> \frac{5}{12}$.

Amazing That Have Exact Result!

Be Amazed Now! And Later!

1. Procedure for 5 muffins, 3 people, smallest piece $\frac{5}{12}$.
2. NO Procedure for 5 muffins, 3 people, smallest piece$> \frac{5}{12}$.

Amazing That Have Exact Result!

Prepare To Be More Amazed! On Next Page!

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece $> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece $> \frac{83}{176}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece$> \frac{64}{143}$.

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece$> \frac{64}{143}$.

All done by hand, no use of a computer

Amazing Results!

1. Procedure for 47 muffins, 9 people, smallest piece $\frac{111}{234}$.
2. NO Procedure for 47 muffins, 9 people, smallest piece$> \frac{111}{234}$.

1. Procedure for 52 muffins, 11 people, smallest piece $\frac{83}{176}$.
2. NO Procedure for 52 muffins, 11 people, smallest piece$> \frac{83}{176}$.

1. Procedure for 35 muffins, 13 people, smallest piece $\frac{64}{143}$.
2. NO Procedure for 35 muffins, 13 people, smallest piece$> \frac{64}{143}$.

All done by hand, no use of a computer

Co-author Erik Metz is a *muffin savant*

General Problem

How can you divide and distribute m muffins to s students so that each students gets $\frac{m}{s}$ AND the MIN piece is MAXIMIZED?

An *(m,s)-procedure* is a way to divide and distribute m muffins to s students so that each student gets $\frac{m}{s}$ muffins.

An (m,s)-procedure is *optimal* if it has the largest smallest piece of any procedure.

$f(m,s)$ be the smallest piece in an optimal (m,s)-procedure.

We have shown $f(5,3) = \frac{5}{12}$.

Note: $f(m,s) \geq \frac{1}{s}$: divide each muffin into s pieces of size $\frac{1}{s}$ and give each student m of them.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote:

$f(3,5) \ge ?$

Clearly $f(3,5) \ge \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \ge \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote:
YES
NO
UNKNOWN TO SCIENCE

$f(3,5) \geq ?$

Clearly $f(3,5) \geq \frac{1}{5}$. Can we get $f(3,5) > \frac{1}{5}$?
Think about it at your desk.
$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better? Vote:
YES
NO
UNKNOWN TO SCIENCE
NO Proof on next slide.

$f(3,5) \le \frac{1}{4}$

There is a procedure for 3 muffins,5 students where each student gets $\frac{3}{5}$ muffins, smallest piece N. We want $N \le \frac{1}{4}$.

Case 0: Some student gets 1 piece, so size $\frac{3}{5}$. Cut that piece in half and give both $\frac{3}{10}$-sized pieces to that student. (Note $\frac{3}{10} > \frac{1}{4}$.) Reduces to other cases.
(**Henceforth:** All students get ≥ 2 pieces.)

Case 1: Some student gets ≥ 3 pieces. Then $N \le \frac{3}{5} \times \frac{1}{3} = \frac{1}{5} < \frac{1}{4}$.
(**Henceforth:** All students get 2 pieces.)

Case 2: All students get 2 pieces. 5 students, so 10 pieces.
Some muffin gets cut into ≥ 4 pieces. Some piece $\le \frac{1}{4}$.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.

3 People, 5 Muffins VS 5 People, 3 Muffins

$f(5,3) \geq \frac{5}{12}$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$f(3,5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3,5)$ proc is $f(5,3)$ proc but swap Divide/Give and mult by $3/5$.
Theorem: $f(m,s) = \frac{m}{s} f(s,m)$.

Floor-Ceiling Theorem (Generalize $f(5,3) \le \frac{5}{12}$)

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2},\frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\le \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\ge \lceil \frac{2m}{s} \rceil$ pieces. $\exists$ piece $\le \frac{m}{s} \times \frac{1}{\lceil 2m/s\rceil} = \frac{\mathbf{m}}{\mathbf{s}\lceil \mathbf{2m/s}\rceil}$.

