# Rectangle Free Coloring of Grids


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**Abstract**

A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi_{n,m} : G_{n,m} \rightarrow [c]$ such that there are no rectangles with all four corners the same color. We address the following question: for which values of $n$ and $m$ is $G_{n,m}$ $c$-colorable? This problem can be viewed as a bipartite Ramsey problem and is related to the Gallai-Witt theorem (also called the multidimensional Van Der Waerden Theorem). We determine (1) *exactly* which grids are 2-colorable, (2) *exactly* which grids are 3-colorable, and (3) (assuming a conjecture) *exactly* which grids are 4-colorable. We use combinatorics and finite fields.



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# 1 Introduction

Let $n, m \in \mathbb{N}$. A two-dimensional *grid* is a set $G_{n,m} = [n] \times [m]$ where $[t] = \{1, \ldots, t\}$ for $t \in \mathbb{N}$. A *rectangle* of $G_{n,m}$ is a subset of the form $\{(a, b), (a+c_1, b), (a+c_1, b+c_2), (a, b+c_2)\}$ for some $a, b, c_1, c_2 \in \mathbb{N}$. A grid $G_{n,m}$ is *c-colorable* if there is a function $\chi_{n,m} : G_{n,m} \to [c]$ such that there are no rectangles with all four corners the same color. Not all grids have $c$-colorings. As an example, for any $c$ clearly $G_{c+1,c^{c+1}+1}$ does not have a $c$-coloring by two applications of the pigeonhole principle. If a grid has a $c$-coloring, we say it is *$c$-colorable.* In this paper, we ask the following question: what are the exact values of $m$ and $n$ for which $G_{n,m}$ is $c$-colorable?

**Def 1.1** Let $n, m, n', m' \in \mathsf{N}$. $G_{m,n}$ *contains* $G_{n',m'}$ if $n' \leq n$ and $m' \leq m$. $G_{m,n}$ *is contained in* $G_{n',m'}$ if $n \leq n'$ and $m \leq m'$. Proper containment means that at least one of the inequalities is strict.

Clearly, if $G_{n,m}$ is $c$-colorable, then all grids that it contains are $c$-colorable. Likewise, if $G_{n,m}$ is not $c$-colorable then all grids that contain it are not $c$-colorable.

**Def 1.2** Fix $c \in \mathbb{N}$, then $\mathrm{OBS}_c$ is the set of all grids $G_{n,m}$ such that $G_{n,m}$ is not $c$-colorable but all grids properly contained in $G_{m,n}$ are $c$-colorable. $\mathrm{OBS}_c$ stands for *Obstruction Sets.* We also call such grids *$c$-minimal.*

We leave the proof of the following theorem to the reader.

**Theorem 1.3** *Fix $c \in \mathbb{N}$. A grid $G_{a,b}$ is $c$-colorable iff it does not contain any element of* $\mathrm{OBS}_c$.

By Theorem 1.3 we can rephrase the question of finding which grids are $c$-colorable: find $\mathrm{OBS}_c$. Note that if $G_{n,m} \in \mathrm{OBS}_c$, then $G_{m,n} \in \mathrm{OBS}_c$.

This problem arises as follows. The Gallai-Witt theorem[1] (also called the multi-dimensional Van Der Waerden theorem) has the following corollary: *For all $c$, there exists $W = W(c)$ such that, for all $c$-colorings of $[W] \times [W]$ there exists a monochromatic square.* The classical proof of the theorem gives very large upper bounds on $W(c)$. Despite some improvements[2] the known bounds on $W(c)$ are still quite large; however, there has been some work on this by [1]. If we relax the problem to seeking a *monochromatic rectangle* then we can obtain far smaller bounds. In fact, in some cases we will obtain exact characterizations of when a grid is $c$-colorable.

Another motivation is the bipartite Ramsey problem: Given $a, c$, what is the least $n$ such that for any $c$-coloring of the edges of $K_{n,n}$ there is a monochromatic $K_{a,a}$? A coloring of $G_{n,n}$ can be viewed as an edge coloring of $K_{n,n}$. A monochromatic rectangle corresponds to a monochromatic $K_{2,2}$. Beineke and Schwenk [3] studied a closely related problem: What is the minimum value of $b$ such that any 2-coloring of $K_{b,b}$ results in a monochromatic $K_{n,m}$? In their work, this minimal value is denoted $R(n, m)$. Later, Hattingh and Henning [8] defined $b(n, m)$ as the minimum $b$ for which any 2-coloring of $K_{b,b}$ contains a monochromatic $K_{m,m}$ or a monochromatic $K_{n,n}$.

In a related paper, Cooper, Fenner, and Purewal [4] generalize the problem to multiple dimensions and obtain upper and lower bounds on the sizes of the obstruction sets.

[1]It was attributed to Gallai in [12] and [13]; Witt proved the theorem in [16].
[2]Both [7] and [5] can be used to obtain better bounds on $W(c)$.

The remainder of this paper is organized as follows. In Sections 2 and 3 we develop tools to show grids *are not* $c$-colorable. In Section 4 we develop tools to show grids *are* $c$-colorable. In Section 5 we obtain upper and lower bounds on $|\mathrm{OBS}_c|$. In Section 6 and 7 we find $\mathrm{OBS}_2$ and $\mathrm{OBS}_3$ respectively. In Section 8 we obtain a most elements of $\mathrm{OBS}_4$, as well as possible elements to complete the set. We also propose a conjecture which, if true, would yield the exact elements of $\mathrm{OBS}_4$. In Section 9 we apply the results to find some new bipartite Ramsey numbers. We conclude with some open questions. The appendix contains some sizes of maximum rectangle free sets (to be defined later).

## 2 Lower Bounds on Uncolorability

A *rectangle-free subset* $A \subseteq G_{n,m}$ is a subset that does not contain a rectangle as defined above. A problem that is closely related to grid-colorability is that of finding a rectangle-free subset of maximum cardinality. This relationship is illustrated by the following lemma.

**Theorem 2.1** *If $G_{n,m}$ is c-colorable, then it contains a rectangle-free subset of size $\lceil \frac{nm}{c} \rceil$.*

**Proof:** A $c$-coloring partitions the elements of $G_{n,m}$ into $c$ rectangle-free subsets. By the pigeon-hole principle, one of these sets must be of size at least $\lceil \frac{nm}{c} \rceil$. ∎

**Def 2.2** Let $n, m \in \mathsf{N}$. $\mathrm{maxrf}(n, m)$ is the size of the maximum rectangle-free $A \subseteq G_{n,m}$.

Finding the maximum cardinality of a rectangle-free subset is equivalent to a special case of a well-known problem of Zarankiewicz [17] (see [6] or [15] for more information). The Zarankiewicz function, denoted $Z_{r,s}(n, m)$, counts the minimum number of edges in a bipartite graph with vertex sets of size $n$ and $m$ that guarantees a subgraph isomorphic to $K_{r,s}$. Zarankiewicz's problem was to determine $Z_{r,s}(n, m)$.

If $r = s$, the function is denoted $Z_r(n, m)$. If one views a grid as an incidence matrix for a bipartite graph with vertex sets of cardinality $n$ and $m$, then a rectangle is equivalent to a subgraph isomorphic to $K_{2,2}$. Therefore the maximum cardinality of a rectangle-free set in $G_{n,m}$ is $Z_2(n, m) - 1$. We will use this lemma in its contrapositive form, i.e., we will often show that $G_{n,m}$ is not $c$-colorable by showing that $Z_2(n, m) \leq \lceil \frac{nm}{c} \rceil$.

Reiman [14] proved the following lemma. Roman [15] later generalized it.

**Lemma 2.3** *Let $m \leq n \leq \binom{m}{2}$. Then $Z_2(n, m) \leq \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n}\right) \right\rfloor + 1$.*

**Corollary 2.4** *Let $m \leq n \leq \binom{m}{2}$. Let $z_{n,m} = \left\lfloor \frac{n}{2} \left(1 + \sqrt{1 + 4m(m-1)/n}\right) \right\rfloor + 1$ be the upper-bound on $Z_2(n, m)$ in Lemma 2.3. If $z_{n,m} \leq \lceil \frac{nm}{c} \rceil$ then $G_{n,m}$ is not c-colorable.*

Corollary 2.4, and some 2-colorings of grids, are sufficient to find $\mathrm{OBS}_2$. To find $\mathrm{OBS}_3$ and $\mathrm{OBS}_4$, we need more powerful tools to show grids are not colorable (along with some 3-colorings and 4-colorings of grids). This next lemma, which has a proof that is very similar to that of the previous lemma gives us two more uncolorability corollaries.

**Def 2.5** Let $n, m, x_1, \ldots, x_m \in \mathsf{N}$. $(x_1, \ldots, x_m)$ is $(n,m)$-*placeable* if there exists a rectangle-free $A \subseteq G_{n,m}$ such that, for $1 \le j \le m$, there are $x_j$ elements of $A$ in the $j^{\text{th}}$ column.

