

The Muffin Problem

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How it Began

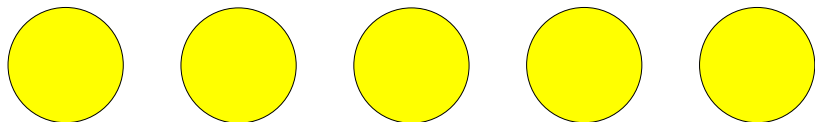
A Recreational Math Conference (Gathering for Gardner) May 2016

I found a pamphlet:

The Julia Robinson Mathematics Festival: A Sample of Mathematical Puzzles Compiled by Nancy Blachman

which had this problem, proposed by Alan Frank:

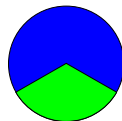
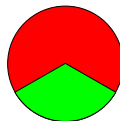
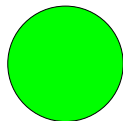
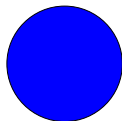
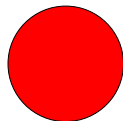
How can you divide and distribute 5 muffins to 3 students so that every student gets $\frac{5}{3}$ where nobody gets a tiny sliver?



Five Muffins, Three Students, Proc by Picture

Person	Color	What they Get
Alice	RED	$1 + \frac{2}{3} = \frac{5}{3}$
Bob	BLUE	$1 + \frac{2}{3} = \frac{5}{3}$
Carol	GREEN	$1 + \frac{1}{3} + \frac{1}{3} = \frac{5}{3}$

Smallest Piece: $\frac{1}{3}$



Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{1}{3}$.

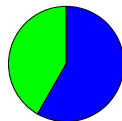
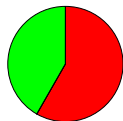
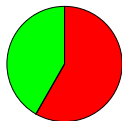
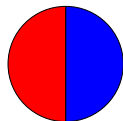
Is there a procedure with a larger smallest piece?

YES WE CAN!

Five Muffins, Three People—Proc by Picture

Person	Color	What they Get
Alice	RED	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Bob	BLUE	$\frac{6}{12} + \frac{7}{12} + \frac{7}{12}$
Carol	GREEN	$\frac{5}{12} + \frac{5}{12} + \frac{5}{12} + \frac{5}{12}$

Smallest Piece: $\frac{5}{12}$



Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

Can We Do Better?

The smallest piece in the above solution is $\frac{5}{12}$.

Is there a procedure with a larger smallest piece?

NO WE CAN'T!

Five Muffins, Three People—Can't Do Better Than $\frac{5}{12}$

There is a procedure for 5 muffins, 3 students where each student gets $\frac{5}{3}$ muffins, smallest piece N . We want $N \leq \frac{5}{12}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both $\frac{1}{2}$ -sized pieces to whoever got the uncut muffin. (Note $\frac{1}{2} > \frac{5}{12}$.) Reduces to other cases.

(**Henceforth:** All muffins are cut into ≥ 2 pieces.)

Case 1: Some muffin is cut into ≥ 3 pieces. Then $N \leq \frac{1}{3} < \frac{5}{12}$.

(**Henceforth:** All muffins are cut into 2 pieces.)

Case 2: All muffins are cut into 2 pieces. 10 pieces, 3 students: **Someone** gets ≥ 4 pieces. He has some piece

$$\leq \frac{5}{3} \times \frac{1}{4} = \frac{5}{12} \quad \text{Great to see } \frac{5}{12}$$

General Problem

$f(m, s)$ be the smallest piece in the best procedure (best in that the smallest piece is maximized) to divide m muffins among s students so that everyone gets $\frac{m}{s}$.

We have shown $f(5, 3) = \frac{5}{12}$ here.

We have shown $f(m, s)$ exists, is rational, and is computable using a Mixed Int Program (in paper).

Amazing Results!/Amazing Theorems!

1. $f(43, 33) = \frac{91}{264}$.
2. $f(52, 11) = \frac{83}{176}$.
3. $f(35, 13) = \frac{64}{143}$.