Someone gets $\le \lfloor \frac{2m}{s} \rfloor$ pieces. $\exists$ piece $\ge \frac{m}{s}\frac{1}{\lfloor 2m/s\rfloor} = \frac{m}{s\lfloor 2m/s\rfloor}$.

The other piece from that muffin is of size $\le 1-\frac{m}{s\lfloor 2m/s\rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

Floor-Ceiling Theorem for optimality.

$f(1,3) = \frac{1}{3}$

$f(3k,3) = 1.$

$f(3k+1,3) = \frac{3k-1}{6k}, \ k \geq 1.$

$f(3k+2,3) = \frac{3k+2}{6k+6}.$

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

$f(4k, 4) = 1$ (easy)

$f(1, 4) = \frac{1}{4}$ (easy)

$f(4k+1, 4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2, 4) = \frac{1}{2}$.

$f(4k+3, 4) = \frac{4k+1}{8k+4}$.

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

$f(4k,4) = 1$ (easy)

$f(1,4) = \frac{1}{4}$ (easy)

$f(4k+1,4) = \frac{4k-1}{8k}$, $k \geq 1$.

$f(4k+2,4) = \frac{1}{2}$.

$f(4k+3,4) = \frac{4k+1}{8k+4}$.

Is FIVE student case a Mod 5 pattern?
VOTE YES or NO
YES but with some exceptions

FIVE Students, $m = 1, \ldots, 11$

$f(1,5) = \frac{1}{5}$ (**easy** or use $f(1,5) = \frac{5}{1} f(5,1)$.)

$f(2,5) = \frac{1}{5}$ (**easy** or use $f(2,5) = \frac{5}{2} f(5,2)$.)

$f(3,5) = \frac{1}{4}$ (use $f(3,5) = \frac{3}{5} f(5,3)$.)

$f(4,5) = \frac{3}{10}$ (use $f(4,5) = \frac{4}{5} f(5,4)$.)

$f(5,5) = 1$ (Easy and fits pattern)

$f(6,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(7,5) = \frac{1}{3}$ (Use **Floor-Ceiling Thm**, NOT pattern)

$f(8,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(9,5) = \frac{2}{5}$ (Use **Floor-Ceiling Thm**, fits pattern)

$f(10,5) = 1$ (Easy and fits pattern)

$f(11,5) =$ (Will come back to this **later**)

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

Floor-Ceiling Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k+1, 5) = \frac{5k+1}{10k+5}$

For $k \geq 2$, $f(5k+2, 5) = \frac{5k-2}{10k}$

For $k \geq 1$, $f(5k+3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k+4, 5) = \frac{5k+1}{10k+5}$

What About FIVE students, ELEVEN muffins?

$f(11,5) \geq \frac{13}{30}$.

Procedure:

1. Divide 8 muffins into $(\frac{13}{30}, \frac{17}{30})$.
2. Divide 2 muffins into $(\frac{14}{30}, \frac{16}{30})$.
3. Divide 1 muffin into $(\frac{15}{30}, \frac{15}{30})$.
4. Give 2 students $[\frac{14}{30}, \frac{13}{30}, \frac{13}{30}, \frac{13}{20}, \frac{13}{20}]$
5. Give 1 student $[\frac{17}{30}, \frac{17}{30}, \frac{16}{30}, \frac{16}{20}]$
6. Give 2 students $[\frac{17}{30}, \frac{17}{30}, \frac{17}{30}, \frac{15}{30}]$.

What About FIVE students, ELEVEN muffins?

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\} \le 0.44.$$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff}= 0.006666\ldots$$

What About FIVE students, ELEVEN muffins?