**Lemma 2.6** *Let $n, m, x_1, \ldots, x_m \in \mathsf{N}$ be such that $(x_1, \ldots, x_m)$ is $(n,m)$-placeable. Then $\sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}$.*

**Proof:** Let $A \subseteq G_{n,m}$ be a set that shows that $(x_1, \ldots, x_m)$ is $(n,m)$-placeable. Let $\binom{A}{2}$ be the set of pairs of elements of $A$. Let $2^{\binom{A}{2}}$ be the powerset of $\binom{A}{2}$.

Define the function $f : [m] \to 2^{\binom{A}{2}}$ as follows. For $1 \le j \le m$,

$$f(j) = \{\{a,b\} : (a,j), (b,j) \in A\}.$$

If $\sum_{j=1}^{m} |f(j)| > \binom{n}{2}$ then there exists $j_1 \ne j_2$ such that $f(j_1) \cap f(j_2) \ne \emptyset$. Let $\{a,b\} \in f(j_1) \cap f(j_2)$. Then

$$\{(a,j_1), (a,j_2), (b,j_1), (b,j_2)\} \subseteq A.$$

Hence $A$ contains a rectangle. Since this cannot happen, $\sum_{j=1}^{m} |f(j)| \le \binom{n}{2}$. Note that $|f(j)| = \binom{x_j}{2}$. Hence $\sum_{i=1}^{m} \binom{x_i}{2} \le \binom{n}{2}$. ∎

**Lemma 2.7** *Let $a, n, m \in \mathsf{N}$. Let $q, r \in \mathbb{N}$ be such that $a = qn + r$ with $0 \le r \le n$. Assume that there exists $A \subseteq G_{n,m}$ such that $|A| = a$ and $A$ is rectangle-free.*

1. *If $q \ge 2$ then*
$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. *If $q = 1$ then*
$$r \le \frac{m(m-1)}{2}.$$

**Proof:** The proof for the $q \ge 2$ and the $q = 1$ case begins the same; hence we will not split into cases yet.

Assume that, for $1 \le j \le m$, the number of elements of $A$ in the $j^{\text{th}}$ column is $x_j$. Note that $\sum_{j=1}^{m} x_j = a$. By Lemma 2.6 $\sum_{j=1}^{m} \binom{x_j}{2} \le \binom{n}{2}$. We look at the least value that $\sum_{j=1}^{n} \binom{x_j}{2}$ can have.

Consider the following question:
Minimize $\sum_{j=1}^{n} \binom{x_j}{2}$

Constraints:

- $\sum_{j=1}^{n} x_j = a$.
- $x_1, \ldots, x_n$ are natural numbers.

One can easily show that this is minimized when, for all $1 \le j \le n$,

$$x_j \in \{\lfloor a/n \rfloor, \lceil a/n \rceil\} = \{q, q+1\}.$$

In order for $\sum_{j=1}^{n} x_j = a$ we need to have $n - r$ many $q$'s and $r$ many $q+1$'s. Hence we obtain

$\sum_{j=1}^{n} \binom{x_j}{2}$ is at least

$$(n-r)\binom{q}{2} + r\binom{q+1}{2}.$$

Hence we have

$$(n-r)\binom{q}{2} + r\binom{q+1}{2} \le \sum_{j=1}^{n} \binom{x_j}{2} \le \binom{m}{2}$$

$$nq(q-1) - rq(q-1) + r(q+1)q \le m(m-1)$$

$$nq(q-1) - rq^2 + rq + rq^2 + rq \le m(m-1)$$

$$nq(q-1) + 2rq \le m(m-1)$$

**Case 1:** $q \ge 2$.

Subtract $2rq$ from both sides to obtain

$$nq(q-1) \le m(m-1) - 2rq.$$

Since $q - 1 \ne 0$ we can divide by $q(q-1)$ to obtain

$$n \le \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

**Case 2:** $q = 1$.

Since $q - 1 = 0$ we get

$$2r \le m(m-1)$$

$$r \le \frac{m(m-1)}{2}.$$

∎

**Corollary 2.8** *Let $m, n \in \mathsf{N}$. If there exists an $r$ where $\frac{m(m-1)}{2} < r \le n$ and $\left\lceil \frac{mn}{c} \right\rceil = n + r$, then $G_{m,n}$ is not $c$-colorable.*

**Corollary 2.9** *Let $n, m \in \mathsf{N}$. Let $\lceil \frac{nm}{c} \rceil = qn + r$ for some $0 \le r \le n$ and $q \ge 2$. If $\frac{m(m-1)-2qr}{q(q-1)} < n$ then $G_{n,m}$ is not c-colorable.*

**Note 2.10** In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

**Corollary 2.11**

1. *Let $c \ge 2$ and $1 \le c' < c$. Let $n > \frac{c}{c'}\binom{c+c'}{2}$. Then $G_{n,c+c'}$ is not c-colorable.*
2. *Let $c \ge 2$ and $1 \le c' < c$. Let $m > \frac{c}{c'}\binom{c+c'}{2}$. Then $G_{c+c',m}$ is not c-colorable. (This follows immediately from part a.)*

**Proof:** Assume, by way of contradiction, that $G_{n,c+c'}$ is $c$-colorable. Then there is a rectangle free set of size

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = \left\lceil n + \frac{c'n}{c} \right\rceil = n + \left\lceil \frac{c'}{c} n \right\rceil.$$

Since $c' < c$ we have

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'}{c} n \right\rceil \le n + \left\lceil \frac{c-1}{c} n \right\rceil = n + \left\lceil n - \frac{n}{c} \right\rceil.$$

The premise of this corollary implies $c < n$. Hence

$$\left\lceil \frac{n(c+c')}{c} \right\rceil \le n + \left\lceil n - \frac{n}{c} \right\rceil \le 2n - 1.$$

Therefore when we divide $n$ into $r = \lceil \frac{c'n}{c} \rceil$.

$$\left\lceil \frac{n(c+c')}{c} \right\rceil = n + \left\lceil \frac{c'n}{c} \right\rceil.$$

We want to apply Corollary 2.8 with $m = c + c'$ and $r = \lceil \frac{c'n}{c} \rceil$. We need

$$\frac{m(m-1)}{2} < r \le n.$$

$$\frac{(c+c')(c+c'-1)}{2} < \left\lceil \frac{c'n}{c} \right\rceil \le n.$$

The second inequality is obvious. The first inequality follows from $n > \frac{c}{c'}\binom{c+c'}{2}$.

∎

**Note 2.12** In the Appendix we use the results of this section to find the sizes of maximum rectangle free sets.

# 3 Tools to Show Sets Contain Rectangles

## 3.1 Conventions

Throughout this section we will have the following notations and conventions.

**Notation 3.1** If $n, m \in \mathsf{N}$ and $A \subseteq G_{n,m}$ then we assume the following.

1. The top row of a grid is row 1.
2. We will denote that $(a, b) \in A$ by putting an $R$ in the $(a, b)$ position.
3. For $1 \leq j \leq m$, $x_j$ is the number of elements of $A$ in column $j$.
4. The rows and columns are reordered so that the following holds (unless we explicitly say otherwise):
   (a) $x_1 \geq x_2 \geq \cdots \geq x_m$.
   (b) The first column has $x_1$ contiguous elements of $A$ starting at row 1.
   (c) The second column has $x_2$ contiguous elements of $A$ (unless we say otherwise).
5. For $1 \leq j \leq m$, $C_j$ is the set of rows $r$ such that $A$ has an element in the $r^{\text{th}}$ row of column $j$. Formally
$$C_j = \{r : (r, j) \in A\}.$$
6. For $1 \leq i \leq k$ let
$$I_i = \sum_{1 \leq j_1 < \cdots < j_i \leq m} |C_{j_1} \cap \cdots \cap C_{j_i}|.$$

**Example 3.2**

| | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | $R$ | | $R$ | | | | | | | | | | $R$ | | $R$ | $R$ |
| 2 | $R$ | $R$ | | | | | | | | $R$ | $R$ | | $R$ | | | | |
| 3 | $R$ | | | | | | | | $R$ | | | | | | $R$ | $R$ | |
| 4 | | | | | | $R$ | | | $R$ | | | $R$ | $R$ | $R$ | | | |
| 5 | | $R$ | $R$ | | | $R$ | | | | | | | | | | | |

$$C_1 = \{2, 3\}, C_2 = \{1, 2, 5\}, C_3 = \{5\}, C_4 = \{1\}, C_5 = \emptyset,$$

$$C_6 = \{4, 5\}, C_7 = \{\}, C_8 = \{\}, C_9 = \{3, 4\}, C_{10} = \{2\},$$

$$C_{11} = \{2\}, C_{12} = \{4\}, C_{13} = \{2, 4\}, C_{14} = \{1, 4\}, C_{15} = \{3\},$$

$$C_{16} = \{1,3\}, C_{17} = \{1\},$$

In this example $A$ is rectangle free. Hence, for all $i < j$ $C_i \cap C_j| \leq 1$. Hence we have the following observations.

