**Open Problems Column**
**Edited by William Gasarch**

**This Issue's Column!** This issue's Open Problem Column is by William Gasarch, Emily Kaplitz, and Erik Metz. It is on the mod behaviour of the sequence

$a_1 = 1$

$(\forall n \geq 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}]$.

**Request for Columns!** I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

**How Does $a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}$ Behave Mod $M$?**

**By William Gasarch[1] and Emily Kaplitz[2] and Erik Metz[3]**

# 1 The Sequence

In the book *Sequences and Mathematical Induction in Mathematical Olympiad Competitions* [1], on page 7, is the following problem:

> *The sequence $a_n$ is defined by*
> $a_1 = 1$
> $(\forall n \geq 2)[a_n = a_{n-1} + a_{\lfloor n/2 \rfloor}]$.
> *Prove that there are infinitely many terms of the sequence that are divided by 7.*

We will henceforth refer to the sequence $a_n$ defined above as *the sequence*.

We give their proof and then make some observations and conjectures.

**Theorem 1.1** $(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod 7]$.

**Proof:**

Throughout this proof $\equiv$ means $\equiv \pmod 7$.

The proof is by induction on $m$.

**Base Case $m = 1$:** Note that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 7$. Hence $i_1 = 5$ suffices.

**Induction Hypothesis:** $(\exists i_1 < \cdots < i_{m-1})[a_{i_1} \equiv \cdots \equiv a_{i_{m-1}} \equiv 0 \pmod 7]$.

**Induction Step:** Let $n = i_{m-1}$. Note that

$a_{2n} = a_{2n-1} + a_n \equiv a_{2n-1}$.

$a_{2n+1} = a_{2n} + a_n \equiv a_{2n}$.

Combining these we obtain that there is an $r \in \{0, \ldots, 6\}$ such that

$$a_{2n-1} \equiv a_{2n} \equiv a_{2n+1} \equiv r.$$

[1] gasarch@umd.edu
[2] ekaplitz@umd.edu
[3] emetz1618@gmail.com

**Case 1:** $r = 0$. Then $a_{2n-1} \equiv 0$ so we can take $i_m = 2n - 1$.

**Case 2:** $r \neq 0$. Look at the following numbers
$a_{4n-3} \equiv a_{4n-3} + 0$ (You will see later why we include this.)
$a_{4n-2} = a_{4n-3} + a_{2n-1} \equiv a_{4n-3} + r$.
$a_{4n-1} = a_{4n-2} + a_{2n-1} \equiv a_{4n-3} + 2r$.
$a_{4n} = a_{4n-1} + a_{2n} \equiv a_{4n-3} + 3r$.
$a_{4n+1} = a_{4n} + a_{2n} \equiv a_{4n-3} + 4r$.
$a_{4n+2} = a_{4n+1} + a_{2n+1} \equiv a_{4n-3} + 5r$.
$a_{4n+3} = a_{4n+2} + a_{2n+1} \equiv a_{4n-3} + 6r$.
Since $r \not\equiv 0$, and 7 is prime, the numbers $0, r, 2r, 3r, 4r, 5r, 6r$ are equivalent to all 7 possibilities mod 7. Hence for some $i \in \{4n-3, \ldots, 4n+3\}$, $a_i \equiv 0$. We set $i_m$ to $i$. ∎

## 2 Other Mods and Empirical Evidence

We leave it to the reader to adapt the proof of Theorem 1.1 to show the following:

**Theorem 2.1** *Let* $r \in \{2, 3, 5, 7\}$. $(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}]$.

The question arises:
*Find all* $r$ *such that*

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

We wrote a program to generate the first million elements of the sequence $a_n \pmod{r}$ for all $r = 2$ to 100. The empirical evidence strongly suggests the following conjecture:

**Conjecture 2.2**

1. *Let* $r \not\equiv 0 \pmod 4$. *Then*

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod{r}].$$

2. *Let* $r \equiv 0 \pmod 4$. *Then*

$$(\forall m \geq 1)(a_m \not\equiv 0 \pmod{r}].$$

*The second part should not be called a conjecture since we prove it in the next section.*

