#### Open Problems Column Edited by William Gasarch

**This Issue's Column!** This issue's Open Problem Column is by William Gasarch, Emily Kaplitz, and Erik Metz. It is on the mod behaviour of the sequence

 $a_1 = 1$ 

 $(\forall n \ge 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}].$ 

**Request for Columns!** I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

How Does  $a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$  Behave Mod M? By William Gasarch<sup>1</sup> and Emily Kaplitz<sup>2</sup> and Erik Metz<sup>3</sup>

## 1 The Sequence

In the book Sequences and Mathematical Induction in Mathematical Olympiad Competitions [?], on page 7, is the following problem:

The sequence  $a_n$  is defined by  $a_1 = 1$   $(\forall n \ge 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}].$ Prove that there are infinitely many terms of the sequence that are divided by 7.

We will henceforth refer to the sequence  $a_n$  defined above as the sequence. We give their proof and then make some observations and conjectures.

**Theorem 1.1**  $(\forall m \ge 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{7}].$ 

#### **Proof:**

Throughout this proof  $\equiv$  means  $\equiv \pmod{7}$ .

The proof is by induction on m.

**Base Case** m = 1: Note that  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_3 = 3$ ,  $a_4 = 5$ ,  $a_5 = 7$ . Hence  $i_1 = 5$  suffices.

Induction Hypothesis:  $(\exists i_1 < \cdots < i_{m-1})[a_{i_1} \equiv \cdots \equiv a_{i_{m-1}} \equiv 0 \pmod{7}].$ 

**Induction Step:** Let  $n = i_{m-1}$ . Note that

 $a_{2n} = a_{2n-1} + a_n \equiv a_{2n-1}.$  $a_{2n+1} = a_{2n} + a_n \equiv a_{2n}.$ Combining these we obtain that there is an  $r \in \{0, \dots, 6\}$  such that

$$a_{2n-1} \equiv a_{2n} \equiv a_{2n+1} \equiv r.$$

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**Case 1:** r = 0. Then  $a_{2n-1} \equiv 0$  so we can take  $i_m = 2n - 1$ .

**Case 2:**  $r \neq 0$ . Look at the following numbers

 $a_{4n-3} \equiv a_{4n-3} + 0$  (You will see later why we include this.)

 $a_{4n-2} = a_{4n-3} + a_{2n-1} \equiv a_{4n-3} + r.$ 

 $a_{4n-1} = a_{4n-2} + a_{2n-1} \equiv a_{4n-3} + 2r.$ 

 $a_{4n} = a_{4n-1} + a_{2n} \equiv a_{4n-3} + 3r.$ 

 $a_{4n+1} = a_{4n} + a_{2n} \equiv a_{4n-3} + 4r.$ 

 $a_{4n+2} = a_{4n+1} + a_{2n+1} \equiv a_{4n-3} + 5r.$ 

 $a_{4n+3} = a_{4n+2} + a_{2n+1} \equiv a_{4n-3} + 6r.$ 

Since  $r \neq 0$ , and 7 is prime, the numbers 0, r, 2r, 3r, 4r, 5r, 6r are equivalent to all 7 possibilities mod 7. Hence for some  $i \in \{4n - 3, \dots, 4n + 3\}$ ,  $a_i \equiv 0$ . We set  $i_m$  to i.

## 2 Other Mods and Empirical Evidence

We leave it to the reader to adapt the proof of Theorem ?? to show the following:

**Theorem 2.1** Let  $r \in \{2, 3, 5, 7\}$ .  $(\forall m \ge 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}]$ .

The question arises: Find all r such that

$$(\forall m \ge 1)(\exists i_1 < \dots < i_m)[a_{i_1} \equiv \dots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

We wrote a program to generate the first million elements of the sequence  $a_n \pmod{r}$  for all r = 2 to 100. The empirical evidence strongly suggests the following conjecture:

#### Conjecture 2.2

1. Let  $r \not\equiv 0 \pmod{4}$ . Then

 $(\forall m \ge 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}].$ 

2. Let  $r \equiv 0 \pmod{4}$ . Then

$$(\forall m \ge 1)(a_m \not\equiv 0 \pmod{r}].$$

The second part should not be called a conjecture since we prove it in the next section.