1. $I_1$ is the number of $R$'s in the grid which is 22.

2. $I_2$ is the number of pairs of columns that intersect. We list all of the intersecting pairs that are nonempty by listing what $C_{j'}$ intersects with $C_j$ where $j' > j$.

   $C_1$ intersects $C_2, C_9, C_{10}, C_{11}, C_{13}, C_{15}, C_{16}$;

   $C_2$ intersects $C_3, C_4, C_6, C_{10}, C_{11}, C_{13}, C_{14}, C_{16}, C_{17}$;

   $C_3$ intersects $C_6$;

   $C_4$ intersects $C_{14}, C_{16}, C_{17}$;

   $C_6$ intersects $C_9, C_{12}, C_{13}, C_{14}$;

   $C_9$ intersects $C_{12}, C_{13}, C_{14}, C_{15}, C_{16}$;

   $C_{10}$ intersects $C_{11}, C_{13}$,

   $C_{11}$ intersects $C_{13}$;

   $C_{12}$ intersects $C_{13}, C_{14}$;

   $C_{13}$ intersects $C_{14}$

   $C_{14}$ intersects $C_{16}, C_{17}$;

   $C_{15}$ intersects $C_{16}$;

   $C_{16}$ intersects $C_{17}$;

   Therefore $I_2 = 39$.

3. $I_3$ is the number of triples of columns that intersect. We list all of the intersecting triplets that are nonempty:

   $(C_1, C_2, C_{10})$, $(C_1, C_2, C_{11})$, $(C_1, C_2, C_{13})$, $(C_1, C_9, C_{15})$, $(C_1, C_9, C_{16})$, $(C_1, C_{10}, C_{11})$, $(C_1, C_{10}, C_{13})$, $(C_1, C_{11}, C_{13})$, $(C_1, C_{15}, C_{16})$;

   $(C_2, C_4, C_{14})$, $(C_2, C_4, C_{16})$, $(C_2, C_4, C_{17})$, $(C_2, C_{10}, C_{11})$, $(C_2, C_{10}, C_{13})$, $(C_2, C_{11}, C_{13})$, $(C_2, C_{14}, C_{16})$, $(C_2, C_{14}, C_{17})$, $(C_2, C_{16}, C_{17})$;

   $(C_4, C_{14}, C_{16})$, $(C_4, C_{14}, C_{17})$, $(C_4, C_{16}, C_{17})$;

   $(C_6, C_9, C_{12})$, $(C_6, C_9, C_{13})$, $(C_6, C_9, C_{14})$, $(C_6, C_{12}, C_{13})$, $(C_6, C_{12}, C_{14})$, $(C_6, C_{13}, C_{14})$;

   $(C_9, C_{15}, C_{16})$, $(C_9, C_{12}, C_{13})$, $(C_9, C_{12}, C_{14})$, $(C_9, C_{13}, C_{14})$;

   $(C_{10}, C_{11}, C_{13})$;

$(C_{12}, C_{13}, C_{14})$;

$(C_{14}, C_{15}, C_{16})$;

Hence $I_3 = 34$.

4. $I_4$ is the number of 4-tuples of columns that intersect. We list all of the intersecting 4-sets that are nonempty:

   $(C_1, C_2, C_{10}, C_{11})$, $(C_1, C_2, C_{10}, C_{13})$, $(C_1, C_9, C_{15}, C_{16})$, $(C_1, C_{10}, C_{11}, C_{13})$;

   $(C_2, C_4, C_{14}, C_{16})$, $(C_2, C_4, C_{14}, C_{17})$, $(C_2, C_4, C_{16}, C_{17})$, $(C_2, C_{10}, C_{11}, C_{13})$, $(C_2, C_{14}, C_{16}, C_{17})$;

   $(C_4, C_{14}, C_{16}, C_{17})$;

   $(C_6, C_9, C_{12}, C_{13})$, $(C_6, C_9, C_{12}, C_{14})$, $(C_6, C_9, C_{13}, C_{14})$;

   $(C_6, C_{12}, C_{13}, C_{14})$;

   $(C_9, C_{12}, C_{13}, C_{14})$;

   Hence $I_4 = 15$.

5. $I_5$ is the number of 5-tuples of columns that intersect. We list all of the intersecting 5-sets that are nonempty:

   $(C_1, C_2, C_{10}, C_{11}, C_{13})$;

   $(C_2, C_4, C_{14}, C_{16}, C_{17})$;

   $(C_6, C_9, C_{12}, C_{13}, C_{14})$;

   Hence $I_5 = 3$.

6. $I_6$ is the number of 6-tuples of columns that intersect. There are none of these, so $I_6 = 0$.

**Def 3.3** Let $n, m \in \mathsf{N}$ and $A \subseteq G_{n,m}$. Let $1 \le i_1 < i_2 \le n$. $C_{i_1}$ and $C_{i_2}$ *intersect* if $C_{i_1} \cap C_{i_2} \neq \emptyset$. The following picture portrays this happening with $C_1$ and $C_2$.

| | 1 | 2 | ... |
|---|---|---|---|
| 1 | $R$ | | ... |
| 2 | $R$ | | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_1 - 1$ | $R$ | | ... |
| $x_1$ | $R$ | $R$ | ... |
| $x_1 + 1$ | | $R$ | ... |
| $x_1 + 2$ | | $R$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_1 + x_2 - 1$ | | $R$ | ... |
| $x_1 + x_2$ | | | ... |
| $x_1 + x_2 + 1$ | | | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | | | ... |

### 3.2 $\sum_{j=1}^{k} x_j$ and $|\bigcup_{j=1}^{k} C_j|$

**Lemma 3.4** *Let $n, m \in \mathsf{N}$. Let $A$ be a rectangle free subset of $G_{n,m}$. Let $1 \le j_1 < j_2 \le n$. Then $|C_{j_1} \cap C_{j_2}| \le 1$.*

**Proof:**

The following picture shows what happens if $|C_1 \cap C_2| \ge 2$. Note that a rectangle is formed. We leave it to the reader to make this into a formal argument.

| | 1 | 2 | ... |
|---|---|---|---|
| 1 | $R$ | | ... |
| 2 | $R$ | | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_1 - 1$ | $R$ | $R$ | ... |
| $x_1$ | $R$ | $R$ | ... |
| $x_1 + 1$ | | $R$ | ... |
| $x_1 + 2$ | | $R$ | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_1 + x_2 - 2$ | | $R$ | ... |
| $x + x_2 - 1$ | | | ... |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n$ | | | ... |

∎

**Lemma 3.5** *Let $n, m \in \mathsf{N}$. Let $1 \le k \le m$. Let $x_1, \ldots, x_M \in \mathsf{N}$. Assume $(x_1, \ldots, x_m)$ is $(n,m)$-placeable via A. (We need not assume that $x_1 \ge \cdots \ge x_m$ and hence can use this for any set of columns.) Then the following two statements are true:*

1. $x_1 + \cdots + x_k \le n + \binom{k}{2}$.
2. *If* $\bigcap_{j=1}^{k} C_j \ne \emptyset$ *then* $\sum_{j=1}^{k} x_j \le n + \sum_{j=2}^{k} (-1)^j \binom{k}{j}$.

**Proof:**

We begin with facts that are useful to prove both parts.

By the law of inclusion-exclusion

$$|\bigcup_{j=1}^{k} C_j| = \sum_{j=1}^{k} |C_j| - I_2 + I_3 - I_4 + \cdots + (-1)^{k+1} I_k.$$

Since $|C_j| = x_j$ we have

$$|\bigcup_{j=1}^{k} C_j| = \sum_{j=1}^{k} x_j - I_2 + I_3 - I_4 + \cdots + (-1)^{k+1} I_k.$$

$$\sum_{j=1}^{k} x_j = |\bigcup_{j=1}^{k} C_j| + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k.$$

Since $|\bigcup_{j=1}^{k} C_j| \le n$ we have

$$\sum_{j=1}^{k} x_j \le n + I_2 - I_3 + I_4 + \cdots + (-1)^{k} I_k.$$

1) Assume $k$ is odd (the case of $k$ even is similar).

$$\sum_{j=1}^{k} x_j \le n + I_2 + (I_4 - I_3) + \cdots + (I_{k-1} - I_{k-2}) - I_k.$$

Since $I_2 \le \binom{k}{2}$ and, for $3 \le j \le k-2$, $(I_{j+1} - I_j) \le 0$ and $-I_k \le 0$ we have

$$\sum_{j=1}^{k} x_j \le n + \binom{k}{2}.$$

2) Since $C_1 \cap \cdots \cap C_k \ne \emptyset$, for all $j$, $I_j = \binom{k}{j}$.

