1 Shawn Ligocki

9.2: Yeah, I would say that the bound is now 432. The question is just about rigor of result. AFAICT, all these results are susceptible to minor bugs b/c implementation is really technical and I don't believe anyone has looked into formal verification for any of this. But from what I've heard, the newer results (up to 432) are similarly believable to the 745 result.

2 Matthew House

BILL:

d) Anyone who is familiar with results about ZFC, PA, etc.

   After discussing Johannes Riebel's Bachelor's Thesis I say

   Find an s-state TM M with $s < 745$ such that M halts IFF ZFC is inconsistent.

   Later I state that Wade showed that if T is ZF $-$ the axiom of regularity. then $N_T \leq 432$ and hence $T_{ZFC} \leq 432$ by Shoenfield Absoluteness.

   SO, the result by Riebel's is still the best known - please verify. See next item.

e) Are all of the N_T bounds obtained as follows:

   FIRST prove there is an s-state TM st M halts iff T is inconsistent.

   SECOND: This yields $N_T \leq s$.

   THIRD: (If needed and applicable) Use Shoenfield absoluteness to get $N_{T'} = N_T \leq s$. I have only seen this used with T'=ZFC.

   Wade's result gives a better bound on N_{ZFC} then Riebel had, but does not improve Riebel's result about ZFC consistent. True?

Regarding d:

Wade's machine halts iff it finds a contradiction in ZF - Regularity. In fact, O'Rear's machine and Riebel's modified version also test for a contradiction in ZF - Reg, instead of the full ZFC! Neither Riebel nor Wade has added or removed any axioms from O'Rear's version.

The idea is that we can prove (in a far weaker metatheory, such as PRA) that

Con(ZF - Reg) iff Con(ZF) and Con(ZF) iff Con(ZFC), so all three statements are really equivalent. Clearly, if Con(ZFC), then Con(ZF - Reg), since a proof of a contradiction in ZF - Reg would also be a proof of a contradiction in ZFC. For the other direction, we prove this via interpretations.

First, we show that if ZF - Reg is consistent, then ZF is consistent. ZF - Reg can interpret ZF via the well-founded von Neumann hierarchy V. For any statement S, we translate it into a statement S* by relativizing all the quantifiers and free variables to range over V. Then, for each axiom A of ZF, there exists a proof in ZF - Reg that A* is true. In particular, the translated axiom of regularity holds within V, and this can be proven even in ZF - Reg.

Now, suppose that ZF is inconsistent. Then there exists a proof P of a contradiction from the ZF axioms. But now, we can create a proof P* from translating each statement in P, then replacing each ZF axiom A in P with the proof of A* from the ZF - Reg axioms. Then P* is a proof of a contradiction in ZF - Reg, thus ZF - Reg is inconsistent. So taking the contrapositive, if ZF - Reg is consistent, then ZF is consistent.

In the same way, we prove that if ZF is consistent, then ZFC is consistent. This time, instead of the von Neumann hierarchy V, we use Gödel's constructible universe L. We translate each statement S into S* by relativizing to L; and for each axiom A of ZFC, there exists a proof in ZF that A* is true, even for the axiom of choice. And we can translate proofs from ZFC into ZF in the same way, so that if ZFC proves a contradiction, then ZF must also prove a contradiction.

None of this relies on Shoenfield absoluteness. It just requires that we know how to translate every proof from the 'stronger' theory into an equivalent proof in the 'weaker' theory, in such a way that any contradiction in the former must also generate a contradiction in the latter. This way, we can say that every *TM that halts iff ZF - Reg is inconsistent* is also a *TM that halts iff ZFC is inconsistent:* there is no distinction between the two. (And in general, it might be hard to pick apart *exactly what axioms* a TM is using, since in principle it could use a formal system radically departing from the traditional syntax and semantics of first-order logic.)

In summary, we can prove that Wade's 432-state machine halts iff ZFC is inconsistent, assuming it has no bugs.

Regarding e:

The 7910-state TM from the 2016 paper of Yedidia & Aaronson relies on a statement that Harvey Friedman claims is independent from ZFC, but is not directly a consistency statement. (As far as I can tell, the main crux of Friedman's arguments come from section 5 of his *Invariant Maximal Cliques and Incompleteness*, where he uses the statement to construct a model of a certain theory of orders that can interpret SRP, of which ZFC is a subtheory; and in turn, the section refers heavily to his Boolean Relation Theory and Incompleteness.)

As far as I'm aware, all later independent TMs are based on O'Rear's machines, which directly test consistency statements. They only depend on equiconsistency arguments like the ones above to show that Con(ZF - Reg) IFF Con(ZF) IFF Con(ZFC); they do not need to invoke Shoenfield absoluteness to obtain $N_{\text{ZF}} = N_{\text{ZFC}}$.