Open Problems Column
Edited by William Gasarch

This issue's Open Problem Column is by Neil Lutz and is on *Some Open Problems in Algorithmic Fractal Geometry*. Neil uses techniques from theoretical computer science to gain insight into problems in pure math. The synergy is awesome!

I invite any reader who has knowledge of some area to contact me and arrange to write a column about open problems in that area. That area can be (1) broad or narrow or anywhere inbetween, and (2) really important or really unimportant or anywhere inbetween.

Some Open Problems in Algorithmic Fractal Geometry
by Neil Lutz

The standard notion of dimension in fractal geometry has a natural interpretation in terms of algorithmic information theory, which enables new, pointwise proof techniques that apply theoretical computer science to pure mathematics. This column outlines recent progress in this area and describes several open problems, with a focus on problems related to Kakeya sets and fractal projections.

1 Classical Fractal Dimensions

There are various ways to define the dimension of a set, but most formalize a familiar intuition: A set's dimension is the number of *free parameters* within the set, i.e., the number of parameters needed to specify a point in the set when the set is already known. For simple geometric figures, this number is obvious: Specifying a point on a known segment requires one parameter, and specifying a point in a known disc requires two parameters.

For certain complex figures like the Koch curve in Figure 1, the answer is less clear. This curve is the limit of the sequence , , , ..., generated by iteratively replacing the middle third of each segment with an equilateral "tooth." The length increases by a factor of 4/3 with every iteration, so the curve has infinite length. Moreover, every portion of the curve contains a scaled copy of the entire curve and therefore also has infinite length. This means that a single parameter will be insufficient to specify a point. On the other hand, it seems inaccurate to say that there are two free parameters; once the x coordinate of a point on the curve is known, the possibilities for its y coordinate are severely limited. Notions of fractal dimension address this ambiguity by allowing the dimension of a set to be any non-negative real value.

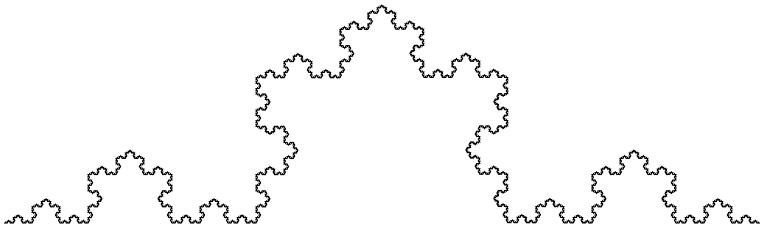

Figure 1: The Koch curve, which has box-counting and Hausdorff dimension $\log_3 4 \approx 1.26$.

One basic notion of fractal dimension is *box-counting dimension*. Given a bounded set $E \subseteq \mathbb{R}^2$, we count the number of squares in an infinite grid of size $\varepsilon > 0$ that have nonempty intersection with E. We then look at the asymptotic behavior of this number, $N_\varepsilon(E)$, as $\varepsilon \to 0$. For example, it is easy to see that $N_\varepsilon(E) = \Theta(\varepsilon^{-1})$ when E is a line segment, and $N_\varepsilon(E) = \Theta(\varepsilon^{-2})$ when E is a disc. In general, we say that E has box-counting dimension d if $N_\varepsilon(E) = \Theta(\varepsilon^{-d})$. This simple definition succeeds at quantifying non-integral dimension for bounded sets and is adequate for many applications, including the Koch curve. Unfortunately, it also has some undesirable mathematical properties. In particular, countable sets can have positive box-counting dimension; consider $E = \{(q, 0) : q \in [0, 1] \text{ is rational}\}$, which is countable but has $N_\varepsilon(E) = \varepsilon^{-1}$.

In contrast, *Hausdorff dimension* is a more sophisticated notion of fractal dimension for which every countable set E has dimension $\dim_H(E) = 0$. More generally, Hausdorff dimension is *countably stable*, meaning that $\dim_H(\cup_i E_i) = \sup_i \dim_H(E_i)$ for any sequence of sets $E_1, E_2, \ldots \subseteq \mathbb{R}^n$. This property is achieved by optimizing over a much larger class of covers for E, compared to the uniform grid covers used in box-counting dimension. This optimization is used to define *s-dimensional Hausdorff measure* H^s for all $s > 0$, which essentially generalizes Lebesgue outer measure. The Hausdorff dimension of a set $E \subseteq \mathbb{R}^n$ is the unique value d such that $H^s(E) = \infty$ for all $s < d$ and $H^s(E) = 0$ for all $s > d$. For this column it is important that every set $E \subseteq \mathbb{R}^n$ with positive Lebesgue measure has $\dim_H(E) = n$, and that the converse does not hold. See Falconer's textbook [12] for formal definitions and more discussion of these quantities.