Hence

$$\sum_{j=1}^{k} x_j \le n + \sum_{j=2}^{k} (-1)^j I_j = n + \sum_{j=2}^{k} (-1)^j \binom{k}{j}.$$

■

It will be convenient to specify the $k = 2$ case of Lemma 3.5.1.

**Lemma 3.6** *Let $n, m \in \mathsf{N}$. Assume $(x_1, \ldots, x_m)$ is $(n, m)$-placeable via $A$. Then $x_1 + x_2 \leq n + 1$.*

## 3.3 Using maxrf

**Lemma 3.7** *Let $n, m \in \mathsf{N}$. Let $x \leq x_1 \leq n$. Assume $(x_1, \ldots, x_m)$ is $(n, m)$-placeable via $A$. Then*

$$|A| \leq x + m - 1 + \text{maxrf}(n - x, m - 1).$$

**Proof:** The following picture portrays what might happen in the case of $n = 12$, $x_1 = 8$. We use double lines to partition the grid in a way that will help be helpful later.

| | 1 | 2 | 3 | 4 | 5 | … | $j$ | ⋯ | $m$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | $R$ | $R$ | | | | ⋯ | | ⋯ | |
| 2 | $R$ | | $R$ | | | ⋯ | | ⋯ | |
| 3 | $R$ | | | $R$ | | ⋯ | | ⋯ | |
| 4 | $R$ | | | | $R$ | ⋯ | | ⋯ | |
| 5 | $R$ | | | | | ⋯ | | ⋯ | |
| 6 | $R$ | | | | | ⋯ | | ⋯ | |
| 7 | $R$ | | | | | ⋯ | | ⋯ | $R$ |
| 8 | $R$ | | | | | ⋯ | $R$ | ⋯ | |
| 9 | | $R$ | $R$ | | | ⋯ | | ⋯ | |
| 10 | | $R$ | | $R$ | | ⋯ | | ⋯ | |
| 11 | | $R$ | | | | ⋯ | | ⋯ | |
| 12 | | $R$ | | | | ⋯ | | ⋯ | |

We view this grid in three parts.

**Part 1:** The first column. This has $x_1$ elements of $A$ in it.

**Part 2:** Consider the grid consisting of rows $1, \ldots, x_1$ and columns $2, \ldots, m$. Look at the $j^{\text{th}}$ column, $2 \leq j \leq m$ in this grid. For each such $j$, this column has at most one element in $A$ (else there would be a rectangle using the first column). Hence the total number of elements of $A$ from this part of the grid is $m - 1$.

**Part 3:** The bottom most $n - x_1$ elements of the right most $m - 1$ columns. This clearly has $\leq \text{maxrf}(n - x_1, m - 1)$ elements in it.

Taking all the parts into account we obtain

$$|A| \leq x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1).$$

We leave it as an exercise to show that, if $x \leq x_1$, then

$$x_1 + (m - 1) + \text{maxrf}(n - x_1, m - 1) \leq x + (m - 1) + \text{maxrf}(n - x, m - 1).$$

∎

# 4 Tools for Finding Proper $c$-colorings

## 4.1 Strong $c$-colorings and Strong $(c, c')$-colorings

**Def 4.1** Let $c, c', n, m \in \mathsf{N}$ and let $\chi : G_{n,m} \rightarrow [c]$. Assume $c' \leq c$.

1. A *half-mono rectangle with respect to* $\chi$ is a rectangle where the left corners are the same color and the right corners are the same color.

2. $\chi$ is a *strong c-coloring* if there are no half-mono rectangles.

3. $\chi$ is a *strong* $(c, c')$*-coloring* if for any half-mono rectangle the color of the left corners and the right corners are (1) different, and (2) in $[c']$.

**Example 4.2**

1. The following is a strong 4-coloring of $G_{5,8}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 4 | 1 | 1 | 4 | 4 |
| 2 | 2 | 4 | 1 | 2 | 4 | 1 | 4 |
| 3 | 4 | 2 | 2 | 4 | 2 | 4 | 1 |
| 4 | 3 | 3 | 3 | 4 | 4 | 2 | 2 |
| 4 | 4 | 4 | 4 | 3 | 3 | 3 | 3 |

2. The following is a strong 3-coloring of $G_{4,6}$.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 3 | 1 | 3 | 3 |
| 2 | 3 | 1 | 3 | 1 | 3 |
| 3 | 2 | 2 | 3 | 3 | 1 |
| 3 | 3 | 3 | 2 | 2 | 2 |

3. The following is a strong $(4, 2)$-coloring of $G_{6,15}$.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 2 | 3 | 3 | 2 | 2 | 2 | 2 |
| 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 4 | 4 | 3 | 3 | 3 | 2 |
| 2 | 1 | 3 | 3 | 2 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 4 | 4 | 3 |
| 2 | 2 | 1 | 4 | 3 | 2 | 1 | 4 | 3 | 1 | 2 | 2 | 1 | 1 | 4 |
| 3 | 3 | 2 | 1 | 4 | 2 | 2 | 1 | 4 | 2 | 1 | 4 | 1 | 2 | 1 |
| 4 | 4 | 4 | 2 | 1 | 4 | 4 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 1 |

4. The following is a strong $(6,2)$-coloring of $G_{8,6}$.

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 6 |
| 1 | 2 | 1 | 2 | 4 | 5 |
| 2 | 1 | 2 | 1 | 5 | 4 |
| 2 | 2 | 1 | 1 | 6 | 3 |
| 3 | 4 | 5 | 6 | 1 | 2 |
| 4 | 5 | 6 | 4 | 1 | 1 |
| 5 | 6 | 3 | 3 | 1 | 2 |
| 6 | 3 | 4 | 5 | 1 | 2 |

5. The following is a strong $(5,3)$-coloring of $G_{8,28}$.

| | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 3 | 2 | 4 | 3 | 4 | 3 | 2 | 3 | 4 | 3 | 2 | 3 | 3 | 2 | 2 | 2 | 2 |
| 1 | 2 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 5 | 4 | 5 | 4 | 3 | 4 | 3 | 4 | 3 | 3 | 4 | 3 | 3 | 3 | 2 |
| 2 | 1 | 3 | 3 | 3 | 3 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 4 | 4 | 3 | 4 | 3 | 4 | 3 |
| 2 | 2 | 1 | 4 | 4 | 4 | 3 | 2 | 1 | 3 | 3 | 3 | 3 | 1 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 4 | 3 | 3 |
| 3 | 3 | 2 | 1 | 5 | 3 | 3 | 2 | 2 | 1 | 4 | 4 | 4 | 2 | 1 | 3 | 3 | 3 | 1 | 2 | 2 | 2 | 1 | 1 | 1 | 5 | 5 | 4 |
| 3 | 4 | 3 | 2 | 1 | 5 | 4 | 3 | 3 | 2 | 1 | 5 | 3 | 2 | 2 | 1 | 5 | 4 | 2 | 1 | 3 | 3 | 1 | 2 | 2 | 1 | 1 | 5 |
| 4 | 3 | 4 | 3 | 2 | 1 | 5 | 3 | 4 | 3 | 2 | 1 | 5 | 3 | 3 | 2 | 1 | 5 | 2 | 2 | 1 | 5 | 2 | 1 | 3 | 1 | 2 | 1 |
| 5 | 5 | 5 | 5 | 3 | 2 | 1 | 4 | 3 | 4 | 3 | 2 | 1 | 3 | 5 | 3 | 2 | 1 | 3 | 3 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 1 |

**Lemma 4.3** *Let $c, c', n, m \in \mathsf{N}$. Let $x = \lfloor c/c' \rfloor$. If $G_{n,m}$ is strongly $(c,c')$-colorable then $G_{n,xm}$ is $c$-colorable.*

**Proof:**
Let $\chi$ be a strong $(c,c')$-coloring of $G_{n,m}$. Let the colors be $\{1,\ldots,c\}$. Let $\chi^i$ be the coloring

$$\chi^i(a,b) = \chi(a,b) + i \pmod{c}.$$

(During calculations mod $c$ we use $\{1,\ldots,c\}$ instead of the more conventional $\{0,\ldots,c-1\}$.)

Take $G_{n,m}$ with coloring $\chi$. Place next to it $G_{n,m}$ with coloring $\chi^{c'}$. Then place next to that $G_{n,m}$ with coloring $\chi^{2c'}$ Keep doing this until you have $\chi^{(x-1)c'}$ placed. The following is an example using the strong $(6,2)$-coloring of $G_{8,6}$ in Example 4.2.4. Since $c' = 2$ and $x = 3$ we will be shifting the colors first by 2 then by 4.