All done by hand, no use of a computer
by Co-author Erik Metz is a **muffin savant** !

Have **General Theorems** from which **upper bounds** follow.

Have **General Procedures** from which **lower bounds** follow.

$$f(3, 5) \geq ?$$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?

$$f(3, 5) \geq ?$$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
 $f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$$f(3, 5) \geq ?$$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
 $f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
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3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

$$f(3, 5) \geq ?$$

Clearly $f(3, 5) \geq \frac{1}{5}$. Can we get $f(3, 5) > \frac{1}{5}$?
 $f(3, 5) \geq \frac{1}{4}$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

Can we do better?

NO

3 People, 5 Muffins VS 5 People, 3 Muffins

$$f(5, 3) \geq \frac{5}{12}$$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$$f(5, 3) \geq \frac{5}{12}$$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$$f(3, 5) \geq \frac{1}{4}$$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

3 People, 5 Muffins VS 5 People, 3 Muffins

$$f(5, 3) \geq \frac{5}{12}$$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$$f(3, 5) \geq \frac{1}{4}$$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3, 5)$ proc is $f(5, 3)$ proc but swap Divide/Give and mult by 3/5.

3 People, 5 Muffins VS 5 People, 3 Muffins

$$f(5, 3) \geq \frac{5}{12}$$

1. Divide 4 muffins $[\frac{5}{12}, \frac{7}{12}]$
2. Divide 1 muffin $[\frac{6}{12}, \frac{6}{12}]$
3. Give 2 students $(\frac{6}{12}, \frac{7}{12}, \frac{7}{12})$
4. Give 1 students $(\frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$

$$f(3, 5) \geq \frac{1}{4}$$

1. Divide 2 muffin $[\frac{6}{20}, \frac{7}{20}, \frac{7}{20}]$
2. Divide 1 muffin $[\frac{5}{20}, \frac{5}{20}, \frac{5}{20}, \frac{5}{20}]$
3. Give 4 students $(\frac{5}{20}, \frac{7}{20})$
4. Give 1 students $(\frac{6}{20}, \frac{6}{20})$

$f(3, 5)$ proc is $f(5, 3)$ proc but swap Divide/Give and mult by 3/5.

Theorem: $f(m, s) = \frac{m}{s} f(s, m)$.

Floor-Ceiling Thm (FC Thm) Generalizes $f(5, 3) \leq \frac{5}{12}$

$$f(m, s) \leq FC(m, s) = \max\left\{\frac{1}{3}, \min\left\{\frac{m}{s \lceil 2m/s \rceil}, 1 - \frac{m}{s \lfloor 2m/s \rfloor}\right\}\right\}.$$

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin, so reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. Some piece $\leq \frac{1}{3}$.

Case 2: Every muffin is cut into 2 pieces, so $2m$ pieces.

Someone gets $\geq \lceil \frac{2m}{s} \rceil$ pieces. \exists piece $\leq \frac{m}{s} \times \frac{1}{\lceil 2m/s \rceil} = \frac{m}{s \lceil 2m/s \rceil}$.

Someone gets $\leq \lfloor \frac{2m}{s} \rfloor$ pieces. \exists piece $\geq \frac{m}{s} \frac{1}{\lfloor 2m/s \rfloor} = \frac{m}{s \lfloor 2m/s \rfloor}$.

The other piece from that muffin is of size $\leq 1 - \frac{m}{s \lfloor 2m/s \rfloor}$.

THREE Students

CLEVERNESS, COMP PROGS for the procedure.

FC Theorem for optimality.

$$f(1, 3) = \frac{1}{3}$$

$$f(3k, 3) = 1.$$

$$f(3k + 1, 3) = \frac{3k-1}{6k}, k \geq 1.$$

$$f(3k + 2, 3) = \frac{3k+2}{6k+6}.$$

Note: A Mod 3 Pattern.

Theorem: For all $m \geq 3$, $f(m, 3) = FC(m, 3)$.