2 Algorithmic Dimension and Randomness

Hausdorff dimension is mathematically robust and is the most studied notion of fractal dimension, but its definition obscures the original intuition that dimension is the number of free parameters. We now discuss a different characterization of Hausdorff dimension that uses algorithmic information theory to reason about "partially free parameters" and preserve that intuition.

Let U be a fixed universal prefix-free Turing machine, and let σ be a binary string. The *Kolmogorov complexity*, or *algorithmic information content*, of σ is

$$K(\sigma) = \min\{|\tau| : U(\tau) = \sigma\},$$

the length of the shortest binary program τ that outputs σ when run on U. Conveniently, this quantity is well defined up to an additive constant without any further specification of the universal machine; see the standard reference [16] for details. Informally, if $K(\sigma)$ is much less than $|\sigma|$, then we consider σ to be very compressible.

In order to translate algorithmic statements into statements with no reference to computation, we will allow U to be an *oracle machine* that can query any bit in some infinite oracle sequence $w \in \{0,1\}^\infty$ as a computational step. We write $K^w(\sigma) = \min\{|\tau| : U^w = \sigma\}$, where U^w is a universal oracle machine with access to w, for the Kolmogorov complexity of σ relative to w. Many results related to Kolmogorov complexity, including Theorems 2 and 3 below, hold relative to arbitrary oracles.

When $x \in \mathbb{R}^n$ and r is a positive integer, we write $x\upharpoonright r$ for the string that interleaves binary representations of x's n coordinates, each truncated r bits to the right of the binary point. Intuitively, $K(x\upharpoonright r)$ is the amount of information needed to describe the location of x up to r bits of precision. The *(effective Hausdorff) dimension* of an individual point $x \in \mathbb{R}^n$ is defined as

$$\dim(x) = \liminf_{r\to\infty} \frac{K(x\upharpoonright r)}{r}.$$

This quantity was originally defined by Jack Lutz [18], and the above characterization was proven by Elvira Mayordomo [26]. Similarly, the dimension of x relative to an oracle $w \in \{0,1\}^\infty$, $\dim^w(x)$ is given by

$$\dim^w(x) = \liminf_{r\to\infty} \frac{K^w(x\upharpoonright r)}{r}.$$

If $y \in \mathbb{R}^m$, then $\dim^y(x)$ denotes $\dim^{w_y}(x)$, where $w_y \in \{0,1\}^\infty$ is some standard binary representation of y.

Almost all points $x \in \mathbb{R}^n$ are *(Martin-Löf) random* or *incompressible*, meaning that there is some constant c such that $K(x\upharpoonright r) \geq nr - c$ for all r [25, 8]. Notice that every random point $x \in \mathbb{R}^n$ has $\dim(x) = n$, and that the converse does not hold.

We can interpret $\dim(x)$ informally as the number of parameters needed to specify the individual point x. To express the dimension of a set E, we take the supremum over all $x \in E$ (since we should be able to specify *any* point in the E with the given number of parameters), and we make the set known by granting access to an oracle w that is optimized for the set E. Jack Lutz and I proved the following *point-to-set principle*, showing that this algorithmic information theoretic formulation of the dimension of a set is identical to Hausdorff dimension.

Theorem 1 (Point-to-set principle for Hausdorff dimension [20])**.** *For every set $E \subseteq \mathbb{R}^n$,*

$$\dim_H(E) = \min_w \sup_{x\in E} \dim^w(x).$$

3 Kakeya Sets and Points on Lines

A *Kakeya set* in $\mathbb{R}^n$ is a set containing line segments of length 1 in all directions. Formally, $E \subseteq \mathbb{R}^n$ is a Kakeya set if, for every point u on the unit sphere S^{n-1}, there is some point $v \in \mathbb{R}^n$ such that the segment $\{tu + v : t \in [0,1]\}$ is contained in E.

