Finite, Infinite, and Large Canonical Ramsey Theorems
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1 Introduction

Henceforth we use *Can* instead of *Canonical*

We prove a general theorem from which one can obtain (1) the infinite 2-ary Can Ramsey Theorem, (2) the finite 2-ary Can Ramsey Theorem, and (3) the large 2-ary Can Ramsey Theorem. Everything in this paper was already known; however (1) to our knowledge, the statement and proof of the large Can Ramsey Theorem never been written down, and (2) our bounds on the large Can Ramsey Number may be new.

Let $LCR_a(k)$ be the large a-ary Can Ramsey number and $LR_a(k)$ be the large a-ary hypergraph Ramsey number. Our proof bounds $LCR_2(k)$ in terms of $LR_3(\frac{k^3}{2})$ (we will define this later). We discuss why it is unlikely to obtain a bound on $LCR_2(k)$ in terims of $LR_2(f(k))$ for reasonable f.

We sketch a general theorem from which one can obtain (1) the infinite a-ary Can Ramsey Theorem, (2) the finite a-ary Can Ramsey Theorem, and (3) the large a-ary Can Ramsey Theorem.

Our proof bounds $LCR_a(k)$ in terms of $LR_{2a-1}(FILLINLATER)$ (we will define this later). We discuss why our reasons to think one cannot bound $LCR_2(k)$ in terms of $LR_2(f(k))$ do not extend. Hence it is plausible that $LCR_a(k)$ can be bounded by $CR_{2a-2}(f(k))$ for some reasonable f.

2 Conventions, Definitions, Notation

Convention 2.1 We will often state a theorem for all $A \subseteq \mathsf{N}$. We are thinking N or $[n]$ or $\{k, \ldots, n\}$.

Def 2.2 Let A be any subset of N. Let $COL\colon \binom{A}{2} \rightarrow \mathsf{N}$. In the cases below $x_1 < y_1$ and $x_2 < y_2$.

1. H is *COL-homog* if

$$(\forall x_1, y_1, x_2, y_2 \in H)[COL(x_1, y_1) = COL(x_2, y_2)]$$

(All the elements of $\binom{H}{2}$ are colored the same.)

2. H is *COL-min homog* if

$$(\forall x_1, y_1, x_2, y_2 \in H)[COL(x_1, y_1) = COL(x_2, y_2) \text{ iff } x_1 = x_2]$$

(The color of a pair x, y depends exactly on $\min\{x, y\}$.)

3. H is *COL-max homog* if

$$(\forall x_1, y_1, x_2, y_2 \in H)[COL(x_1, y_1) = COL(x_2, y_2) \text{ iff } y_1 = y_2]$$

(The color of a pair x, y depends exactly on $\max\{x, y\}$.)

4. H is *COL-rainbow* if

$$(\forall x_1, y_1, x_2, y_2 \in H)[COL(x_1, y_1) = COL(x_2, y_2) \text{ iff } (x_1 = x_2 \wedge y_1 = y_2)]$$

(All the elements of $\binom{H}{2}$ are colored differently.)

5. H is *COL-cool* if H is either COL-homog, COL-min-homog, COL-max-homog or COL-rainbow.

We may drop the prefix of COL when the coloring is understood.

Def 2.3 Let $A \subseteq \mathsf{N}$.

1. A is *large* if $|A| > \min(A)$.

2. Let $f\colon \mathsf{N} \to \mathsf{N}$ be a monotone increasing function. A is *f-large* if $|A| > f(\min(A))$.

We state three *2-ary Can Ramsey Theorems*. Henceforth we will use the phrase *Can Ramsey*.

Theorem 2.4

1. *(Infinite Can Ramsey) For every coloring $COL\colon \binom{\mathsf{N}}{2} \to \mathsf{N}$ there exists an infinite H that is cool.*

2. *(Finite Can Ramsey) For every k there exists n such that for every coloring $COL\colon \binom{[n]}{2} \to \mathsf{N}$ there exists an H, $|H| \geq k$, that is cool. We denote this n by $CR_2(k)$.*

3. *(Large Can Ramsey) For every k there exists n such that for every coloring $COL\colon \binom{\{k,\ldots,n\}}{2} \to \mathsf{N}$ there exists an H, H large, that is cool. We denote this n by $LCR_2(k)$.*

3 Needed Lemmas

We state four Ramsey Theorems in one lemma. First we need a definition.