FOUR Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

$$f(4k, 4) = 1 \text{ (easy)}$$

$$f(1, 4) = \frac{1}{4} \text{ (easy)}$$

$$f(4k + 1, 4) = \frac{4k-1}{8k}, k \geq 1.$$

$$f(4k + 2, 4) = \frac{1}{2}.$$

$$f(4k + 3, 4) = \frac{4k+1}{8k+4}.$$

Note: A Mod 4 Pattern.

Theorem: For all $m \geq 4$, $f(m, 4) = FC(m, 4)$.

FC-Conjecture: For all m, s with $m \geq s$, $f(m, s) = FC(m, s)$.

FIVE Students

CLEVERNESS, COMP PROGS for procedures.

FC Theorem for optimality.

For $k \geq 1$, $f(5k, 5) = 1$.

For $k = 1$ and $k \geq 3$, $f(5k + 1, 5) = \frac{5k+1}{10k+5}$. $f(11, 5)$?

For $k \geq 2$, $f(5k + 2, 5) = \frac{5k-2}{10k}$. $f(7, 5) = FC(7, 5) = \frac{1}{3}$

For $k \geq 1$, $f(5k + 3, 5) = \frac{5k+3}{10k+10}$

For $k \geq 1$, $f(5k + 4, 5) = \frac{5k+1}{10k+5}$

Note: A Mod 5 Pattern.

Theorem: For all $m \geq 5$ **except $m=11$** , $f(m, 5) = FC(m, 5)$.

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lceil 22/5 \rceil}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!

What About FIVE students, ELEVEN muffins?

1. We have a procedure which shows $f(11, 5) \geq \frac{13}{30}$.
2. $f(11, 5) \leq \max\{\frac{1}{3}, \min\{\frac{11}{5\lfloor 22/5 \rfloor}, 1 - \frac{11}{5\lfloor 22/5 \rfloor}\}\} = \frac{11}{25}$.

So

$$\frac{13}{30} \leq f(11, 5) \leq \frac{11}{25} \quad \text{Diff} = 0.006666\dots$$

If $f(5, 11) < \frac{11}{25}$ then FC-conjecture is false!

$$\text{WE SHOW: } f(11, 5) = \frac{13}{30}$$

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Muffins

There is a procedure for 11 muffins, 5 students where each student gets $\frac{11}{5}$ muffins, smallest piece N . We want $N \leq \frac{13}{30}$.

Case 0: Some muffin is uncut. Cut it $(\frac{1}{2}, \frac{1}{2})$ and give both halves to whoever got the uncut muffin. Reduces to other cases.

Case 1: Some muffin is cut into ≥ 3 pieces. $N \leq \frac{1}{3} < \frac{13}{30}$.

(**Negation of Case 0 and Case 1:** All muffins cut into 2 pieces.)

$f(11, 5) = \frac{13}{30}$, Easy Case Based on Students

Case 2: Some student gets ≥ 6 pieces.

$$N \leq \frac{11}{5} \times \frac{1}{6} = \frac{11}{30} < \frac{13}{30}.$$

Case 3: Some student gets ≤ 3 pieces.

One of the pieces is

$$\geq \frac{11}{5} \times \frac{1}{3} = \frac{11}{15}.$$

Look at the muffin it came from to find a piece that is

$$\leq 1 - \frac{11}{15} = \frac{4}{15} < \frac{13}{30}.$$

(Negation of Cases 2 and 3: Every student gets 4 or 5 pieces.)

$f(11, 5) = \frac{13}{30}$, Fun Cases

Case 4: Every muffin is cut in 2 pieces, every student gets 4 or 5 pieces. Number of pieces: 22. Note ≤ 11 pieces are $> \frac{1}{2}$.

- ▶ s_4 is number of students who get 4 pieces
- ▶ s_5 is number of students who get 5 pieces

$$4s_4 + 5s_5 = 22$$

$$s_4 + s_5 = 5$$

$s_4 = 3$: There are 3 students who have 4 shares.

