Open Problems Column
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This Issue's Column!

This issue's Open Problem Column is by William Gasarch and is *A Known Problem in Ramsey Theory: Ramsey Multiplicity.*

Request for Columns!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

A Known Problem in Ramsey Theory:
Ramsey Multiplicity
by William Gasarch

1 Introduction

In this column we state a class of open problems that are well known in Ramsey Theory but probably not to my non-Ramsey readers. Nothing I present is original.

The problem is as follows: Let G be a graph. Fill in the blank: For every 2-coloring of the edges of K_n (the complete graph on n vertices) there exist BLANK monochromatic copies of G. The key words to Google are *Ramsey Multiplicity.*

In Section 2 we state a motivating question. In Section 3 we look at K_n such that you always get 2 monochromatic copies of K_3. In Section 4 we show that, for all 2-colorings of the edges of K_n there exist $\frac{n^3}{24} - O(n^2)$ monochromatic K_3's. Our upper bound is precise and matches the lower bound; however we do not prove this. In Section 5 we state some of what is known. In Section 6 we state a class of open problems.

2 A Motivating Question

We abbreviate *monochromatic* by *mono* throughout.

Recall the first theorem one usually hears in Ramsey Theory:

For all 2-colorings of the edges of K_6 there is a mono triangle.

From this we obtain a trivial theorem:

For all 2-colorings of the edges of K_{12} there are 2 mono triangles.

How big does n have to be such that any 2-coloring of K_n has 2 mono triangles? The answer is in the next section and may surprise you.

3 Two Monochromatic K_3

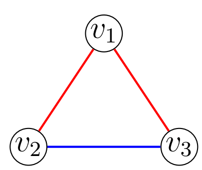
Theorem 3.1 *For all 2-colorings of edges of K_6 there are 2 mono triangles.*

Proof: Let COL be a 2-coloring of the edges of K_6. Let R, B, M, be the sets of RED, BLUE, and MIXED (having both RED and BLUE edges) triangles, respectively. Clearly

$$|R| + |B| + |M| = \binom{6}{3} = 20.$$

We show that $|M| \le 18$, so $|R| + |B| \ge 2$.

Let T be a mixed triangle. It looks like this:

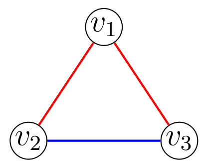


Note that there will be two vertices that have both a RED and a BLUE edge coming out of them.

- (v_2, v_1) is red, (v_2, v_3) is blue. View this as $(v_2, \{v_1, v_3\})$.
- (v_3, v_1) is red, (v_3, v_2) is blue. View this as $(v_3, \{v_1, v_2\})$.

Def 3.2 A *Zan* is an element $(v, \{u, w\}) \in V \times \binom{V}{2}$ such that $v \notin \{u, w\}$ and $\mathrm{COL}(v, u) \neq \mathrm{COL}(v, w)$. ZAN is the set of all Zans.

Map ZAN to M by mapping $(v, \{u, w\})$ to triangle (v, u, w). This mapping is exactly 2-to-1: every element of M has two Zans mapping to it. The Zans that map to



are $(v_2, \{v_1, v_3\})$ and $(v_3, \{v_1, v_2\})$.

Since there is a 2-to-1 map from ZAN to M, $|M| = |\text{ZAN}|/2$. Now we want to bound $|\text{ZAN}|$. Look at how much each vertex can contribute to ZAN. Note that each vertex has degree 5.

Cases:

1. v has $\deg_R(v) = 5$ and $\deg_B(v) = 0$: v contributes 0.
2. v has $\deg_R(v) = 4$ and $\deg_B(v) = 1$: v contributes 4.
3. v has $\deg_R(v) = 3$ and $\deg_B(v) = 2$: v contributes 6. Max.

There are 6 vertices, each contribute ≤ 6, $|M| \leq |\text{ZAN}|/2 \leq 6 \times 6/2 = 18$

$$|R| + |B| = 20 - |M| \geq 2$$

∎

4 Many Mono Triangles

The following theorem was first proven by Goodman [5]; however, we give an easier proof given by Schwenk [9], as presented by Dorwart and Finkbeiner [2].