## 3 $r \equiv 0 \pmod 4 \implies (\forall m \geq 1)[a_m \not\equiv 0 \pmod r)]$

In this section we will prove that, for all $m$, $a_m \not\equiv 0 \pmod 4$. A trivial corollary is

$$r \equiv 0 \pmod 4 \implies (\forall m \geq 1)[a_m \not\equiv 0 \pmod r)].$$

The first few terms of the sequence mod 4 are

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ | $a_8$ | $a_9$ | $a_{10}$ | $a_{11}$ | $a_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 3 | 2 | 1 | 2 | 3 | 2 | 1 | 3 |

This pattern indicates three things:

- If $n$ is odd then $a_n \equiv 1, 3 \pmod 4$.
- If you remove the 2's from the sequence you get $1, 3, 1, 3, 1, 3$
- $a_n \not\equiv 0 \pmod 4$.

We prove all three.

**Theorem 3.1** *All $\equiv$ are mod 4.*

1. $(\forall n \geq 1)$[*If $n$ is odd then* $a_n \equiv 1, 3 \pmod 4$]
2. $a_1 \not\equiv 0$ *and* $a_2 \not\equiv 0$. *(We separate these two cases since Part 2 only covers $n \geq 3$.)*
3. $(\forall n \geq 3)$
   - *(a) If $a_n \equiv 1$ then either $a_{n-1} \equiv 3$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 3$.*
   - *(b) If $a_n \equiv 3$ then either $a_{n-1} \equiv 1$ or $a_{n-1} \equiv 2$ and $a_{n-2} \equiv 1$.*
   - *(c) $a_n \not\equiv 0$.*

**Proof:** The following equations will be used throughout and are easily verified.
EQ1: $a_{2m-1} = a_{2m-2} + a_{m-1}$
EQ2: $a_{2m} = a_{2m-1} + a_m$
EQ3: $a_{2m+1} = a_{2m} + a_m$
EQ4: $a_{2m+1} = a_{2m-1} + 2a_m$.

1) We prove this by induction by a triple induction on $n$.
**Base Case** $n = 1$. $a_1 = 1 \equiv 1$.
**IH** For all $1 \leq n' < n$, if $n'$ is odd then $a_{n'} \equiv 1, 3$.
**IS** If $n$ is even there is nothing to prove, so we take $n$ to be odd. Let $n = 2m + 1$. By the IH, $a_{2m-1} \equiv 1, 3$. By EQ4 $a_{2m+1} = a_{2m-1} + 2a_m$, hence since $a_{2m-1} \equiv 1, 3$ we have $a_{2m+1} \equiv 1, 3$.

2) This is obvious.

3) We prove this by induction on $n$. We will assume the theorem for all $3 \leq n' \leq n - 1$ and $n$ even, and prove it for $n$ and $n + 1$.

**Base Case** The theorem starts at $n = 3$. From the table of $a_i$'s before this theorem one can see that the theorem holds for $n = 3, 4, 5, 6, 7$. So the proof below needs to work for $n \geq 8$. You will see at the proof of 3a why we needed to start at $n = 8$.

**IH** For all $n'$, $3 \leq n' \leq n-1$, all three parts of the theorem holds. Note that $3 < n-1$ since $n \geq 8$.

**IS** Let $n = 2m$ and $n \geq 8$. Note that $m \geq 4$. We prove 3a for $a_{2m}$. The proof for $3b$ is similar.

3a) By Part 1, $a_{2m-1} \not\equiv 0$, hence $a_{2m-1} \equiv 1, 3$. We will do the $a_{2m-1} \equiv 1$ case and leave the $a_{2m-1} \equiv 3$ case to the reader. We have cases based on $a_m$. By the IH (Part 3c), $a_m \equiv 1, 2, 3$. (Need that $m \geq 1$ to use IH, and we have $m \geq 4$.)