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 2 | 3 | 6 | 3 | 3 | 4 | 4 | 5 | 2 | 5 | 5 | 6 | 6 | 1 | 4 |
| 1 | 2 | 1 | 2 | 4 | 5 | 3 | 4 | 3 | 4 | 6 | 1 | 5 | 6 | 5 | 6 | 2 | 3 |
| 2 | 1 | 2 | 1 | 5 | 4 | 4 | 3 | 4 | 3 | 1 | 6 | 6 | 5 | 6 | 5 | 3 | 2 |
| 2 | 2 | 1 | 1 | 6 | 3 | 4 | 4 | 3 | 3 | 2 | 5 | 6 | 6 | 5 | 5 | 4 | 1 |
| 3 | 4 | 5 | 6 | 1 | 2 | 5 | 6 | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 5 | 6 |
| 4 | 5 | 6 | 4 | 1 | 1 | 6 | 1 | 2 | 6 | 3 | 3 | 2 | 3 | 4 | 2 | 5 | 5 |
| 5 | 6 | 3 | 3 | 1 | 2 | 1 | 2 | 5 | 5 | 3 | 4 | 3 | 4 | 1 | 1 | 5 | 6 |
| 6 | 3 | 4 | 5 | 1 | 2 | 2 | 5 | 6 | 1 | 3 | 4 | 4 | 1 | 2 | 3 | 5 | 6 |

We claim that the construction always creates a $c$-coloring of $G_{m,xn}$.

We show that there is no rectangle with the two leftmost points from the first $G_{n,m}$. From this, to show that there are no rectangles at all is just a matter of notation.

Assume that in column $i_1$ there are two points colored $R$ (in this proof $1 \le R, B, G \le c$.) We call these *the $i_1$-points*. The points cannot form a rectangle with any other points in $G_{n,m}$ since $\chi$ is a $c$-coloring of $G_{n,m}$. The $i_1$-points cannot form a rectangle with points in columns $i_1+m$, $i_1+2m$, …, $i_1+(c-1)m$ since the colors of those points are $R+c' \pmod{c}$, $R+2c' \pmod{c}$, …, $R+(x-1)c' \pmod{c}$, all of which are not equal to $R$. Is there a $j$, $1 \le j \le x-1$ and a $i_2$, $1 \le i_2 \le m$ such that the $i_1$-points form a rectangle with points in column $i_2+jm$?

Since $\chi$ is a strong $(c,c')$-coloring, points in column $i_2$ and on the same row as the $i_1$-points are either colored *differently*, or both colors are in $[c']$. We consider both of these cases.

**Case 1:** In column $i_2$ the colors are $B$ and $G$ where $B \ne G$ (it is possible that $B = R$ or $G = R$ but not both). By the construction, the points in column $i_2+jm$ are colored $B+jc' \pmod{c}$ and $G+jc' \pmod{c}$. These points are colored differently, hence they cannot form a rectangle with the $i_1$-points.

| $\cdots$ | $i_1$ | $\cdots$ | $i_2$ | $\cdots$ | $\cdots$ | $i_1+jm$ | $\cdots$ | $i_2+jm$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $\cdots$ | $R$ | $\cdots$ | $B$ | $\cdots$ | $\cdots$ | $R+jc'$ | $\cdots$ | $B+jc'$ | $\cdots$ |
| $\cdots$ | $R$ | $\cdots$ | $G$ | $\cdots$ | $\cdots$ | $R+jc'$ | $\cdots$ | $G+jc'$ | $\cdots$ |

**Case 2:** In column $i_2$ the colors are both $B$.

| $\cdots$ | $i_1$ | $\cdots$ | $i_2$ | $\cdots$ | $\cdots$ | $i_1+jm$ | $\cdots$ | $i_2+jm$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $\cdots$ | $R$ | $\cdots$ | $B$ | $\cdots$ | $\cdots$ | $R+jc'$ | $\cdots$ | $B+jc'$ | $\cdots$ |
| $\cdots$ | $R$ | $\cdots$ | $B$ | $\cdots$ | $\cdots$ | $R+jc'$ | $\cdots$ | $B+jc'$ | $\cdots$ |

We have $R, B \in [c']$. By the construction, the points in column $i_2+jm$ are both colored $B+jc' \pmod{c}$. We show that $R \not\equiv B+jc' \pmod{c}$. Since $1 \le j \le x-1$ we have

$$c' \le jc' \le (x-1)c'.$$

Hence

$$B+c' \le B+jc' \le B+(x-1)c'.$$

Since $B \in [c']$ we have $B + (x-1)c' \le xc'$. Hence

$$B + c' \le B + jc' \le xc'.$$

By the definition of $x$ we have $xc' \le c$. Since $B \in [c']$ we have $B + c' \ge c' + 1$. Hence

$$c' + 1 \le B + jc' \le c.$$

Since $R \in [c']$ we have that $R \not\equiv B + jc'$. ▮

## 4.2 Using Combinatorics and Strong $(c, c')$-colorings

**Theorem 4.4** *Let $c \ge 2$.*

1. *There is a strong $c$-coloring of $G_{c+1,\binom{c+1}{2}}$.*

2. *There is a $c$-coloring of $G_{c+1,m}$ where $m = c\binom{c+1}{2}$.*

**Proof:**
1) We first do an example of our construction. In the $c = 5$ case we obtain the following coloring.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 5 | 5 | 5 | 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 5 | 5 | 5 | 5 | 2 | 2 | 2 | 2 | 2 | 2 |
| 1 | 5 | 2 | 2 | 2 | 5 | 2 | 2 | 2 | 5 | 5 | 5 | 3 | 3 | 3 |
| 2 | 2 | 5 | 3 | 3 | 2 | 5 | 3 | 3 | 5 | 3 | 3 | 5 | 5 | 4 |
| 3 | 3 | 3 | 5 | 4 | 3 | 3 | 5 | 4 | 3 | 5 | 4 | 5 | 4 | 5 |
| 4 | 4 | 4 | 4 | 5 | 4 | 4 | 4 | 5 | 4 | 4 | 5 | 4 | 5 | 5 |

Here is our general construction. Index the columns by $\binom{[c+1]}{2}$. Color rows of column $\{x, y\}$, $x < y$, as follows.

1. Color rows $x$ and $y$ with color $c$.

2. On the other spots use the colors $\{1, 2, 3, \ldots, c-1\}$ in increasing order (the actual order does not matter).

We call the coloring $\chi : G_{n,m} \to [c]$. We show that ther are no half-mono rectangles. Let $RECT = \{p_1, p_2, q_1, q_2\}$ be a rectangle with $p_1, p_2$ in column $\{x, y\}$ and $q_1, q_2$ in column $\{x', y'\}$.

If any of $p_1, p_2, q_1, q_2$ have a color in $\{1, \ldots, c-1\}$ then $RECT$ cannot be a half-mono rectangle since the colors $\{1, \ldots, c-1\}$ only appear once in each column.

If $\chi(p_1) = \chi(p_2) = \chi(q_1) = \chi(q_2) = c$ then $p_1$ and $p_2$ are in rows $x$ and $y$, and $q_1$ and $q_2$ are in rows $x'$ and $y'$. Since $RECT$ is a rectangle $\{x, y\} = \{x', y'\}$. Hence $p_1, p_2, q_1, q_2$ are all in the same column. This contradicts $RECT$ being a rectangle.

2) This follows from Lemma 4.3 with $c = c$ and $c' = 1$, and Part (1) of this theorem. ∎

The next theorem generalizes Theorem 4.4.

**Theorem 4.5** *Let $c, c' \in \mathsf{N}$ with $c \geq 2$ and $1 \leq c' \leq c$.*

1. *There is a strong $(c, c')$-coloring of $G_{c+c',m}$ where $m = \binom{c+c'}{2}$.*
2. *There is a $c$-coloring of $G_{c+c',m'}$ where $m' = \lfloor c/c' \rfloor \binom{c+c'}{2}$.*

To prove Theorem 4.5, we will use a partition of $\binom{[2n]}{2}$ into perfect matchings of $[2n]$ for certain values of $n$. Each perfect matching thus has size $n$.

We first give some examples and then a general lemma.

**Example 4.6**

1. If $n = 3$, $2n = 6$, $2n - 1 = 5$. We show a partition of $\binom{[6]}{2}$ into 5 parts of size 3. We first pair up the elements as follows, each number in the top row being paired with the number below it:

   | | | |
   |---|---|---|
   | 1 | 2 | 3 |
   | 6 | 5 | 4 |

   This corresponds to $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$. This is our first part of size 3.