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\} \le 0.44.$$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

VOTE:

1. $f(11,5) = \frac{13}{30}$: Needs NEW methods to bound $f(m,s)$.
2. $f(11,5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11,5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**

What About FIVE students, ELEVEN muffins?

$$f(m,s) \le \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s\lceil 2m/s\rceil}, 1-\frac{m}{s\lfloor 2m/s\rfloor}\right\}\right\} \le 0.44.$$

So

$$\frac{13}{30} \le f(11,5) \le \frac{11}{25} \qquad \text{Diff} = 0.006666\ldots$$

VOTE:

1. $f(11,5) = \frac{13}{30}$: Needs NEW methods to bound $f(m,s)$.
2. $f(11,5) = \frac{11}{25}$: Needs NEW better procedure.
3. $f(11,5) = \alpha$ where $\frac{13}{30} < \alpha < \frac{11}{25}$. Needs both:
4. **UNKNOWN TO SCIENCE!**

KNOWN: $\mathbf{f(11,5) = \frac{13}{30}}$

HAPPY: New opt tech more interesting than new proc.

$f(11,5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N. We want $N \le \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \le \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11,5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.
One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(**Negation of Cases 2 and 3:** Every student gets 4 or 5 pieces.)

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- s_4 is number of students who get 4 pieces
- s_5 is number of students who get 5 pieces

$$\begin{aligned} 4s_4 + 5s_5 &= 22 \\ s_4 + s_5 &= 5 \end{aligned}$$

$s_4 = 3$: There are 3 students who have 4 pieces.
$s_5 = 2$: There are 2 students who have 5 pieces.

$f(11,5) = \frac{13}{30}$, Fun Cases

$$\diamond \quad \diamond \quad \diamond \quad \diamond \quad \diamond \qquad \text{(Sums to } 11/5\text{)}$$
$$\diamond \quad \diamond \quad \diamond \quad \Diamond \quad \Diamond \qquad \text{(Sums to } 11/5\text{)}$$

$$\circ \quad \circ \quad \circ \quad \circ \qquad \text{(Sums to } 11/5\text{)}$$
$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$
$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$

Case 4.1: One of (say)

$$\circ \quad \circ \quad \circ \quad \bigcirc \qquad \text{(Sums to } 11/5\text{)}$$

is $\le \frac{1}{2}$. Then there is a piece

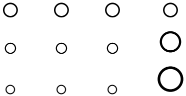
$$\ge \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$

The other piece from the muffin is

$$\le 1 - \frac{17}{30} = \frac{13}{30} \qquad \text{Great to see } \frac{13}{30}.$$

$f(11,5) = \frac{13}{30}$, Fun Cases

Case 4.2: All



(Sums to $11/5$)
(Sums to $11/5$)
(Sums to $11/5$)

are $> \frac{1}{2}$.
There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

The Techniques Generalizes- We Do $f(24, 11)$

The technique for $f(11, 5) \le \frac{13}{30}$ has a generalization with a bakers dozen subcases. We do one concrete example:

Definition: Assume we have a protocol where all muffins are cut into two pieces. If x is a piece then the other piece in the muffin it came from is its **buddy**. Note that $B(x) = 1 - x$.

$f(24,11) \le \frac{19}{44}$

Theorem: $f(24,11) \le \frac{19}{44}$ ($\ge$ also known)

Assume $(24,11)$-procedure with smallest piece $> \frac{19}{44}$.
Can assume all muffin cut in two and all student gets ≥ 2 shares.
We show that there is a piece $\le \frac{19}{44}$.

Case 1: A student gets ≥ 6 shares. Some piece $\le \frac{24}{11\times 6} < \frac{19}{44}$.

Case 2: A student gets ≤ 3 shares. Some piece $\ge \frac{24}{11\times 3} = \frac{8}{11}$.
Buddy of that piece $\le 1 - \frac{8}{11} \le \frac{3}{11} < \frac{19}{44}$.

Case 3: Every muffin is cut in 2 pieces and every student gets either 4 or 5 shares. Total number of shares is 48.

How many students get 4? 5? Where are the Shares?