Def 3.1 Let A be any subset of N. Let $a, c \in \mathsf{N}$. Let $COL\colon \binom{A}{a} \rightarrow [c]$. A set $H \subseteq A$ is *homogeneous* if COL restricted to $\binom{H}{a}$ is constant (so all the hyperedges of H are the same color). Henceforth we refer to homogeneous as homog.

Lemma 3.2

1. *(a-ary infinite Ramsey) For every $a, c \in \mathsf{N}$, for every coloring $COL\colon \binom{\mathsf{N}}{a} \rightarrow [c]$ there exists an infinite homog H.*
2. *(a-ary finite Ramsey) For every $a, c \in \mathsf{N}$, for every k there exists n such that for every coloring $COL\colon \binom{[n]}{a} \rightarrow [c]$ there exists an H, $|H| \geq k$, that is homog. We denote this n by $R_a(k, c)$.*
3. *(a-ary Large Ramsey) For every $a, c \in \mathsf{N}$, for every k there exists n such that for every coloring $COL\colon \binom{\{k,\ldots,n\}}{a} \rightarrow [c]$ there exists an H, H large, that is homog. We denote this n by $LR_a(k, c)$.*
4. *(Extended a-ary Large Ramsey) Let f be a monotone increasing function. For every $a, c \in \mathsf{N}$, for every k there exists n such that for every coloring $COL\colon \binom{\{k,\ldots,n\}}{a} \rightarrow [c]$ there exists an H, H f-large, that is homog. We denote this n by $LR_a(f(k), c)$.*

We will have lemmas that will have as corollaries both Infinite and Finite Can Ramsey Theorems. Hence we need the following conventions.

Convention 3.3 If $X \subseteq \mathsf{N}$ and $r \in \mathsf{Q}$ then

$$|X|^r = \begin{cases} \infty & \text{if } |X| \text{ is infinite} \\ |X|^r & \text{if } |X| \text{ is finite} \end{cases} \tag{1}$$

We use the same convention for other functions such as $(2|X|)^r$ or $|X| - 1$,

4 Infinite, Finite, and Large 2-ary Can Ramsey

In this section we use the infinite, finite, and large 3-ary Hypergraph Ramsey Theorem to get infinite, finite, and large 2-ary Can Ramsey Theorem. This proof is a tweek of the proof of 2-ary Can Ramsey from Lefmann and Rödl [2].

Def 4.1 Let A be any subset of N. Let $COL\colon \binom{A}{2} \to \mathsf{N}$. Let $c \in \mathsf{N}$.

1. If $v \in A$ then $\deg_c(v)$ is the number of edges colored c that have v as an endpoint.
2. A set $M \subseteq A$ is *maximal rainbow* if (1) M is rainbow, (2) if $v \notin M$ then $M \cup \{v\}$ is not rainbow.

Lemma 4.2 *Let $A \subseteq \mathsf{N}$. Let COL be a coloring of $\binom{A}{2}$. Assume that for all $v \in A$, and all colors c, $\deg_c(v) \le 1$. If M is a maximal rainbow set then $|M| \ge (2|A|)^{1/3}$.*

Proof:
Let M be a maximal rainbow set. This means that,

$$(\forall y \in A - M)[M \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in A - M$. Why is $y \notin M$? One of the following must occur:

1. There exists $u, u_1, u_2 \in M$ such that $u_1 \ne u_2$ and $COL(y,u) = COL(u_1,u_2)$. (It is possible for $u = u_1$ or $u = u_2$.)
2. There exists $u_1 \ne u_2 \in M$ such that $COL(y,u_1) = COL(y,u_2)$. This cannot happen since then y has some color degree ≥ 2.

We map $A - M$ to $M \times \binom{M}{2}$ by mapping $y \in A - M$ to $(u, \{u_1,u_2\})$ as indicated in item 1 above. This map is injective since if y_1 and y_2 both map to $(u, \{u_1,u_2\})$ then $COL(y_1,u) = COL(y_2,u)$.

This map has domain of size $|A| - |M|$ and co-domain of size $|M|\binom{|M|}{2}$. Hence

$$|A - M| \le |M|\binom{|M|}{2}$$

1. If A is infinite and M is finite then the above inequality is a contradiction. Hence if A is infinite then M is infinite.