   We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

   | | | |
   |---|---|---|
   | 1 | 6 | 2 |
   | 5 | 4 | 3 |

   | | | |
   |---|---|---|
   | 1 | 5 | 6 |
   | 4 | 3 | 2 |

   | | | |
   |---|---|---|
   | 1 | 4 | 5 |
   | 3 | 2 | 6 |

   | | | |
   |---|---|---|
   | 1 | 3 | 4 |
   | 2 | 6 | 5 |

   Note that the first pair went $\{1, 5\}$, $\{1, 4\}$, $\{1, 3\}$, $\{1, 2\}$. That is, 1 was fixed but the other element decreased by 1. Also note that the second and third pair had both elements decrease by 1 except 2 goes to 6. This partition is a special case of a general construction we will have later. The same applies to the next example.

2. If $n = 4$, $2n = 8$, $2n - 1 = 7$. Here is a partition of $\binom{[8]}{2}$ into 7 parts of size 4.

   We keep 1 fixed and keep rotating the other numbers clockwise to obtain the following parts.

$$\begin{array}{|c|c|c|c|} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 8 & 2 & 3 \\ 7 & 6 & 5 & 4 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 7 & 8 & 2 \\ 6 & 5 & 4 & 3 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 5 & 6 & 7 \\ 4 & 3 & 2 & 8 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 4 & 5 & 6 \\ 3 & 2 & 8 & 7 \end{array}$$

$$\begin{array}{|c|c|c|c|} 1 & 3 & 4 & 5 \\ 2 & 8 & 7 & 6 \end{array}$$

The next lemma shows that such partitions always exist. The lemma (and the examples above) is based on the Wikipedia entry on Round Robin tournaments. We present a proof for completeness.

**Lemma 4.7** *Let $n \in \mathsf{N}$. $\binom{[2n]}{2}$ can be partitioned into $2n-1$ sets $P_1, \ldots, P_{2n-1}$, each of size $n$, such that each $P_i$ is itself a partition of $[2n]$ into pairs (i.e., a perfect matching).*

**Proof:** Following the examples above, we define the cyclic permutation $\rho$ on $\{2, 3, \ldots, 2n\}$ as follows:

$$\rho(x) = \begin{cases} x - 1 & \text{if } 2 < x \leq 2n, \\ 2n & \text{if } x = 2, \end{cases}$$

for all $x \in \{2, 3, \ldots, 2n\}$. Then for each $1 \leq i \leq 2n-1$, we define

$$P_i = \{\ \{1, 2n - i + 1\}\ \} \cup \left\{\ \{\rho^{(i-1)}(j), \rho^{(i-1)}(2n - j + 1)\} \mid 2 \leq j \leq n \right\},$$

noting that $2n - i + 1 = \rho^{(i-1)}(2n)$. It is not too hard to see that each $P_i$ contains exactly $n$ pairwise disjoint pairs, so it suffices to show that no pair appears in two different $P_i$. Clearly, no pair of the form $\{1, x\}$ can appear in more than one $P_i$. Suppose $\{x, y\}$ appears in both $P_i$ and $P_j$ for some $i < j$, where $2 \le x, y \le 2n$. Then without loss of generality, $x$ appears in the top row of $P_i$ with $y$ just below it. If $x$ is still in the top row of $P_j$, then $x$ has shifted to the right and $y$ to the left, and so $x$ and $y$ are not vertically aligned in $P_j$, which means that $\{x, y\} \notin P_j$. So it must be that $x$ is on the bottom row of $P_j$ with $y$ just above it. But for this to happen, $x$ and $y$ would have to rotate different amounts from $P_i$ to $P_j$ (one an even distance and the other an odd distance), but they rotate the same amount, namely, $j - i$ spaces—contradiction. Thus the $P_i$ are as required. ∎

**Proof:** [Proof of Theorem 4.5]
1) Here is our general construction. We split into two cases.

**Case 1: $c + c'$ is even.** Then $c + c' = 2n$ for some $n$. Since $c' \le c$, we also have $c' \le n$. Let $P_1, \ldots, P_{2n-1}$ be the partition of $[2n]$ of Lemma 4.7. Index the elements of each $P_i$ as $p_{i,j}$ for $1 \le j \le n$, that is, $P_i = \{p_{i,1}, p_{i,2}, \ldots, p_{i,n}\}$. We break up the columns into $2n - 1$ blocks of $n$ columns each (note that $n(2n-1) = \binom{2n}{2}$). We color the $j^{\text{th}}$ column in the $i^{\text{th}}$ block as follows:

- Assign color 1 to the two elements of $p_{i,(j+1) \bmod n}$,
- Assign color 2 to the two elements of $p_{i,(j+2) \bmod n}$,
- $\vdots$
- Assign color $c'$ to the two elements of $p_{i,(j+c') \bmod n}$,
- Assign one each of the colors $c' + 1, \ldots, c$ one each to the rest of the elements in the column in increasing order.

Suppose some pair $p_{i,k} = \{x, y\}$ is monochrome in two separate columns. Then both these columns must be in the $i^{\text{th}}$ block, the $j_1^{\text{st}}$ column (colored $c_1$) and $j_2^{\text{nd}}$ column (colored $c_2$), say. Then we must have

$$k = (j_1 + c_1) \bmod n = (j_2 + c_2) \bmod n.$$

Since $j_1 \ne j_2$, we must have $c_1 \ne c_2$.

**Case 2: $c + c'$ is odd.** Then we choose a simpler partition. Let $c + c' = 2n + 1$ for some $n$. Since $c' < c$, we also have $c' \le n$. For $1 \le i \le 2n + 1$ and $1 \le j \le n$, define

$$p_{i,j} = \{(i + j) \bmod (2n + 1), (i - j) \bmod (2n + 1)\} \in \binom{[2n+1]}{2},$$

and let

$$P_i = \{p_{i,1}, \ldots, p_{i,n}\}.$$

It is not too hard to see that all the pairs within the same $P_i$ are pairwise disjoint and that no pair is contained in more than one $P_i$.

We now proceed with exactly the same recipe as in Case 1, except that $\binom{2n+1}{2} = n(2n+1)$, we group the columns into $2n+1$ blocks of $n$ columns each. We get a strong $(c, c')$-coloring just as in Case 1.

2) This follows from Lemma 4.3 and Part (1) of this theorem. ∎

**Corollary 4.8** *For all $c \geq 2$, there is a $c$-coloring of $G_{2c,2c^2-c}$.*

## 4.3 Using Finite Fields and Strong $c$-colorings

**Def 4.9** Let $X$ be a finite set and $q \in \mathsf{N}$, $q \geq 3$. Let $P \subseteq \binom{X}{q}$.

$$\text{pairs}(P) = \{\{a_1, a_2\} \in \binom{X}{2} : (\exists a_3, \ldots, a_q)[\{a_1, \ldots, a_q\} \in P]\}.$$

**Example 4.10** Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $q = 3$.

1. Let $P = \{\{1, 2, 6\}, \{1, 8, 9\}, \{2, 4, 6\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 6\}, \{2, 6\}, \{1, 8\}, \{1, 9\}, \{8, 9\}, \{2, 4\}, \{4, 6\}\}$$

2. Let $P = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$. Then

$$\text{pairs}(P) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}, \{7, 8\}, \{7, 9\}, \{8, 9\}\}.$$

**Lemma 4.11** *Let $c, m, r \in \mathsf{N}$. Assume there exist $P_1, \ldots, P_m \subseteq \binom{[cr]}{r}$ such that the following hold.*

- *For all $1 \leq j \leq m$, $P_j$ is a partition of $[cr]$ into $c$ parts of size $r$.*
- *For all $1 \leq j_1 < j_2 \leq m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$.*

*Then*

1. *$G_{cr,m}$ is strongly $c$-colorable.*
2. *$G_{cr,cm}$ is $c$-colorable.*

**Proof:**
1)

We define a strong $c$-coloring $COL$ of $G_{cr,m}$ using $P_1, \ldots, P_m$.

Let $1 \le j \le m$. Let

$$P_j = \{L_j^1, \ldots, L_j^c\}$$

where each $L_j^i$ is a subset of $r$ elements from $[cr]$.

Let $1 \le i \le cr$ and $1 \le j \le m$. Since $P_j$ is a partition of $[cr]$ there exists a unique $u$ such that $i \in L_j^u$. Define

$$COL(i, j) = u.$$

We show that this is a strong $c$-coloring. Assume, by way of contradiction, that there exists $1 \le i_1 < i_2 \le 2k$ and $1 \le j_1 < j_2 \le 2k - 1$ such that $COL(i_1, j_1) = COL(i_1, j_2) = u$ and $COL(i_2, j_1) = COL(i_2, j_2) = v$. By definition of the coloring we have

$$i_1 \in L_{j_1}^u, i_1 \in L_{j_2}^u, i_2 \in L_{j_1}^v, i_2 \in L_{j_2}^v$$

Then

$$\{i_1, i_2\} \in \text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}),$$

contradicting the second premise on the $P$'s.

2) This follows from Part (1) and Lemma 4.3 with $c = c$ and $c' = 1$. ∎

The Round Robin partition of Lemma 4.7 is an example of a partition satisfying the premises of Lemma 4.11, where $c = n$, $r = 2$, and $m = 2n - 1 = 2c - 1$. The next theorem yields partitions with bigger values of $r$.