$s_5 = 2$: There are 2 students who have 5 shares.

We call a share that goes to a person who gets 4 shares a **4-share**.

We call a share that goes to a person who gets 5 shares a **5-share**.

$$f(11, 5) = \frac{13}{30}, \text{ Fun Cases}$$

Case 4.1: is $\leq \frac{1}{2}$. Then there is a piece

$$\geq \frac{(11/5) - (1/2)}{3} = \frac{17}{30}.$$

The other piece from the muffin is

$$\leq 1 - \frac{17}{30} = \frac{13}{30} \quad \text{Great to see } \frac{13}{30}.$$

Case 4.2: All 4-shares are $> \frac{1}{2}$. So there are $4s_4 = 12$ 4-shares.
There are ≥ 12 pieces $> \frac{1}{2}$. Can't occur.

Essence of the Interval Method

1. Every muffin cut into two pieces.
2. Find L such that some students get either L or $L + 1$ pieces.
3. Find how many students get L ($L + 1$) pieces.
4. Find intervals that these pieces must be in.
5. Find how many pieces are in an interval
6. Get a contradiction out of this.

Note: Can turn Interval Theorem into a function INT such that $f(m, s) \leq INT(m, s)$.

FC CONJECTURE STILL SORT OF TRUE

FC Conj: For all $m \geq s$, $f(m, s) = FC(m, s)$. FALSE

Theorem: For fixed s , for $m \geq \frac{s^3+2s^2+s}{2}$ $f(m, s) = FC(m, s)$.

Statistics: For $3 \leq s \leq 50$, $s + 1 \leq m \leq 59$:

$f(m, s) = FC(m, s)$ in 683 cases

$f(m, s) = INT(m, s)$ in 194 cases

Still 108 cases left. Need new technique!

The Buddy-Match Method! (BM)

Can FC and INT do everything?

No.

They are very good when $\frac{2m}{s} > 3$ but NOT so good otherwise.

We do a concrete example of **The Buddy-Match Method**

$$f(43, 39) \leq \frac{53}{156}$$

(We have matching lower bound also)

Definition: Assume we have a protocol where all students get 2 or 3 shares. If x is a 2-share then the other share that student has is the shares **match**. Note that $M(x) = \frac{m}{s} - x$.

Warning: We will apply M to intervals. These intervals have to have only 2-shares in them! But they will!

$$f(43, 39) \leq \frac{53}{156}$$

Theorem $f(43, 39) \leq \frac{53}{156}$ (\geq also known).

Assume there is an $(43, 39)$ -procedure with smallest piece $> \frac{53}{156}$.

Can assume all muffins cut in 2 pieces, all students get ≥ 2 shares.

Case 1: A student gets ≥ 4 shares. Some share $\leq \frac{43}{39 \times 4} < \frac{53}{156}$.

Case 2: A student gets ≤ 1 shares. Can't occur.

Case 3: Every muffin is cut in 2 pieces and every student gets either 2 or 3 shares. The total number of shares is 86.

How Many Students get Two Shares? Three Shares?

Let s_2 (s_3) be the number of 2-students (3-students).

$$2s_2 + 3s_3 = 86$$

$$s_2 + s_3 = 39 \text{ Get } s_2 = 31 \text{ and } s_3 = 8$$

Case 3.1, 3.2, 3.3, 3.4:

(\exists) 3-share $\geq \frac{66}{156}$. Rm. Now 2-shares $\geq \frac{43}{39} - \frac{66}{156} = \frac{53}{78}$.

So some share $\leq \frac{53}{156}$.