Let s_4 (s_5) be the number of 4-students (5-students).

$$\begin{aligned} 4s_4 + 5s_5 &= 48 \\ s_4 + s_5 &= 11 \qquad \text{Get } s_4 = 7 \text{ and } s_5 = 4 \end{aligned}$$

Case 3.1: $(\exists)$ 4-sh $\le \frac{21}{44}$. Rm. Now: 3 shares $\ge \frac{24}{11} - \frac{21}{44}$. $(\exists)$ share

$$\ge \frac{(24/11) - (21/44)}{3} = \frac{25}{44}.$$

Buddy is

$$\le 1 - \frac{25}{44} = \frac{19}{44}.$$

SO can assume all 4-shares are $> \frac{21}{44}$.
By similar reasoning:
Case 3.2: 4-shares in $(\frac{21}{44}, \frac{25}{44})$, 5-shares in $(\frac{19}{44}, \frac{20}{44})$.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

More Refined Picture of What is Going On

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad \text{0 shs} \quad \underset{\frac{21}{44}}{](} \quad \text{28 4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 1: There are no shares $x \in [\frac{23}{44}, \frac{24}{44}]$.

If there was such a share then $B(x) \in [\frac{20}{44}, \frac{21}{44}]$.
The following picture captures what we know so far.

$$\underset{\frac{19}{44}}{(} \quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[} \quad 0 \quad \underset{\frac{21}{44}}{](} \quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[} \quad 0 \quad \underset{\frac{24}{44}}{](} \quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

$$\underset{\frac{19}{44}}{(}\quad \text{20 5-shs} \quad \underset{\frac{20}{44}}{)[}\quad 0 \quad \underset{\frac{21}{44}}{](}\quad \text{8 S4-shs} \quad \underset{\frac{23}{44}}{)[}\quad 0 \quad \underset{\frac{24}{44}}{](}\quad \text{20 L4-shs} \quad \underset{\frac{25}{44}}{)}$$

Claim 2: Every 4-student has at least 3 L4 shares.

If a 4-student had ≤ 2 L4 shares then he has

$$< 2 \times \left(\frac{23}{44}\right) + 2 \times \left(\frac{25}{44}\right) = \frac{24}{11}.$$

Contradiction: There are at least $3 \times s_4 = 3 \times 7 = 21$ L4 shares. But there are only 20.

What Else do we Have -Concrete

1. Formulas for $f(m,6)$ and $f(m,7)$.
2. For $s = 8, \ldots, 100$ conjectures for $f(m,s)$. $f(m,s)$ seems to be a mod s pattern.
3. Formulas for $f(s+1,s)$, $f(s+2,s)$, $f(s+3,s)$, $f(s+4,s)$. $f(s+d,s)$ seems to have a mod $3d$ pattern.
4. A computer program that, on input m, s uses our theorems to find α with $f(m,s) \le \alpha$ and then tries to prove $f(m,s) \ge \alpha$ using linear algebra.
5. For $1 \le m, s \le 50$ have all $f(m,s)$ (Need to check that.)
6. Mixed integer program that always solves the problem but it is slow and has not been that useful.

What Else do we have-Abstract

1. For fixed s, for $m \geq \frac{s^3+2s^2+s}{2}$ $f(m,s)$ matches the Floor-ceiling bound.
2. $f(m,s)$ always exists and is rational. Provable by compactness argument OR a large number of Linear Programs, OR one MIP. The last two proofs also give that $f(m,s)$ is computable. Nice synergy – applied math tools helping us prove theorems in pure math!

Open Problems-Complexity

Consider:
Given m, s in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. Given m, s is there a bound on the denominators of the sizes of shares used?

Open Problems-Misc

1. Show that for all $m \geq s$, $f(m,s) \geq \frac{1}{3}$.
2. Prove that we ALWAYS get mod s pattern for $f(m,s)$ (true for large enough m).
3. Prove that we ALWAYS get mod $3d$ pattern for $f(s+d,s)$.

Accomplishment I Am Most Proud of

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Accomplishment I Am Most Proud of:

Convinced

- 4 High School students (Guang, Naveen, Naveen, Sunny)
- 3 college student (Erik, Jacob, Daniel)
- 1 professor (John D)

that the most important field of Mathematics is **Muffinry.**