**Open Problems Column**
**Edited by William Gasarch**
**This Issue's Column!**

This issue's Open Problem Column is by William Gasarch and is *A Known Problem in Ramsey Theory: Ramsey Multiplicity*.

**Request for Columns!**

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

**A Known Problem in Ramsey Theory:**
**Ramsey Multiplicity**
by William Gasarch

# 1 Introduction

In this column we state a class of open problems that are well known in Ramsey Theory but probably not to my non-Ramsey readers. Nothing I present is original.

The problem is as follows: Let $G$ be a graph. Fill in the blank: For every 2-coloring of the edges of $K_n$ (the complete graph on $n$ vertices) there exist BLANK monochromatic copies of $G$. The key words to Google are *Ramsey Multiplicity*.

In Section 2 we state a motivating question. In Section 3 we look at $K_n$ such that you always get 2 monochromatic copies of $K_3$. In Section 4 we show that, for all 2-colorings of the edges of $K_n$ there exist $\frac{n^3}{24} - O(n^2)$ monochromatic $K_3$'s. Our upper bound is precise and matches the lower bound; however we do not prove this. In Section 5 we state some of what is known. In Section 6 we state a class of open problems.

# 2 A Motivating Question

We abbreviate *monochromatic* by *mono* throughout.

Recall the first theorem one usually hears in Ramsey Theory:

*For all 2-colorings of the edges of $K_6$ there is a mono triangle.*

From this we obtain a trivial theorem:

*For all 2-colorings of the edges of $K_{12}$ there are 2 mono triangles.*

How big does $n$ have to be such that any 2-coloring of $K_n$ has 2 mono triangles? The answer is in the next section and may surprise you.

# 3 Two Monochromatic $K_3$

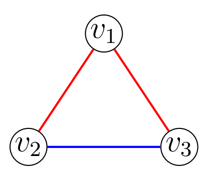
**Theorem 3.1** *For all 2-colorings of edges of $K_6$ there are 2 mono triangles.*

**Proof:** Let COL be a 2-coloring of the edges of $K_6$. Let $R$, $B$, $M$, be the sets of RED, BLUE, and MIXED (having both RED and BLUE edges) triangles, respectively. Clearly

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \le 18$, so $|R| + |B| \ge 2$.

Let $T$ be a mixed triangle. It looks like this:

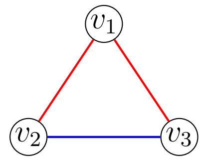



Note that there will be two vertices that have both a RED and a BLUE edge coming out of them.

- $(v_2, v_1)$ is red, $(v_2, v_3)$ is blue. View this as $(v_2, \{v_1, v_3\})$.
- $(v_3, v_1)$ is red, $(v_3, v_2)$ is blue. View this as $(v_3, \{v_1, v_2\})$.

**Def 3.2** A *Zan* is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and $\mathrm{COL}(v, u) \ne \mathrm{COL}(v, w)$. ZAN is the set of all Zans.

Map ZAN to $M$ by mapping $(v, \{u, w\})$ to triangle $(v, u, w)$. This mapping is exactly 2-to-1: every element of $M$ has two Zans mapping to it. The Zans that map to




are $(v_2, \{v_1, v_3\})$ and $(v_3, \{v_1, v_2\})$.

Since there is a 2-to-1 map from ZAN to $M$, $|M| = |\text{ZAN}|/2$. Now we want to bound $|\text{ZAN}|$. Look at how much each vertex can contribute to ZAN. Note that each vertex has degree 5.

**Cases:**

1. $v$ has $\deg_R(v) = 5$ and $\deg_B(v) = 0$: $v$ contributes 0.
2. $v$ has $\deg_R(v) = 4$ and $\deg_B(v) = 1$: $v$ contributes 4.
3. $v$ has $\deg_R(v) = 3$ and $\deg_B(v) = 2$: $v$ contributes 6. Max.

There are 6 vertices, each contribute $\le 6$, $|M| \le |\text{ZAN}|/2 \le 6 \times 6/2 = 18$

$$|R| + |B| = 20 - |M| \ge 2$$

∎

# 4 Many Mono Triangles

The following theorem was first proven by Goodman [5]; however, we give an easier proof given by Schwenk [9], as presented by Dorwart and Finkbeiner [2].