Theorem 4.1

1. *Assume* $n \equiv 1 \pmod 4$. *For all 2-colorings of the edges of* K_n *there are at least* $\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$ *mono triangles.*
2. *Assume* $n \equiv 3 \pmod 4$. *For all 2-colorings of the edges of* K_n *there are at least* $\frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}$ *mono triangles.*
3. *Assume* $n \equiv 0 \pmod 2$. *For all 2-colorings of the edges of* K_n *there are at least* $\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}$ *mono triangles*

Proof:

A porism to the proof of Theorem 3.1 is that

$$|R| + |B| = \binom{n}{3} - |\text{ZAN}|/2.$$

Hence we will upper bound |ZAN|.

Case 1: $n \equiv 1 \pmod 2$. The degree of each vertex is $n - 1 \equiv 0 \pmod 2$. To maximize |ZAN| we would, at each vertex, color half of the edges RED and half BLUE. So each vertex contributes $(\frac{n-1}{2})^2$, and there are n vertices, so we have $|\text{ZAN}| \le \frac{(n-1)^2 n}{4}$. Since $|\text{ZAN}|/2 = M \in \mathsf{N}$ we have

$$|\text{ZAN}|/2 \le \left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor$$

hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor$$

Case 1a: $n \equiv 1 \pmod 4$ so $(n-1)^2 \equiv 0 \pmod{16}$. Hence:

$$\left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor = \frac{(n-1)^2 n}{8}.$$

Hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24}$$

Case 1b: $n \equiv 3 \pmod 4$. One can easily show that

$$\left\lfloor \frac{(n-1)^2 n}{8} \right\rfloor = \frac{(n-1)^2 n - 4}{8}.$$

Hence

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{(n-1)^2 n - 4}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{5n}{24} + \frac{1}{2}$$

Case 2: $n \equiv 0 \pmod 2$. The degree of each vertex is $n - 1 \equiv 1 \pmod 2$. To maximize |ZAN| we would, at each vertex, color $\frac{n-2}{2}$ of the edges RED and color $\frac{n}{2}$ of the edges BLUE. So each vertex contributes $\frac{n(n-2)}{4}$, and there are n vertices, so we have $|\text{ZAN}| \le \frac{n^2(n-2)}{4}$. Since $|\text{ZAN}|/2 = M \in \mathsf{N}$ we have

$$|\text{ZAN}|/2 \le \left\lfloor \frac{n^2(n-2)}{8} \right\rfloor.$$

Since $n \equiv 0 \pmod 2$, $n^2(n-2) \equiv 0 \pmod 8$, so the bound on $|\text{ZAN}|/2$ is always in N. Hence we do not need the floor. Hence

$$|\text{ZAN}|/2 \le \frac{n^2(n-2)}{8}$$

$$|R| + |B| \ge \frac{n(n-1)(n-2)}{6} - \frac{n^2(n-2)}{8} = \frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}.$$

$\blacksquare$

Note 4.2 The bounds given in Theorem 4.1 are tight. This was proven by Goodman [5] and Savvy [8]. They view the problem differently. They asked *given that you want t triangles, how big must n be?*

5 What is Already Known

Def 5.1 $R(k)$ is the least number n such that any 2-coloring of the edges of K_n has a mono K_k. By Ramsey's theorem for graphs, for all k, $R(k)$ exists. It is known that $R(3) = 6$ and $R(4) = 18$; however, all that is known about $R(5)$ is $43 \le R(5) \le 48$. It is a standard result that $R(k) \le 4^{k - c\log k}$ for some constant c.

1. Stanislaw Radziszowski and Konrad Piwakowski [6] proved the following: All 2-colorings of K_{18} have 9 mono K_4's. This is tight. This was proven with the help of a computer.

2. Erdős [3] proved the following:

   Theorem 5.2 *Let $k \in \mathsf{N}$. Let $R = R(k)$. Let N be large, so large that $R \ll N$. Let COL be a 2-coloring of the edges of K_N. Then there are $\ge \frac{N^k}{4^{(1+o(1))k^2}}$ mono K_k's.*

We give the proof because we can.

Proof: Let $A_1, \ldots, A_{\binom{N}{R}}$ be a list of all the R-subsets of $[N]$.

Note that by the definition of R and the A_i's, every A_i has a mono K_k. We can't just say there are $\binom{N}{R}$ mono K_k's since it may be that two A_i's produce the same mono K_k.