By similar reasoning (Case 3.2, 3.3, 3.4) we have:

$$\left(\frac{53}{156} \text{ 24 3-shs} \right) \left[\frac{66}{156} \text{ 0 shs} \right] \left(\frac{69}{156} \text{ 62 2-shs} \right) \left[\frac{103}{156} \right]$$

The Buddy-Match Method

$$\left(\begin{array}{c} 24 \text{ 3-shs} \\ \frac{53}{156} \end{array} \right) \left[\begin{array}{c} 0 \text{ shs} \\ \frac{66}{156} \end{array} \right] \left(\begin{array}{c} 62 \text{ 2-shs} \\ \frac{69}{156} \end{array} \right) \left(\begin{array}{c} \\ \frac{103}{156} \end{array} \right)$$

$$\left| \left(\frac{53}{156}, \frac{69}{156} \right) \right| = 24$$

$$\left| B \left(\frac{53}{156}, \frac{69}{156} \right) \right| = \left| \frac{87}{156}, \frac{103}{156} \right| = 24$$

$$\left| M \left(\frac{87}{156}, \frac{103}{156} \right) \right| = \left| \frac{69}{156}, \frac{85}{156} \right| = 24$$

$$\left| \left(\frac{53}{156}, \frac{69}{156} \right) \cup \left(\frac{69}{156}, \frac{85}{156} \right) \cup \left(\frac{87}{156}, \frac{103}{156} \right) \right| = 24 \times 3 = 72$$

$$\left| \left(\frac{85}{156}, \frac{87}{156} \right) \right| = 86 - 72 = 14.$$

More Buddy-Match Method

$$|(\frac{85}{156}, \frac{87}{156})| = 14. \text{ Buddy-Match yields } |(\frac{53}{156}, \frac{55}{156})| = 14$$

$$|[\frac{66}{156}, \frac{69}{156}]| = 0. \text{ Buddy-Match yields } |[\frac{55}{156}, \frac{58}{156}]| = 0.$$

The following picture captures what we know so far about 3-shares.

$$\left(\begin{array}{cc} & 14 \\ \frac{53}{156} & \frac{55}{156} \end{array} \right) [\begin{array}{cc} 0 & \\ \frac{58}{156} & \frac{66}{156} \end{array}] \left(\begin{array}{cc} 10 & \\ \frac{58}{156} & \frac{66}{156} \end{array} \right)$$

Big Shares and Small Shares

$$\left(\frac{53}{156}, \frac{14}{156} \right) \left[\frac{55}{156}, 0 \right] \left(\frac{10}{156}, \frac{66}{156} \right)$$

- ▶ Shares in $\left(\frac{53}{156}, \frac{55}{156} \right)$ are *small shares*;
- ▶ Shares in $\left(\frac{58}{156}, \frac{66}{156} \right)$ are *large shares*;

Notation d_i is num of students who have i small shares ($3 - i$ large shares).

$$d_0 = 0 \text{ since } 3 \times \frac{58}{156} = \frac{174}{156} > \frac{172}{156} = \frac{43}{39}.$$

$$d_3 = 0 \text{ since } 3 \times \frac{55}{156} = \frac{165}{156} < \frac{172}{156} = \frac{43}{39}.$$

SO there are NO d_0 -students or d_3 -students.

d_1 and d_2 Students Cause a Gap!

$$\left(\begin{array}{c} 14 \\ \frac{53}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{156} \end{array} \right] \left(\begin{array}{c} 10 \\ \frac{58}{156} \end{array} \right) \left(\begin{array}{c} 66 \\ \frac{66}{156} \end{array} \right)$$

d_1 : If a d_1 -student has a large shares $\geq \frac{61}{156}$ then he will have

$$> \frac{53}{156} + \frac{58}{156} + \frac{61}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of d_1 -student are in $(\frac{58}{156}, \frac{61}{156})$.

d_2 : If a d_2 -student has a large shares $\leq \frac{62}{156}$ then he will have

$$< \frac{55}{156} + \frac{55}{156} + \frac{62}{156} = \frac{172}{156} = \frac{43}{39}.$$

Upshot: Large shares of a d_2 -student are in $(\frac{62}{156}, \frac{66}{156})$.