# $3 \quad r \equiv 0 \pmod{4} \implies (\forall m \ge 1)[a_m \not\equiv 0 \pmod{r}]$

In this section we will prove that, for all  $m, a_m \not\equiv 0 \pmod{4}$ . A trivial corollary is

$$r \equiv 0 \pmod{4} \implies (\forall m \ge 1)[a_m \not\equiv 0 \pmod{r}]$$

The first few terms of the sequence mod 4 are

$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$	$a_9$	$a_{10}$	$a_{11}$	$a_{12}$
1	2	3	1	3	2	1	2	3	2	1	3

This pattern indicates three things:

- If n is odd then  $a_n \equiv 1, 3 \pmod{4}$ .
- If you remove the 2's from the sequence you get 1, 3, 1, 3, 1, 3
- $a_n \not\equiv 0 \pmod{4}$ .

We prove all three.

**Theorem 3.1**  $All \equiv are \mod 4$ .

- 1.  $(\forall n \geq 1)$  [If n is odd then  $a_n \equiv 1, 3 \pmod{4}$ ]
- 2.  $a_1 \neq 0$  and  $a_2 \neq 0$ . (We separate these two cases since Part 2 only covers  $n \geq 3$ .)
- 3.  $(\forall n \geq 3)$ 
  - (a) If  $a_n \equiv 1$  then either  $a_{n-1} \equiv 3$  or  $a_{n-1} \equiv 2$  and  $a_{n-2} \equiv 3$ .
  - (b) If  $a_n \equiv 3$  then either  $a_{n-1} \equiv 1$  or  $a_{n-1} \equiv 2$  and  $a_{n-2} \equiv 1$ .
  - (c)  $a_n \not\equiv 0$ .

**Proof:** The following equations will be used throughout and are easily verified.

EQ1:  $a_{2m-1} = a_{2m-2} + a_{m-1}$ EQ2:  $a_{2m} = a_{2m-1} + a_m$ EQ3:  $a_{2m+1} = a_{2m} + a_m$ EQ4:  $a_{2m+1} = a_{2m-1} + 2a_m$ .

1) We prove this by induction by a triple induction on n.

Base Case n = 1.  $a_1 = 1 \equiv 1$ .

**IH** For all  $1 \le n' < n$ , if n' is odd then  $a_{n'} \equiv 1, 3$ .

**IS** If n is even there is nothing to prove, so we take n to be odd. Let n = 2m + 1. By the IH,  $a_{2m-1} \equiv 1, 3$ . By EQ4  $a_{2m+1} = a_{2m-1} + 2a_m$ , hence since  $a_{2m-1} \equiv 1, 3$  we have  $a_{2m+1} \equiv 1, 3$ .

2) This is obvious.

3) We prove this by induction on n. We will assume the theorem for all  $3 \le n' \le n-1$  and n even, and prove it for n and n+1.

**Base Case** The theorem starts at n = 3. From the table of  $a_i$ 's before this theorem one can see that the theorem holds for n = 3, 4, 5, 6, 7. So the proof below needs to work for  $n \ge 8$ . You will see at the proof of 3a why we needed to start at n = 8.

**IH** For all n',  $3 \le n' \le n-1$ , all three parts of the theorem holds. Note that 3 < n-1 since  $n \ge 8$ .

**IS** Let n = 2m and  $n \ge 8$ . Note that  $m \ge 4$ . We prove 3a for  $a_{2m}$ . The proof for 3b is similar.

3a) By Part 1,  $a_{2m-1} \neq 0$ , hence  $a_{2m-1} \equiv 1, 3$ . We will do the  $a_{2m-1} \equiv 1$  case and leave the  $a_{2m-1} \equiv 3$  case to the reader. We have cases based on  $a_m$ . By the IH (Part 3c),  $a_m \equiv 1, 2, 3$ . (Need that  $m \geq 1$  to use IH, and we have  $m \geq 4$ .)

Case 1  $a_m \equiv 1$ .

EQ2:  $a_{2m} = a_{2m-1} + a_m \equiv 1 + 1 \equiv 2$ .