The study of Kakeya sets is nearly a century old, and it has strong connections to other areas of mathematics, including harmonic analysis, arithmetic combinatorics, and PDE [14, 32, 5]. Within computer science, Kakeya sets have been shown to emerge from optimal strategies in certain pursuit games [1], and studying finite-field versions of Kakeya sets has led to improved randomness mergers and extractors [11, 10]. The famous *Kakeya conjecture* asserts that, for all $n \geq 2$, every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n. Davies [6] proved that this statement holds for $n = 2$. For $n \geq 3$, it is an open and actively studied problem, as are several variants. Dvir [9] proved a version of the conjecture for sets in finite fields.

The algorithmic dimensional approach to this problem is to pursue bounds on the dimension of individual points on individual lines rather than directly bounding the Hausdorff dimension of a set. This allows us to disregard the global structure of the set. For example, Jack Lutz and I proved the following theorem, which concerns only a single point in $\mathbb{R}^2$.

Theorem 2 ([20])**.** *Let $a, b, x \in \mathbb{R}$. If a is random and x is random relative to (a, b), then* $\dim(x, ax + b) = 2$.

This may be considered a strong pointwise version of Davies's theorem stating that every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2. In fact, Davies's theorem follows easily from Theorems 1 and 2. The proof is included here as a simple example of the point-to-set principle in action.

Alternative proof of Davies's theorem. Let $E \subseteq \mathbb{R}^2$ be a Kakeya set, and let w be the minimizing oracle of Theorem 1. Let a be random relative to w, and let b be such that the intersection of E with the line $y = ax + b$ contains a segment. Choose x random relative to (a, b, w) such that $(x, ax + b) \in E$. Then

$$\dim_H(E) = \sup_{z \in E} \dim^w(z) \geq \dim^w(x, ax + b)\,,$$

which is 2 by Theorem 2, applied relative to the oracle w. □

This proof is very different from Davies's original argument and served as a "proof of concept," demonstrating that algorithmic dimension techniques can be used to prove non-trivial results in classical fractal geometry. It also seems to be more susceptible to generalization, which has motivated pursuing more general bounds on $\dim(x, ax + b)$ for $a, b, x \in \mathbb{R}$.

It is trivial to show that $\dim(x, ax + b) \leq \dim(a, b, x)$, and given Theorem 2, we might hope to show that $\dim(x, ax + b) \geq \dim(a, x)$. In fact, no such result can hold, at least not relative to all oracles, because this pointwise problem is related to the Hausdorff dimension of *Furstenberg sets*, which are variants of Kakeya sets in $\mathbb{R}^2$. We first describe *generalized Furstenberg sets*, as introduced by Molter and Rela [27].

Instead of containing segments in all directions, a generalized Furstenberg set contains α-dimensional subsets of lines in a β-dimensional set of directions, for some parameters $\alpha, \beta \in (0, 1]$. Formally, $F \subseteq \mathbb{R}^2$ is an (α, β)-Furstenberg set if there is some $J \subseteq S^1$ with $\dim_H(J) \geq \beta$ such that, for all $u \in J$, there is a line $\mathcal{L}_u$ in direction u with $\dim_H(\mathcal{L}_u \cap F) \geq \alpha$. (Non-generalized) α-Furstenberg sets are $(\alpha, 1)$-Furstenberg sets with $J = S^1$, and Kakeya sets are a special case of 1-Furstenberg sets.

Molter and Rela [27] showed that that (α, β)-Furstenberg sets must have Hausdorff dimension at least $\alpha + \max\{\frac{\beta}{2}, \alpha + \beta - 1\}$, and it is known that α-Furstenberg sets may have Hausdorff dimension as low as $\frac{1+3\alpha}{2}$ [33]. Now suppose that $\dim(x, ax + b) \geq \dim(a, x)$ held for all $a, b, x \in \mathbb{R}$ and relative to all oracles. Then the point-to-set principle could be used to show that all α-Furstenberg sets have Hausdorff dimension at least $1 + \alpha$, which would contradict the $\frac{1+3\alpha}{2}$ upper bound. Hence, we should aim for more modest lower bounds on $\dim(x, ax + b)$. Don Stull and I proved the following.

Theorem 3 ([22])**.** *For all $a, b, x \in \mathbb{R}$,* $\dim(x, ax + b) \geq \dim^{a,b}(x) + \min\{\dim^{a,b}(x), \dim(a, b)\}$.