Upshot Upshot: There are NO shares in $[\frac{61}{156}, \frac{62}{156}]$

Even More Buddy Match

The following picture captures what we know so far about 3-shares.

$$\left(\begin{array}{c} 14 \\ \frac{53}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{156} \end{array} \right] \left(\begin{array}{c} x \\ \frac{58}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{61}{156} \end{array} \right] \left(\begin{array}{c} y \\ \frac{62}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{66}{156} \end{array} \right]$$

Use Buddy-Match to show that $\left| \left(\frac{61}{156}, \frac{62}{156} \right) \right| = \left| \left(\frac{62}{156}, \frac{63}{156} \right) \right|$. So:

$$\left(\begin{array}{c} 14 \\ \frac{53}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{156} \end{array} \right] \left(\begin{array}{c} x \\ \frac{58}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{61}{156} \end{array} \right] \left(\begin{array}{c} y \\ \frac{63}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{66}{156} \end{array} \right]$$

$$x + y = 10.$$

Use Buddy-Match to show that $\left| \left(\frac{58}{156}, \frac{61}{156} \right) \right| = \left| \left(\frac{63}{156}, \frac{66}{156} \right) \right|$ so they are are both 5.

$$\left(\begin{array}{c} 14 \\ \frac{53}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{55}{156} \end{array} \right] \left(\begin{array}{c} 5 \\ \frac{58}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{61}{156} \end{array} \right] \left(\begin{array}{c} 5 \\ \frac{63}{156} \end{array} \right) \left[\begin{array}{c} 0 \\ \frac{66}{156} \end{array} \right]$$

Equations

$$\left(\frac{53}{156}, 14\right) \cup \left(0, \frac{55}{156}\right) \cup \left(\frac{58}{156}, 5\right) \cup \left(\frac{61}{156}, 0\right) \cup \left(\frac{63}{156}, 5\right) \cup \left(\frac{66}{156}, \right)$$

Only the d_2 -students use $\left(\frac{63}{156}, \frac{66}{156}\right)$. Every d_2 student uses one share from that interval:

$$d_2 = 5.$$

Each d_i student uses i shares from $\left(\frac{53}{156}, \frac{55}{156}\right)$:

$$1 \times d_1 + 2 \times d_2 = 14 : \text{ So } d_1 = 4$$

There are 8 3-students:

$$d_1 + d_2 = 8 : \text{ So } 5 + 4 = 8. \text{CONTRADICTION!}$$

The Essence of The Buddy-Match Method

1. Works when $\lceil \frac{2m}{s} \rceil = 3$: Just 2-shares and 3-shares.
2. $2m$ pieces, s_2 students get 2 shares, s_3 students get 3 shares.
3. Find a GAP
4. Using BM Sequence on 3-shares-interval find intervals that cover **almost** the entire interval. Missing an interval (a, b) .
5. Use BM on (a, b) to get info on an initial interval of 3-shares.
6. Use BM on GAP to get GAPS within the 3-shares.
7. Set up linear equations relating intervals and types of students.
8. Show that system has no solution in \mathbf{N} .

Note: Can turn BM technique into a function $BM(m, s)$ such that $f(m, s) \leq BM(m, s)$.

Statistics

For $3 \leq s \leq 50$, $s + 1 \leq m \leq 59$:

$f(m, s) = FC(m, s)$ in 683 cases

$f(m, s) = INT(m, s)$ in 193 cases

$f(m, s) = BM(m, s)$ in 85 cases.

$f(m, s) = ERIK(m, s)$ in 5 cases.

$f(m, s)$ is OPEN in 18 cases.

~ 98% of the cases solved!

A Guess that Works. But Why?

1) We suspected there was a constant X such that:

$$(\forall k \geq 1) \left[f(21k + 11, 21k + 4) \leq \frac{7k + X}{21k + 4} \right]$$

2) We knew that $f(11, 4) = \frac{9}{20}$ so we conjectured $X = \frac{9}{5}$.

3) We prove the result with $X = \frac{9}{5}$ and $k \geq 1$ using BM. We prove matching lower bound for several k .

4) But the proof for $f(11, 4)$ ($k = 0$) **cannot use BM and is totally unrelated to the proof for $k \geq 1$.**

Note: This technique always worked!

