The Erdos Distinct Distance Problem: Given \( n \) what is the minimum number of distinct distances between \( n \) points in the plane? We call this value \( g(n) \). Julia Gariabldi et al. [5] wrote an excellent book on the topic.

1. Paul Erdosns [4] showed that \( \sqrt{n} - O(1) \leq g(n) \leq O\left( \frac{n}{\sqrt{\log n}} \right) \). The first inequality would make a good high school math competition problem were it not so well known. We leave it to you. The second inequality is obtained by placing the points on a regular \( \sqrt{n} \times \sqrt{n} \) grid and using some number theory. Erdos made no conjecture about \( g(n) \) in this paper.

2. Leo Moser [10] showed \( g(n) = \Omega(n^{2/3}) \). This paper claims that Erdos conjectured \( (\forall \epsilon > 0)[g(n) \geq \Omega(n^{1-\epsilon})] \). It is likely that Erdos made this conjecture in a talk or in a paper listing open problems.

3. Fan Chung [1] showed \( g(n) = \Omega(n^{5/7}) \). This paper also claims that Erdos conjectured \( (\forall \epsilon > 0)[g(n) \geq \Omega(n^{1-\epsilon})] \).

4. Fan Chung et al. [2] showed \( g(n) = \Omega\left( \frac{n^{4/5}}{\log n} \right) \). This paper, as do all subsequent papers, claims that Erdos conjectured \( g(n) = \Theta\left( \frac{n}{\sqrt{\log n}} \right) \). It is likely that Erdos made this stronger conjecture in a talk or in a paper listing open problems.

5. Laszlo Szekeres [12] showed \( g(n) = \Omega(n^{4/5}) \). This proof is easier than the proofs in the papers by Chung and Chung et al. Moreover, it was a big breakthrough for techniques.

6. Jozsef Solymosi and Casaba D. Toth [8] showed \( g(n) = \Omega(n^{6/7}) \) (Note that 6/7 = 0.857142857142\ldots). This was also a big breakthrough for techniques. Later papers were refinements of this method. In their book on combinatorial geometry, on page 116, Janos Pach and Micha Sharir [11] state:

A close inspection of the Solymosi-Toth proof show that without any additional geometric ideas, it can never lead to a lower bound greater than \( \Omega(n^{8/9}) \).
Note that $8/9 = 0.888\cdots$.

7. Gabor Tardos [13] showed that, for all $\epsilon > 0$, $g(n) = \Omega(n^{(4e/(5e-1)-\epsilon)}).
Note that $\frac{4e}{5e-1} \approx 0.86353538281$. We paraphrase a comment they make on page 9:

The Ruzsa construction shows that one cannot use this technique to prove an $\Omega(n^{8/9})$ lower bound on the number of distinct distances. A modification of the proof of our theorem may help but the Ruzsa construction still shows that, for all $\epsilon$, a lower bound of $\Omega(n^{(8/9)+\epsilon})$ is out or reach.

They reference Ruzsa as private communication. I have not been able to find a paper by Ruzsa with this result.

8. Nets Katz and Gabor Tardos [9] showed that, for all $\epsilon > 0$, $g(n) = \Omega(n^{((48e-14e)/(55-16e))-\epsilon})$. Note that $\frac{48e-14e}{55-16e} \approx 0.86413751027$ (which is transcendental so it’s not going to have a repeating pattern).

As noted above Janos Pach, Micha Sharir, and Gabor Tardos all said that $\Omega(n^{8/9})$ was as far as the Solymosi-Toth techniques could go. I heard Solymosi say the same at a conference. The work of Ruzsa formalizes this intuition. This barrier result did not become an end in itself. There are no papers asking what other open problems (interesting or not) could this barrier on proof techniques be applied to.

So what did the community do? They found new techniques!

1. Larry Guth and Nets Katz [6] solved The Joints Conjecture: given any set of $n$ lines in $\mathbb{R}^3$ there are at most $O(n^{3/2})$ joints – points which are the intersection of three lines that are linearly independent (there is an example of $n$ lines with $\Omega(n^{3/2})$ joints). They used techniques from algebraic geometry.

2. Gyorgy Elekes and Micha Shari [3] used the techniques of Larry Guth and Nets Katz to set up a framework to study $g(n)$ and other problems. They solved some problems with their framework, though they did not get better lower bounds on $g(n)$.
3. Larry Guth and Nets Katz [7] used the framework of Gyorgy Elekes and Micha Sharir to obtain

\[ g(n) = \Omega \left( \frac{n}{\log n} \right). \]

The first conjecture, \( g(n) = \Omega(n^{1-\epsilon}) \) is now solved. The second conjecture, \( g(n) = \Omega \left( \frac{n}{\sqrt{\log n}} \right) \) is still open.

References