2. If A is finite then

$$|A-M| \le |M|\binom{|M|}{2} = |M|^2(|M|-1)/2 = \frac{|M|^3-|M|^2}{2} \le \frac{|M^3}{2} - |M|$$

$$|A| \le \frac{|M|^3}{2}.$$

$$|M| \ge (2|A|)^{1/3}.$$

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Theorem 4.3 *Let A be any subset of* N. *Let $COL\colon \binom{A}{2} \to \mathsf{N}$. There exists $COL'\colon \binom{A}{3} \to [5]$ such that if H is a COL'-homog set of size at least 5 then one of the following holds:*

- *H is* COL-*homog,* COL-*min-homog, or* COL-*max-homog.*
- *Every maximal* COL-*rainbow subset of H has size* $\ge (2|H|)^{1/3}$.

Proof: We are given $COL\colon \binom{A}{2} \to \mathsf{N}$. We use COL to define $COL'\colon \binom{A}{3} \to [5]$. As we define COL' we will say what happens if the COL'-homog set H is of the color indicated.

In the cases below $x_1 < x_2 < x_3$.

1. If $\mathrm{COL}(x_1, x_2) = COL(x_1, x_3) = \mathrm{COL}(x_2, x_3)$ then $\mathrm{COL}'(x_1, x_2, x_3) = 1$.

   Assume H is colored 1. By renumbering we assume that $H = \{1, 2, 3, \ldots, \}$ (possibly finite). We show that H is homog by showing that element of $\binom{H}{2}$ is colored $\mathrm{COL}(1, 2)$.

   Let $(c, d) \in \binom{H}{2}$ with $c \le d$.

   (a) If $c = 1$ then $\mathrm{COL}(c, d) = \mathrm{COL}(1, 2)$ by taking $x_1 = 1 = c$, $x_2 = 2$, $x_3 = d$.

(b) If $c = 2$ then $\mathrm{COL}(c, d) = \mathrm{COL}(1, 2)$ by taking $x_1 = 1$, $x_2 = 2 = c$, $x_3 = d$.

(c) If $c \ge 3$ then $\mathrm{COL}(1, 2) = \mathrm{COL}(3, 4) = \cdots = \mathrm{COL}(c, c+1) = \mathrm{COL}(c, d)$.

2. If $\mathrm{COL}(x_1, x_2) = COL(x_1, x_3)$ and $\mathrm{COL}(x_1, x_2) \ne COL(x_2, x_3)$ then $COL'(x_1, x_2, x_3) = 2$.

   Assume H is colored 2. We show that H is min-homog. Let $x_1 < x_2$ $y_1 < y_2$. We can also assume $x_1 \le y_1$.

   We need to show

   $$\mathrm{COL}(x_1, x_2) = \mathrm{COL}(y_1, y_2) \text{ iff } x_1 = y_1.$$

   Clearly if $x_1 = y_1$ then $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(y_1, y_2)$.

   If $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(y_1, y_2)$ then look at $\mathrm{COL}(x_1, y_1)$. Note that

   $$\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_1, y_1) = \mathrm{COL}(x_1, y_2).$$

   Hence they all equal $\mathrm{COL}(y_1, y_2)$. In particular

   $$\mathrm{COL}(x_1, y_1) = \mathrm{COL}(y_1, y_2).$$

   This is only possible of $x_1 = y_1$.

3. If $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_2, x_3)$ and $\mathrm{COL}(x_1, x_2) \ne \mathrm{COL}(x_1, x_3)$ then $COL'(x_1, x_2, x_3) = 3$.

   Assume H is colored 3. The reader can show that H is max-homog in a manner similar to the proof of min-homog in the last part.

4. If $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_2, x_3)$ and $\mathrm{COL}(x_1, x_2) \ne \mathrm{COL}(x_1, x_3)$ then $\mathrm{COL}'(x_1, x_2, x_3) = 4$.

   We show that any if $|H| \le 4$. Assume, by way of contradiction, that $|H| \ge 5$. Renumber so that $\{1, 2, 3, 4, 5\} \subseteq H$. Then:

   $$\mathrm{COL}(1, 2) = \mathrm{COL}(2, 3) = \mathrm{COL}(3, 4) = \mathrm{COL}(4, 5)$$

and

$$\mathrm{COL}(1,3) = \mathrm{COL}(3,4).$$

Hence $\mathrm{COL}(1,2) = \mathrm{COL}(1,3)$, which contradicts the coloring.