Another Guess that Works But we Don't Know Why

Want to know $f(41, 19)$. Can't use BM.

$41 - 19 = 22$. So try to prove, diff d is always Mod $3d$ pattern.

Need X :

$$(\forall k \geq 1) \left[f(66k + 41, 66k + 19) \leq \frac{22k + X}{66k + 19} \right]$$

Find X using BM and linear algebra (have program for that).

Get conj: $f(41, 19) = \frac{X}{19}$.

Note: This seems to always work but have not been able to use to get new results yet.

Open Problems: $1 \leq s \leq 40, s + 1 \leq 59$

M	S	LB	UB	Method for UB
41	19	980/2280	983/2280	<i>ERIK</i>
41	23	195/483	196/4839	<i>FC</i>
54	25	215/500	216/500/	<i>FC</i>
59	26	134/312	135/312	<i>FC</i>
47	29	140/348	141/348	<i>FC</i>
49	30	340/840	343/840	<i>FC</i>
52	31	152/372	153/372	<i>INT</i>
55	31	75/186	76/186	<i>FC</i>
59	33	159/396	160/396	<i>FC</i>
55	34	164/408	165/408	<i>FC</i>
57	35	56/140	57/140	<i>FC</i>
47	36	74/216	75/216	<i>FC</i>
48	37	512/1480	515/1480	<i>BM</i>

Open Problems: $41 \leq s \leq 50$, $s + 1 \leq 59$

M	S	LB	UB	Method for UB
55	42	86/252	87/252	<i>FC</i>
53	43	183/516	184/516	<i>INT</i>
55	43	90/258	91/258	<i>INT</i>
56	43	59/172	60/172	<i>FC</i>
59	45	92/270	93/270	<i>FC</i>

Programs

We have a program that on input (m, s) :

1. We we used FC, INT, BM to get upper bounds.
2. BM method is a theorem generator.
3. Use linear algebra to try to find a lower bound (a procedure).

Results

1. FC, INT, and BM upper bounds on $f(m, s)$
2. For fixed s , for $m \geq \sim s^3$, $f(m, s) = FC(m, s)$.
3. For all $m \geq s$ $f(m, s) \geq \frac{1}{3}$.
4. For $1 \leq s \leq 7$ have proven formulas for $f(m, s)$. Mod s pattern
5. For $s = 8, \dots, 100$ conjectures for $f(m, s)$. $f(m, s)$ seems to be a mod s pattern.
6. For $1 \leq d \leq 7$ have proven formulas for $f(s + d, s)$. Mod $3d$ pattern.
7. For all d conjecture that our Theorem Generator gives $f(s + d, s)$.
8. Conjecture that for all a, d there exists X such that

$$(\forall k \geq 0) \left[f(3dk + a + d, 3dk + a) \leq \frac{dk + X}{3dk + a} \right]$$

Open Problems-Complexity

Consider:

Given m, s in binary, compute $f(m, s)$.

1. Is the problem in P? We keep on finding techniques that we think cover all cases (so it would be in P) but then finding a case not covered.
2. Is it in NP? The procedure might be very large compared to the input.
3. Is it NP-complete or NP-hard?
4. The problem IS in FPT: $(\forall m \geq s^3)[f(m, s) = FC(m, s)]$.

Conjectures That are Surely True

1. $f(m, s)$ has mod s pattern with a few exceptions (known for large m).
2. $f(s + d, s)$ has mod $3d$ pattern with no exceptions.
3. $f(m, s)$ only depends on m/s .
4. If $f(m, s) = \frac{a}{b}$ then s divides b .

Accomplishment I Am Most Proud of

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Accomplishment I Am Most Proud of:

Convinced

- ▶ 4 High School students (Guang, Naveen, Naveen, Sunny)
- ▶ 3 college student (Erik, Jacob, Daniel)
- ▶ 1 professor (John D)

that the most important field of Mathematics is **Muffinry**.