We now produce many monochromatic K_k's.

(a) $X = \left\{A_i : 1 \le i \le \binom{N}{R}\right\}$

(b) $Y = \emptyset$. Y will contain many mono K_k's.

(c) If $X \ne \emptyset$ do the following (else terminate). Let i be the least number such that $A_i \in X$. It has a mono K_k. Call it C.

   i. Add C to Y (we will soon see that C is not already in Y).

   ii. Remove from X all A_j's that have C in them. Hence we are removing $\binom{N-k}{R-k}$ A_j's.

   iii. Goto Step c

In every iteration X loses $\binom{N-k}{R-k}$ A_j's. Hence the number of mono K_k's that this process produces is at least

$$\frac{\binom{N}{R}}{\binom{N-k}{R-k}} = \frac{N!}{R!(N-R)!} \times \frac{(N-R)!(R-k)!}{(N-k)!} = \frac{N!}{(N-k)!} \times \frac{(R-k)!}{R!}.$$

We need to lower bound this quantity. We use $\frac{(R-k)!}{R!} \ge \frac{1}{R^k} \ge \frac{1}{4^{k^2}}$. The other inequality we need will be more delicate.

$$\frac{N!}{(N-k)!} \ge (N-k+1)^k = \frac{N^k}{(N/(N-k+1))^k}$$

We look at the denominator

$$\left(\frac{N}{N-k+1}\right)^k = \left(1+\frac{k-1}{N-k+1}\right)^k \sim e^{((k-1)k)/(N-k+1)} \sim 4^{(c(k-1)k)/(N-k+1)}$$

for some constant c. Hence the number of mono K_k's is at least

$$\frac{N^k}{4^{k^2+(c(k-1)k/(N-k+1))}} = \frac{N^k}{4^{k^2(1+o(1))}}.$$

■

3. David Conlon [1] proved the following: Fix t. For n large, for any 2-coloring of K_n there are $\frac{n^t}{C^{(1+o(1))t^2}}$ mono K_t's where $C \sim 2.18$. Note that Conlon's result is an improvement of Theorem 5.2.

4. Jacob Fox [4] looks at the problem for target graphs G other than K_t.

5. For more results (1) look at the bibliographies of the papers above, (2) Google *Ramsey Multiplicity*, and (3) watch this cool lecture by David Conlon:

   `https://www.ima.umn.edu/2014-2015/W9.8-12.14/21327`

6 Open Problems

Here is a large class of open problems with the same theme as Theorem 4.1. First we need some notation.

Notation 6.1 Let $k \geq 1$.

1. C_k is the cycle graph on k vertices. Vertex i has an edge to vertices $i-1 \pmod{k}$ and $i+1 \pmod{k}$ but no other vertices.

2. P_k is the path graph on k vertices. If $i \leq k-1$ then vertex i has an edge to vertex $i+1$.

3. W_k is the wheel graph on k vertices. It is C_{k-1} with one more vertex that has an edge to all other vertices.

4. $K_{1,k}$ is the star graph.

5. If G is a graph on k vertices, $R(G)$ is the least number n such that any 2-coloring of the edges of K_n has a mono G. By Ramsey's theorem for graphs, for all G, $R(G)$ exists; however, for many graphs, $R(G)$ is much lower than $R(k)$.

6. The values of $R(G)$ are known for $G \in \{C_k, P_k, W_k, K_{1,k}\}$. See Stanislaw Radziszowski's survey of Small Ramsey Numbers [7].

We now state some open problems.

1. Fix k. Find the function f such that, for all 2-colorings of K_n there are $f(n)$ mono K_k's. Try to make $f(n)$ as large as possible. The $k = 3$ case is solved. The $k = 4$ case is not solved, but it is plausible that it will be. The $k = 5$ case is harder than finding $R(5)$ and hence is unlikely to be solved ... ever.

2. Replace K_k in the last item with C_k, P_k, W_k, $K_{1,k}$ or whatever your favorite parameterized set of graphs is. When $R(G)$ is known there is hope of solving this problem. If $R(G)$ is unknown, then note that the problem of finding $f(n)$ is harder than finding $R(G)$.

3. Let G be your favorite graph and L be your favorite number. Find the least n (or at least a non-obvious k) such that every 2-coloring of the edges of K_n yields L copies of a mono G.

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