**Theorem 4.1**

1. *Assume $n \equiv 1 \pmod 4$. For all 2-colorings of the edges of $K_n$ there are at least $\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$ mono triangles.*
2. *Assume $n \equiv 3 \pmod 4$. For all 2-colorings of the edges of $K_n$ there are at least $\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}$ mono triangles.*
3. *Assume $n \equiv 0 \pmod 2$. For all 2-colorings of the edges of $K_n$ there are at least $\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}$ mono triangles*

**Proof:**

A porism to the proof of Theorem 3.1 is that

$$|R| + |B| = \binom{n}{3} - |\mathrm{ZAN}|/2.$$

Hence we will upper bound $|\mathrm{ZAN}|$.

**Case 1:** $n \equiv 1 \pmod 2$. The degree of each vertex is $n-1 \equiv 0 \pmod 2$. To maximize $|\mathrm{ZAN}|$ we would, at each vertex, color half of the edges RED and half BLUE. So each vertex contributes $(\frac{n-1}{2})^2$, and there are $n$ vertices, so we have $|\mathrm{ZAN}| \le \frac{(n-1)^2 n}{4}$. Since $|\mathrm{ZAN}|/2 = M \in \mathsf{N}$ we have

$$|\mathrm{ZAN}|/2 \le \left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor$$

hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor$$

**Case 1a:** $n \equiv 1 \pmod 4$ so $(n-1)^2 \equiv 0 \pmod{16}$. Hence:

$$\left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor = \frac{(n-1)^2 n}{8}.$$

Hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

**Case 1b:** $n \equiv 3 \pmod 4$. One can easily show that

$$\left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor = \frac{(n-1)^2 n - 4}{8}.$$

Hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n - 4}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}$$

**Case 2:** $n \equiv 0 \pmod 2$. The degree of each vertex is $n-1 \equiv 1 \pmod 2$. To maximize $|\mathrm{ZAN}|$ we would, at each vertex, color $\frac{n-2}{2}$ of the edges RED and color $\frac{n}{2}$ of the edges BLUE. So each vertex contributes $\frac{n(n-2)}{4}$, and there are $n$ vertices, so we have $|\text{ZAN}| \le \frac{n^2(n-2)}{4}$. Since $|\text{ZAN}|/2 = M \in \mathsf{N}$ we have

$$|\text{ZAN}|/2 \le \left\lfloor \frac{n^2(n-2)}{8} \right\rfloor.$$

Since $n \equiv 0 \pmod 2$, $n^2(n-2) \equiv 0 \pmod 8$, so the bound on $|\text{ZAN}|/2$ is always in $\mathsf{N}$. Hence we do not need the floor. Hence

$$|\text{ZAN}|/2 \le \frac{n^2(n-2)}{8}$$

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{n^2(n-2)}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}.$$

■

**Note 4.2** The bounds given in Theorem 4.1 are tight. This was proven by Goodman [5] and Savvy [8]. They view the problem differently. They asked *given that you want $t$ triangles, how big must $n$ be?*

# 5 What is Already Known

**Def 5.1** $R(k)$ is the least number $n$ such that any 2-coloring of the edges of $K_n$ has a mono $K_k$. By Ramsey's theorem for graphs, for all $k$, $R(k)$ exists. It is known that $R(3) = 6$ and $R(4) = 18$; however, all that is known about $R(5)$ is $43 \le R(5) \le 48$. It is a standard result that $R(k) \le 4^{k - c\log k}$ for some constant $c$.

1. Stanislaw Radziszowski and Konrad Piwakowski [6] proved the following: All 2-colorings of $K_{18}$ have 9 mono $K_4$'s. This is tight. This was proven with the help of a computer.

2. Erdős [3] proved the following:

   **Theorem 5.2** *Let $k \in \mathsf{N}$. Let $R = R(k)$. Let $N$ be large, so large that $R \ll N$. Let $COL$ be a 2-coloring of the edges of $K_N$. Then there are $\ge \frac{N^k}{4^{(1+o(1))k^2}}$ mono $K_k$'s.*

We give the proof because we can.

**Proof:** Let $A_1, \ldots, A_{\binom{N}{R}}$ be a list of all the $R$-subsets of $[N]$.

Note that by the definition of $R$ and the $A_i$'s, every $A_i$ has a mono $K_k$. We can't just say there are $\binom{N}{R}$ mono $K_k$'s since it may be that two $A_i$'s produce the same mono $K_k$.

We now produce many monochromatic $K_k$'s.

(a) $X = \left\{A_i : 1 \le i \le \binom{N}{R}\right\}$

(b) $Y = \emptyset$. $Y$ will contain many mono $K_k$'s.