Notice that Theorem 2 follows immediately from Theorem 3. Theorem 3 also yielded the first instance of a new result in classical fractal geometry via algorithmic dimension techniques: The following corollary improves on the previous lower bound for (α, β)-Furstenberg sets whenever $\alpha, \beta < 1$ and $\alpha > \beta/2$.

Corollary 4 ([22]). *If $F \subseteq \mathbb{R}^2$ is an (α, β)-Furstenberg set, then $\dim_H(F) \geq \alpha + \min\{\alpha, \beta\}$.*

Improving the bound in Theorem 3 (or proving that it is tight) is a general goal. The following problem asks about a specific case and is a pointwise version of a question that Katz and Tao [15] asked about α-Furstenberg sets.

Problem. *Is there some $c > 0$ such that, for all $a, b, x \in \mathbb{R}$ with a random and $\dim^{a,b}(x) = \frac{1}{2}$, we have $\dim(x, ax + b) > 1 + c$?*

Finding bounds on $\dim(x, ax + b)$ is closely related to describing the *dimension spectrum* of the line $\mathcal{L}_{ab} = \{(x, ax + b) : x \in \mathbb{R}\}$, defined as $\mathrm{sp}(\mathcal{L}_{ab}) = \{\dim(x) : x \in \mathcal{L}_{ab}\}$. Don Stull and I [23] showed that this spectrum must have infinite cardinality, and that it contains a unit interval whenever $\dim(a, b)$ is equal to the *effective packing dimension* $\mathrm{Dim}(a, b)$, i.e., whenever $\lim_{r\to\infty} \frac{K((a,b)\upharpoonright r)}{r}$ exists.

Problem. *Does $\mathrm{sp}(\mathcal{L}_{ab})$ contain a unit interval for all $a, b \in \mathbb{R}$?*

Although the above problems concern refinements of Theorem 2 in $\mathbb{R}^2$, the original motivation for the present line of research was the hope that Theorem 2 can eventually be extended to higher dimensions.

Problem. *Let $n \geq 2$, $a, b \in \mathbb{R}^{n-1}$, and $x \in \mathbb{R}$. If a is random and x is random relative to (a, b), then must $\dim(x, ax + b) = n$?*

If the answer is *yes* relative to all oracles, then the same argument used above for Davies's theorem would imply that the classical Kakeya conjecture also holds. So far very little is known about the dimension of points on lines in higher-dimensional Euclidean spaces. Any non-trivial results on that subject would be interesting.

4 Resource-Bounded Measure of Kakeya Sets

While the minimum dimension of Kakeya sets in $\mathbb{R}^n$ is still an open question for $n \geq 3$, their minimum Lebesgue measure is not; Besicovitch [2, 3] constructed a Kakeya set of Lebesgue measure zero and decades later [4] constructed a set of Lebesgue measure zero that contains an entire line in every direction, which we call a *Besicovitch set*. Notice that in both cases it was sufficient to construct the set in $\mathbb{R}^2$, since taking the Cartesian product with $\mathbb{R}^{n-2}$ yields a set with the desired property in $\mathbb{R}^n$. This section discusses strengthening Besicovitch's results by showing that they hold even for weaker, resource-bounded versions of Lebesgue measure.

These measures are closely related to weak, complexity-theoretic notions of randomness, which are defined in terms of a sequence's *unpredictability* instead of its incompressibility. Very informally, a sequence $w \in \{0, 1\}^\infty$ is *computably random* if no computable function that places fair bets on successive bits of w can win infinite money. This intuition is formalized using betting strategies called *martingales* and can be modified to define computable randomness for points in $\mathbb{R}^n$; see [19] for details. Computable randomness was introduced by Schnorr [29, 30]. The martingale formalism also establishes a notion of *computable measure*, and a set $E \subseteq \mathbb{R}^n$ has computable measure zero if and only if it contains no computably random point.

Computable randomness is strictly weaker than Martin-Löf randomness, meaning that every computably random point is Martin-Löf random and that the converse does not hold. Jack Lutz [17] defined even weaker randomness notions (and corresponding measures) by placing resource bounds on the bettor. For instance, w is *exponential time random* or *exp-random* if no exponential time-computable function can win infinite money by placing fair bets on w, and $E \subseteq \mathbb{R}^n$ has *exp-measure zero* if and only if it contains no exp-random point.