5. (The reader can show that this is the only case left.) If $COL'(x_1,x_2) \neq COL'(x_1,x_3)$, $COL'(x_1,x_2) \neq COL'(x_2,x_3)$ and $COL'(x_1,x_3) \neq COL'(x_2,x_3)$ then $COL'(x_1,x_2,x_3) = 5$.

   Assume H is colored 5. Clearly, for all $v \in H$ and for all colors c, $\deg_c(v) \le 1$. By Lemma 4.2 every maximal rainbow subset of H is $\ge (2|H|)^{1/3}$.

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Corollary 4.4

1. *For every coloring* $COL\colon \binom{\mathsf{N}}{2} \to \mathsf{N}$ *there exists an infinite* H *that is cool.*

2. *For every* k *there exists* n *such that for every coloring* $COL\colon \binom{[n]}{2} \to \mathsf{N}$ *there exists an* H, $|H| \ge k$, *that is cool. We can take* $n = R_3(\frac{k^3}{2}, 5)$. *Hence* $CR_2(k) \le R_3(\frac{k^3}{2}, 5)$.

3. *For every* k *there exists* n *such that for every coloring* $COL\colon \binom{\{k,\ldots,n\}}{2} \to \mathsf{N}$ *there exists an* H, H *large, that is cool. We can take* $n = LR_3(\frac{k^3}{2}, 5)$. *Hence* $LCR_2(k) \le LR_3(\frac{k^3}{2}, 5)$.

Proof:
1) Let $COL\colon \binom{\mathsf{N}}{2} \to \mathsf{N}$. By Theorem 4.3 with $A = \mathsf{N}$ there exists a coloring $COL'\colon \binom{\mathsf{N}}{3} \to [5]$ such that if H is COL'-homog then one of the following holds:

- H is COL-homog, COL-min-homog, or COL-max-homog.

- Every maximal COL-rainbow subset of H is of size $\ge (2|H|)^{1/3}$.

By Lemma 3.2.1 there is an *infinite* COL$'$-homog set. Hence there is an infinite cool set.
2) Let $COL\colon \binom{[R_3(\frac{k^3}{2},5)]}{2} \to \mathsf{N}$. By Theorem 4.3 with $A = [R_3(\frac{k^3}{2},5)]$ there exists a coloring $COL'\colon \binom{R_3(\frac{k^3}{2},5)}{3} \to [5]$ such that if H is COL'-homog then one of the following holds:

- H is COL-homog, COL-min-homog, or COL-max-homog.
- Every maximal COL-rainbow subset of H is of size $\ge (2|H|)^{1/3}$.

By Lemma 3.2.2 there is an COL$'$-homog set H such that $|H| \ge \frac{k^3}{2}$. In the first case H is a COL-homog or COL-min-homog or COL-max-homog set of size $\ge \frac{k^3}{2} \ge k$. In the second case H has a rainbow subset of size $\ge (2(\frac{k^3}{2})^{1/3} = k$.

3) Let $COL\colon \binom{\{k,\ldots,LR_3(\frac{k^3}{2},5)\}}{2} \to \mathsf{N}$. By Theorem 4.3 with $A = [\{k,\ldots,LR_3(\frac{k^3}{2},5)\}]$ there exists a coloring $COL'\colon \binom{\{k,\ldots,LR_3(\frac{k^3}{2},5)\}}{4} \to [5]$ such that if H is COL'-homog then one of the following holds:

- H is COL-homog, COL-min-homog, or COL-max-homog.
- Every maximal COL-rainbow subset of H is of size $\ge (2|H|)^{1/3}$.

By Lemma 3.2.3 there is a COL$'$-homog set H such that $|H| \ge \frac{\min(H)^3}{2}$. In the first case H is a large set (since $|H| \ge \frac{\min(H)^3}{2} \ge \min(H)$) that is either COL-homog, COL-min-homog, or COL-max-homog.
In the second case we have:

- $|H| \ge \frac{\min(H)^3}{2}$.
- *every* maximal COL-rainbow subset of H is of size

$$\ge (2|H|)^{1/3} \ge 2\frac{\min(H)^3}{2} = \min(H).$$

By a greedy algorithm we can obtain a maximal COL-rainbow subset H' of H that has $\min(H)$ in it. Hence $\min(H') \le \min(H)$. Note that

$$|H'| \ge \min(H) = \min(H')$$

hence H' is a large COL-rainbow set. ∎

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References

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