**Theorem 4.12** *Let $p$ be a prime and $s, d \in \mathsf{N}$.*

1. *$G_{p^{ds}, \frac{p^{ds}-1}{p-1}}$ is strongly $p^{ds-s}$-colorable.*
2. *$G_{p^{ds}, \frac{p^{ds}-1}{p-1} p^{ds-s}}$ is $p^{ds-s}$-colorable.*

**Proof:** Let $c = p^{ds-s}$, $r = p^s$, and $m = \frac{p^{ds}-1}{p^s-1}$. We show that there exists $P_1, \ldots, P_m$ satisfying the premise of Lemma 4.11. The result follows immediately.

Let $F$ be the finite field on $p^s$ elements. We identify $[cr]$ with the set $F^d$.

**Def 4.13**

1. Let $\vec{x} \in F^d$, $\vec{y} \in F^d - \{0^d\}$. Then

$$L_{\vec{x},\vec{y}} = \{\vec{x} + f\vec{y} \mid f \in F\}.$$

Sets of this form are called *lines*. Note that for all $\vec{x}, \vec{y}, a \in F$ with $a \neq 0$,

$$L_{\vec{x},\vec{y}} = L_{\vec{x},a\vec{y}}.$$

2. Two lines $L_{\vec{x},\vec{y}}$, $L_{\vec{z},\vec{w}}$ *have the same slope* if $\vec{y}$ is a multiple of $\vec{w}$.

The following are easy to prove and well-known.

- If $L$ and $L'$ are two distinct lines that have the same slope, then $L \cap L' = \emptyset$.
- If $L$ and $L'$ are two distinct lines with different slopes, then $|L \cap L'| \le 1$.
- If $L$ is a line then there are exactly $r = p^s$ points on $L$.
- If $L$ is a line then there are exactly $c = p^{ds-s}$ lines that have the same slope as $L$ (this includes $L$ itself).
- There are exactly $\frac{p^{ds}-1}{p^s-1}$ different slopes.

We define $P_1, \ldots, P_m$ as follows.

1. Pick a line $L$. Let $P_1$ be the set of lines that have the same slope as $L$.
2. Assume that $P_1, \ldots, P_{j-1}$ have been defined and that $j \le m$. Let $L$ be a line that is not in $P_1 \cup \cdots \cup P_{j-1}$. Let $P_j$ be the set of all lines that have the same slope as $L$.

We need to show that $P_1, \ldots, P_m$ satisfies the premises of Lemma 4.11
a) For all $1 \le j \le m$, $P_j$ is a partition of $[cr]$ into $c$ parts of size $r$. Let $L \in P_j$. Note that $P_j$ is the set of all lines with the same slope as $L$. Clearly this partitions $F^d$ which is $[cr]$.
b) For all $1 \le j_1 < j_2 \le m$, $\text{pairs}(P_{j_1}) \cap \text{pairs}(P_{j_2}) = \emptyset$. Let $L_1$ be any line in $P_{j_1}$ and $L_2$ be any line in $P_{j_2}$. Since $|L_1 \cap L_2| \le 1 < 2$ we have the result.

Note that each $P_j$ has $c = p^{ds-s}$ sets (lines) in it, each set (line) has $r = p^s$ numbers (points), and there are $m = \frac{p^{ds}-1}{p^s-1}$ many $P$'s. Hence the premises of Lemma 4.11 are satisfied. ∎

It is convenient to state the $s = 1$, $d = 2$ case of Theorem 4.12.

**Corollary 4.14** *Let $p$ be a prime.*

1. *There is a strong $p$-coloring of $G_{p^2,p+1}$.*
2. *There is a $p$-coloring of $G_{p^2,p^2+p}$.*

**Note 4.15** It would be of interest to obtain a lemma similar to Theorem 4.12 that does not need prime powers and possibly yields strong $(c, c')$-colorings.

## 4.4 Using Finite Fields for the Square and Almost Square Case

Can Theorem 4.12 be used to get that, if $c$ is a prime power, $G_{c^2,c^2}$ is $c$-colorable. Not quite. If $d = 2$ one obtains that a grid of dimensions $p^{2s} \times \frac{p^s-1}{p-1}p^s$ is $c$-colorable.

Ken Berg and Quimey Vivas have both shown (independently) that if $c$ is a prime power then $G_{c^2,c^2}$ is $c$-colorable. (They both emailed me their proofs.) Ken Berg extended this to show that if $c$ is a prime power then $G_{c^2,c^2+c}$ is $c$-colorable. We present both proofs. This result is orthogonal to Theorem 4.12 in that there are results you can get from either that you cannot get from the other.

**Theorem 4.16** *If $c$ is a prime power then $G_{c^2,c^2}$ is $c$-colorable.*

**Proof:**

Let $F$ be a field of $c$ elements. We view the elements of $G_{c^2,c^2}$ as indexed by $(F \times F) \times (F \times F)$. The colorings is

$$COL((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2 + y_2.$$

Note that all of this arithmetic takes place in the field $F$.

Assume, by way of contradiction, that there is a monochromatic rectangle. Then there exists $w_1, w_2, x_1, x_2, y_1, y_2, z_1, z_2 \in F$ such that $((w_1, w_2), (x_1, x_2))$, $((w_1, w_2), (y_1, y_2))$, $((z_1, z_2), (x_1, x_2))$, and $((z_1, z_2), (y_1, y_2))$ are all distinct and

$$COL((w_1, w_2), (x_1, x_2)) = COL((w_1, w_2), (y_1, y_2)) = COL((z_1, z_2), (x_1, x_2)) = COL((z_1, z_2), (y_1, y_2)).$$

Since $COL((w_1, w_2), (x_1, x_2)) = COL((w_1, w_2), (y_1, y_2))$

$$\begin{aligned} w_1x_1 + w_2 + x_2 &= w_1y_1 + w_2 + y_2 \\ w_1(x_1 - y_1) &= y_2 - x_2 \end{aligned}$$

Since $COL((z_1, z_2), (x_1, x_2)) = COL((z_1, z_2), (y_1, y_2))$

$$\begin{aligned} z_1x_1 + z_2 + x_2 &= z_1y_1 + z_2 + y_2 \\ z_1(x_1 - y_1) &= y_2 - x_2 \end{aligned}$$

Combining these two we get

$$\begin{aligned} w_1(x_1 - y_1) &= z_1(x_1 - y_1) \\ (w_1 - z_1)(x_1 - y_1) &= 0 \end{aligned}$$

Since the arithmetic takes place in a field we obtain that either $w_1 = z_1$ or $x_1 = y_1$.

**Case 1:** $w_1 = z_1$

Since $COL((w_1, w_2), (x_1, x_2)) = COL((z_1, z_2), (x_1, x_2))$. Hence

$$\begin{aligned} w_1x_1 + w_2 + x_2 &= z_1x_1 + z_2 + x_2 \\ z_1x_1 + w_2 + x_2 &= z_1x_1 + z_2 + x_2 \text{ Since } w_1 = z_1. \\ w_2 &= z_2 \end{aligned}$$

Since $w_1 = z_1$ and $w_2 = z_2$ the four points are not distinct. This is a contradiction.

**Case 2:** $x_1 = y_1$**.** Similar to Case 1.

▮

**Theorem 4.17** *If $c$ is a prime power then $G_{c^2,c^2+c}$ is c-colorable.*

**Proof:**

Let $F$ be a field of $c$ elements. Let $*$ be a symbol to which we assign no meaning to. We view the elements of $G_{c^2,c^2+c}$ as indexed by $(F \times F) \times (F \cup \{*\} \times F)$.

We describe the coloring. Assume $x_1, x_2, y_1, y_2 \in F$.

$$\begin{aligned} COL((x_1, x_2), (y_1, y_2)) &= x_1 y_1 + x_2 + y_2 \\ COL((x_1, x_2), (*, y_2)) &= x_1 + y_2 \end{aligned}$$

Note that all of this arithmetic takes place in the field $F$.

Assume, by way of contradiction, that there is a monochromatic rectangle. Then there exists $w_1, w_2, x_2, , y_2, z_1, z_2 \in F$ and $x_1, y_1 \in F \cup \{*\}$ such that $((w_1, w_2), (x_1, x_2))$, $((w_1, w_2), (y_1, y_2))$, $((z_1, z_2), (x_1, x_2))$, $((z_1, z_2), (y_1, y_2))$ are all distinct and

$$COL((w_1, w_2), (x_1, x_2)) = COL((w_1, w_2), (y_1, y_2)) = COL((z_1, z_2), (x_1, x_2)) = COL((z_1, z_2), (y_1, y_2)).$$

By the proof of Theorem 4.16 at least one $x_1, y_1$ is $*$. We can assume $x_1 = *$. There are two cases.