By effectivizing Schoenberg's simplified construction of a Kakeya set with measure zero [31], Jack Lutz and I [19] showed that there are Kakeya sets of double exponential time measure zero, meaning that in every direction there are segments of arbitrary length that contain no double exponential time random point. In the same work [19], by effectivizing the original Besicovitch set construction [4], we showed that in every direction there are lines that contain no computably random point. A positive solution to the following problem would improve on both of those results, demonstrating that Besicovitch sets and Kakeya sets have measure zero in a very strong sense.

Problem. *Is it true that for all $a \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that $\mathcal{L}_{ab}$ contains no exp-random point?*

5 Fractal Projections

As described in the survey of Falconer, Fraser, and Jin [13], Marstrand's projection theorems have been central fixtures in fractal geometry in recent decades. We approach them here from an algorithmic dimensional perspective. For any set $E \subseteq \mathbb{R}^2$ and any $\theta \in [0, \pi)$, let $\text{proj}_\theta E$ be the orthogonal projection of E onto a line through the origin in direction θ.

Theorem 5 (Marstrand's projection theorems [24])**.** *If $E \subseteq \mathbb{R}^2$ is analytic, then for almost all $\theta \in [0, \pi)$,*

(a) $\dim_H(\text{proj}_\theta E) = \min\{1, \dim_H(E)\}$.

(b) if $\dim_H(E) > 1$*, then* $\text{proj}_\theta E$ *has positive Lebesgue measure.*

By the fact that dimension is preserved by locally bi-Lipschitz computable bijections [28] and a simple geometric duality argument, Theorem 3 implies the following pointwise analogue to Theorem 5(a).

Corollary 6. *If $z \in \mathbb{R}^2$ and $\theta \in [0, \pi)$ is random relative to z, then* $\dim(\theta, \text{proj}_\theta z) = 1 + \min\{1, \dim(z)\}$.

This correspondence suggests two directions of inquiry. First, can Corollary 6 be used to strengthen Theorem 5(a)? There is some precedent to suggest that it might. For example, I recently used algorithmic dimensional techniques to extend another fundamental result in fractal geometry, Marstrand's slicing theorem [24], from analytic sets to arbitrary sets. Given a set $E \subseteq \mathbb{R}^2$, this theorem concerns E's vertical slices, $E_x = \{y : (x, y) \in E\}$:

Theorem 7 ([21])**.** *If $E \subseteq \mathbb{R}^2$ is any set, then for almost all $x \in \mathbb{R}$,* $\dim_H(E_x) \geq \max\{0, \dim_H(E) - 1\}$.

Unfortunately, the situation for projections is more delicate; it was shown by Davies [7], using the continuum hypothesis, that neither part of Theorem 5 can be extended even to arbitrary Hausdorff-measurable sets. More modest refinements or extensions to classes of non-analytic sets are hopefully still possible.

Second, can we prove a pointwise analogue to Theorem 5(b)? This is equivalent to the following question.

Problem. *If $a, b \in \mathbb{R}$ with* $\dim(a, b) > 1$ *and $x \in \mathbb{R}$ is random relative to (a, b), then is $(x, ax + b)$ random?*

6 More Algorithmic Fractal Geometry

Although this column is focused on Kakeya-like sets, the results stated above should be considered early steps in a larger program of applying ideas from the theory of computation to problems in classical fractal geometry. Direct progress on those problems, such as Corollary 4, is obviously a desirable outcome. There are other ways to advance this program, though, and some progress in the following broad categories seems likely to be relatively straightforward.

First, we can pursue more generalizations in the vein of Theorem 7. The analytical techniques used in classical fractal geometry often depend on set properties like measurability or compactness, and pointwise techniques seem well-suited to circumvent these restrictions in some cases. Second, these techniques can be used to simplify classical proofs. The proof of an important theorem about the Hausdorff dimension of Cartesian products, for example, is much simpler when algorithmic dimension is used [21]. Third, we can ask non-classical questions about classical sets. Dimension spectra (in the sense described above) and resource-bounded measure are both algorithmic concepts, but they allow us to ask more nuanced questions about dimension and measure, which may lead to further structural insights. Finally, we can establish further pointwise, algorithmic characterizations of classical set properties, along the lines of Theorem 1. Packing dimension, for example, admits such a characterization [20]. Results in any of these categories would contribute to a more robust and intuitive foundation for fractal geometry.

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