(c) If $X \neq \emptyset$ do the following (else terminate). Let $i$ be the least number such that $A_i \in X$. It has a mono $K_k$. Call it $C$.

   i. Add $C$ to $Y$ (we will soon see that $C$ is not already in $Y$).

   ii. Remove from $X$ all $A_j$'s that have $C$ in them. Hence we are removing $\binom{N-k}{R-k}$ $A_j$'s.

   iii. Goto Step c

In every iteration $X$ loses $\binom{N-k}{R-k}$ $A_j$'s. Hence the number of mono $K_k$'s that this process produces is at least

$$\frac{\binom{N}{R}}{\binom{N-k}{R-k}} = \frac{N!}{R!(N-R)!} \times \frac{(N-R)!(R-k)!}{(N-k)!} = \frac{N!}{(N-k)!} \times \frac{(R-k)!}{R!}.$$

We need to lower bound this quantity. We use $\frac{(R-k)!}{R!} \ge \frac{1}{R^k} \ge \frac{1}{4^{k^2}}$. The other inequality we need will be more delicate.

$$\frac{N!}{(N-k)!} \ge (N-k+1)^k = \frac{N^k}{(N/(N-k+1))^k}$$

We look at the denominator

$$\left(\frac{N}{N-k+1}\right)^k = \left(1+\frac{k-1}{N-k+1}\right)^k \sim e^{((k-1)k)/(N-k+1)} \sim 4^{(c(k-1)k)/(N-k+1)}$$

for some constant $c$. Hence the number of mono $K_k$'s is at least

$$\frac{N^k}{4^{k^2+(c(k-1)k/(N-k+1))}} = \frac{N^k}{4^{k^2(1+o(1))}}.$$

$\blacksquare$

3. David Conlon [1] proved the following: Fix $t$. For $n$ large, for any 2-coloring of $K_n$ there are $\frac{n^t}{C^{(1+o(1))t^2}}$ mono $K_t$'s where $C \sim 2.18$. Note that Conlon's result is an improvement of Theorem 5.2.

4. Jacob Fox [4] looks at the problem for target graphs $G$ other than $K_t$.

5. For more results (1) look at the bibliographies of the papers above, (2) Google *Ramsey Multiplicity*, and (3) watch this cool lecture by David Conlon:

   `https://www.ima.umn.edu/2014-2015/W9.8-12.14/21327`

# 6 Open Problems

Here is a large class of open problems with the same theme as Theorem 4.1. First we need some notation.

**Notation 6.1** Let $k \geq 1$.

1. $C_k$ is the cycle graph on $k$ vertices. Vertex $i$ has an edge to vertices $i-1 \pmod{k}$ and $i+1 \pmod{k}$ but no other vertices.

2. $P_k$ is the path graph on $k$ vertices. If $i \leq k-1$ then vertex $i$ has an edge to vertex $i+1$.

3. $W_k$ is the wheel graph on $k$ vertices. It is $C_{k-1}$ with one more vertex that has an edge to all other vertices.

4. $K_{1,k}$ is the star graph.

5. If $G$ is a graph on $k$ vertices, $R(G)$ is the least number $n$ such that any 2-coloring of the edges of $K_n$ has a mono $G$. By Ramsey's theorem for graphs, for all $G$, $R(G)$ exists; however, for many graphs, $R(G)$ is much lower than $R(k)$.

6. The values of $R(G)$ are known for $G \in \{C_k, P_k, W_k, K_{1,k}\}$. See Stanislaw Radziszowski's survey of Small Ramsey Numbers [7].

We now state some open problems.

1. Fix $k$. Find the function $f$ such that, for all 2-colorings of $K_n$ there are $f(n)$ mono $K_k$'s. Try to make $f(n)$ as large as possible. The $k = 3$ case is solved. The $k = 4$ case is not solved, but it is plausible that it will be. The $k = 5$ case is harder than finding $R(5)$ and hence is unlikely to be solved ... ever.

2. Replace $K_k$ in the last item with $C_k$, $P_k$, $W_k$, $K_{1,k}$ or whatever your favorite parameterized set of graphs is. When $R(G)$ is known there is hope of solving this problem. If $R(G)$ is unknown, then note that the problem of finding $f(n)$ is harder than finding $R(G)$.

3. Let $G$ be your favorite graph and $L$ be your favorite number. Find the least $n$ (or at least a non-obvious $k$) such that every 2-coloring of the edges of $K_n$ yields $L$ copies of a mono $G$.

# 7 Acknowledgments

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