EQ3:  $a_{2m+1} = a_{2m} + a_m \equiv 2 + 1 \equiv 3$ .

Case 2  $a_m \equiv 2$ .

EQ2: 
$$a_{2m} = a_{2m-1} + a_m \equiv 1 + 2 \equiv 3$$

EQ3: 
$$a_{2m+1} = a_{2m} + a_m \equiv 3 + 2 \equiv 1$$
.

**Case 3**  $a_m \equiv 3$ . We show this case cannot occur. Since  $a_m \equiv 3$ , by the IH, either (1)  $a_{m-1} \equiv 1$  or (2)  $a_{m-1} \equiv 2$  and  $a_{m-2} \equiv 1$ . (Need  $m \geq 3$  to use the IH, and we have  $m \geq 4$ .) **Case 3.1**  $a_{m-1} \equiv 1$ .

EQ1: 
$$a_{2m-1} = a_{2m-2} + a_{m-1}$$
  
 $1 \equiv a_{2m-2} + 1$   
 $a_{2m-2} \equiv 0$  which contradicts the IH.

(Need  $2m - 2 \ge 3$  to use the IH, and we have  $m \ge 4$ .) **Case 3.2**  $a_{m-1} \equiv 2$  and  $a_{m-2} \equiv 1$ . We recap and extend what we know. We are assuming  $a_{2m-1} \equiv 1$ . From EQ1,

$$a_{2m-1} = a_{2m-2} + a_{m-1}.$$

Putting in  $a_{2m-1} \equiv 1$  and  $a_{m-1} \equiv 2$ , we get

$$a_{2m-2} \equiv 3.$$

From the recurrence, we have

$$a_{2m-2} = a_{2m-3} + a_{m-1}.$$

Putting in  $a_{2m-2} \equiv 3$  and  $a_{m-1} \equiv 2$ , we have

$$3 \equiv a_{2m-3} + 2$$

 $\mathbf{SO}$ 

 $a_{2m-3} \equiv 1.$ 

From the recurrence, we have

$$a_{2m-3} = a_{2m-4} + a_{m-2}.$$

Putting in  $a_{2m-3} \equiv 1$  and  $a_{m-2} \equiv 1$ , we get

$$1 = a_{2m-4} + 1,$$

 $\mathbf{so}$ 

$$a_{2m-4} \equiv 0.$$

This contradicts the IH. (Need  $2m - 4 \ge 3$  to use the IH, and we have  $m \ge 4$ . Note that  $m \ge 3$  would not have sufficed.)

- 3b) The proof is similar to that of Part 3a.
- 3c) We prove this in the reverse order: we first show  $a_{n+1} \neq 0$  and then that  $a_n \neq 0$ .  $a_{n+1}$ : Since *n* is even, n+1 is odd. By Part 1  $a_{n+1} \equiv 1, 3$ , hence  $a_{n+1} \neq 0$ .

 $a_n$ : Since  $a_{n+1} \equiv 1, 3$ , by Part 3b (not the IH, but what I just proved in the IS),  $a_n \equiv 1, 2, 3$ , so  $a_n \neq 0$ .

## 4 Conclusion

We restate the part of our conjecture that is still unproven:

**Conjecture 4.1** Let  $r \not\equiv 0 \pmod{4}$ . Then

$$(\forall m \ge 1)(\exists i_1 < \dots < i_m)[a_{i_1} \equiv \dots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

The following questions can also be considered

1. Let  $r \not\equiv 0 \pmod{4}$ . Let  $0 \leq i \leq r - 1$ . What is the density of

$$\{a_n \colon a_n \equiv i \pmod{r}\}.$$

- 2. Let  $A, B, C \in \mathbb{Z}$  and  $r \geq 2$ . What is the behaviour of  $a_1 = A$ .
  - $a_n = Ba_{n-1} + Ca_{\lfloor n/2 \rfloor} \pmod{r}.$

## References

[1] Z. Feng, editor. Sequences and Mathematical Induction In Mathematical Olympiad Competitions. World Scientific, Singapore, 2020. Translated by Feng Ma and Youren Wang.