**Case 1** $a_m \equiv 1$.

$$\text{EQ2: } a_{2m} = a_{2m-1} + a_m \equiv 1 + 1 \equiv 2.$$

$$\text{EQ3: } a_{2m+1} = a_{2m} + a_m \equiv 2 + 1 \equiv 3.$$

**Case 2** $a_m \equiv 2$.

$$\text{EQ2: } a_{2m} = a_{2m-1} + a_m \equiv 1 + 2 \equiv 3.$$

$$\text{EQ3: } a_{2m+1} = a_{2m} + a_m \equiv 3 + 2 \equiv 1.$$

**Case 3** $a_m \equiv 3$. We show this case cannot occur. Since $a_m \equiv 3$, by the IH, either (1) $a_{m-1} \equiv 1$ or (2) $a_{m-1} \equiv 2$ and $a_{m-2} \equiv 1$. (Need $m \geq 3$ to use the IH, and we have $m \geq 4$.)

**Case 3.1** $a_{m-1} \equiv 1$.

$$\begin{aligned} \text{EQ1: } \quad a_{2m-1} &= a_{2m-2} + a_{m-1} \\ 1 &\equiv a_{2m-2} + 1 \\ a_{2m-2} &\equiv 0 \text{ which contradicts the IH.} \end{aligned}$$

(Need $2m - 2 \geq 3$ to use the IH, and we have $m \geq 4$.)

**Case 3.2** $a_{m-1} \equiv 2$ and $a_{m-2} \equiv 1$.

We recap and extend what we know.

We are assuming $a_{2m-1} \equiv 1$.

From EQ1,

$$a_{2m-1} = a_{2m-2} + a_{m-1}.$$

Putting in $a_{2m-1} \equiv 1$ and $a_{m-1} \equiv 2$, we get

$$a_{2m-2} \equiv 3.$$

From the recurrence, we have

$$a_{2m-2} = a_{2m-3} + a_{m-1}.$$

Putting in $a_{2m-2} \equiv 3$ and $a_{m-1} \equiv 2$, we have

$$3 \equiv a_{2m-3} + 2,$$

so

$$a_{2m-3} \equiv 1.$$

From the recurrence, we have

$$a_{2m-3} = a_{2m-4} + a_{m-2}.$$

Putting in $a_{2m-3} \equiv 1$ and $a_{m-2} \equiv 1$, we get

$$1 = a_{2m-4} + 1,$$

so

$$a_{2m-4} \equiv 0.$$

This contradicts the IH. (Need $2m - 4 \geq 3$ to use the IH, and we have $m \geq 4$. Note that $m \geq 3$ would not have sufficed.)

3b) The proof is similar to that of Part 3a.

3c) We prove this in the reverse order: we first show $a_{n+1} \not\equiv 0$ and then that $a_n \not\equiv 0$.
$a_{n+1}$: Since $n$ is even, $n+1$ is odd. By Part 1 $a_{n+1} \equiv 1, 3$, hence $a_{n+1} \not\equiv 0$.
$a_n$: Since $a_{n+1} \equiv 1, 3$, by Part 3b (not the IH, but what I just proved in the IS), $a_n \equiv 1, 2, 3$, so $a_n \not\equiv 0$. ∎

## 4 Conclusion

We restate the part of our conjecture that is still unproven:

**Conjecture 4.1** *Let* $r \not\equiv 0 \pmod 4$. *Then*

$$(\forall m \geq 1)(\exists i_1 < \cdots < i_m)[a_{i_1} \equiv \cdots \equiv a_{i_m} \equiv 0 \pmod r].$$

The following questions can also be considered

1. Let $r \not\equiv 0 \pmod 4$. Let $0 \leq i \leq r-1$. What is the density of

$$\{a_n : a_n \equiv i \pmod r\}.$$

2. Let $A, B, C \in \mathbb{Z}$ and $r \geq 2$. What is the behaviour of
   $a_1 = A$.
   $a_n = Ba_{n-1} + Ca_{\lfloor n/2 \rfloor} \pmod r$.

## References

[1] Z. Feng, editor. *Sequences and Mathematical Induction In Mathematical Olympiad Competitions.* World Scientific, Singapore, 2020. Translated by Feng Ma and Youren Wang.