**Case 1:** $y_1 = *$. Since

$$COL((w_1, w_2), (*, x_2)) = COL((w_1, w_2), (*, y_2))$$

we have

$$w_1 + x_2 = w_1 + y_2$$

so $x_2 = y_2$. Hence $(x_1, x_2) = (y_1, y_2)$ so the points are not distinct.

**Case 2:** $y_1 \neq *$. Since

$$COL((w_1, w_2), (*, x_2)) = COL((z_1, z_2), (*, x_2))$$

we have

$$w_1 + x_2 = z_1 + x_2$$

so $w_1 = z_1$. Since

$$COL((w_1, w_2), (y_1, y_2)) = COL((z_1, z_2), (y_1, y_2))$$

we have

$$w_1 y_1 + w_2 + y_2 = z_1 y_1 + z_2 + y_2.$$

Since $w_1 = z_1$ we have $w_2 = z_2$. Hence we have $(w_1, w_2) = (z_1, z_2)$ so the points are not distinct.

▮

# 5 Bounds on the Sizes of Obstruction Sets

## 5.1 An Upper Bound

Using the uncolorability bounds, we can obtain an upper-bound on the size of a $c$-colorable grid.

**Theorem 5.1** *For all $c > 0$, $G_{c^2+c,c^2+c}$ is not $c$-colorable.*

**Proof:** We apply Corollary 2.9 with $m = c^2 + c$ and $n = c^2 + c$. Note that

$$\begin{aligned}\left\lceil \frac{nm}{c} \right\rceil &= \left\lceil \frac{(c^2+c)(c^2+c)}{c} \right\rceil \\ &= (c+1)(c^2+c).\end{aligned}$$

Letting $q = c + 1$ and $r = 0$, we have

$$\begin{aligned}\frac{m(m-1) - 2qr}{q(q-1)} &= \frac{(c^2+c)(c^2+c-1)}{(c+1)c} \\ &= c^2 + c - 1 \\ &< c^2 + c \\ &= n.\end{aligned}$$

▮

Using this, we can obtain an upper-bound on the size of an obstruction set.

**Theorem 5.2** *If $c > 0$, then $|\mathrm{OBS}_c| \le 2c^2$.*

**Proof:** For each $r$, there can be at most one $c$-minimal grid of the form $G_{r,n}$. Likewise, there can be at most one $c$-minimal grid of the form $G_{n,r}$. If $r \le c$ then for all $n$, $G_{r,n}$ and $G_{n,r}$ are trivially $c$-colorable and are, therefore, not $c$-minimal. Theorem 5.1 shows that for all $n, m > c^2 + c$, $G_{n,m}$ is not $c$-minimal. It follows that there can be at most two $c$-minimal grids for each integer $r$ where $c < r \le c^2 + c$. Therefore there are at most $2c^2$ $c$-minimal grids in $\mathrm{OBS}_c$. ▮

## 5.2 A Lower Bound

To get a lower bound on $|\mathrm{OBS}_c|$, we will combine Corollary 2.11 and Theorem 4.5(2) with the following lemma:

**Lemma 5.3** *Suppose that $G_{m_1,n}$ is c-colorable and $G_{m_2,n}$ is not c-colorable. Then there exists a grid $G_{x,y} \in \mathrm{OBS}_c$ such that $m_1 < x \le m_2$ (and in addition, $y \le n$).*

**Proof:** Given $n$, let $x$ be the least integer such that $G_{x,n}$ is not $c$-colorable. Clearly, $m_1 < x \le m_2$. Now given $x$ as above, let $y$ be least such that $G_{x,y}$ is not $c$-colorable. Clearly, $y \le n$ and $G_{x,y} \in \mathrm{OBS}_c$. ▮

**Theorem 5.4** $|\mathrm{OBS}_c| \ge 2\sqrt{c}(1 - o(1))$.

**Proof:** For any $c \ge 2$ and any $1 \le c' \le c$ we can summarize Corollary 2.11 and Theorem 4.5(2) as follows:

$$G_{c+c',n} \text{ is } \begin{cases} c\text{-colorable} & \text{if } n \le \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}, \\ \text{not } c\text{-colorable} & \text{if } n > \frac{c}{c'}\binom{c+c'}{2}. \end{cases}$$

(We won't use the fact here, but note that this is tight if $c'$ divides $c$.)

Suppose $c' > 1$ and

$$\frac{c}{c'}\binom{c+c'}{2} < \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}. \tag{1}$$

Then letting $n := \left\lfloor \frac{c}{c'-1} \right\rfloor \binom{c+c'-1}{2}$, we see that $G_{c+c'-1,n}$ is $c$-colorable, but $G_{c+c',n}$ is not. Then by Lemma 5.3, there is a grid $G_{c+c',y} \in \mathrm{OBS}_c$ for some $y$. So there are at least as many elements of $\mathrm{OBS}_c$ as there are values of $c'$ satisfying Inequality (1)—actually twice as many, because $G_{n,m} \in \mathrm{OBS}_c$ iff $G_{m,n} \in OBS_c$.

Fix any real $\varepsilon > 0$. Clearly, Inequality (1) holds provided

$$\frac{c}{c'}\binom{c+c'}{2} \le \left(\frac{c}{c'-1} - 1\right)\binom{c+c'-1}{2}.$$

A rather tedious calculation reveals that if $2 \le c' \le (1-\varepsilon)\sqrt{c}$, then this latter inequality holds for all large enough $c$. Including the grid $G_{c+1,n} \in \mathrm{OBS}_c$ where $n = c\binom{c+1}{2} + 1$, we then get $|\mathrm{OBS}_c| \ge \lfloor (1-\varepsilon)\sqrt{c} \rfloor$ for all large enough $c$, and since $\varepsilon$ was arbitrary, we therefore have $|\mathrm{OBS}_c| \ge \sqrt{c}(1 - o(1))$.

To double the count, we notice that $c + c' \le \left\lfloor \frac{c}{c'} \right\rfloor \binom{c+c'}{2}$, hence $G_{c+c',c+c'}$ is $c$-colorable by Theorem 4.5(2). This means that $G_{c+c',y} \in \mathrm{OBS}_c$ for some $y > c + c'$, and so we can count $G_{y,c+c'} \in \mathrm{OBS}_c$ as well without counting any grids twice. ▮

# 6 Which Grids Can be Properly 2-Colored?

**Lemma 6.1**

1. *$G_{7,3}$ and $G_{3,7}$ are not 2-colorable*
2. *$G_{5,5}$ is not 2-colorable.*
3. *$G_{7,2}$ and $G_{2,7}$ are 2-colorable (this is trivial).*
4. *$G_{6,4}$ and $G_{4,6}$ are 2-colorable.*

**Proof:**

We only consider grids of the form $G_{n,m}$ where $n \geq m$.

1,2)

In the following table we show that $G_{7,3}$ and $G_{5,5}$ are not 2-colorable. For each $(n,m)$ we plan to use either Corollary 2.8 or 2.9. In the table we give, for each $(n,m)$, the value of $\lceil \frac{nm}{2} \rceil$, the $q,r$ such that $\lceil \frac{nm}{2} \rceil = qn + r$ with $0 \leq r \leq n-1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

| $m$ | $n$ | $\lceil \frac{nm}{2} \rceil$ | $q$ | $r$ | Use | Prem | Arith |
|---|---|---|---|---|---|---|---|
| 3 | 7 | 11 | 1 | 4 | Cor 2.8 | $\frac{m(m-1)}{2} < r \leq n1$ | $3 < 4 \leq 7$ |
| 5 | 5 | 13 | 2 | 3 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n1$ | $4 < 5$ |

Table 1: Uncolorability values for $c = 2$

3) $G_{7,2}$ is clearly 2-colorable.

4) $G_{6,4}$ is 2-colorable by Corollary 4.14 with $p = 2$.

∎

**Theorem 6.2** $\text{OBS}_2 = \{G_{7,3}, G_{5,5}, G_{3,7}\}$.

**Proof:**

$G_{7,3}$ is not 2-colorable by Lemma 6.1. $G_{6,3}$ is 2-colorable by Lemma 6.1. $G_{7,2}$ is 2-colorable by Lemma 6.1. Hence $G_{7,3}$ is 2-minimal. The proof for $G_{3,7}$ is similar.

$G_{5,5}$ is not 2-colorable by Lemma 6.1. $G_{5,4}$ and $G_{4,5}$ are 2-colorable by Lemma 6.1. Hence $G_{5,5}$ is 2-minimal.

We need to show that $G_{7,3}$, $G_{5,5}$, and $G_{3,7}$ are the only 2-minimal grids. We consider the different possible values of $n$ with $m \leq n$ and then use symmetry.

| $n$ | $m \le n$ | comment |
|---|---|---|
| $1, 2, 3, 4$ | any $m \le n$ | $G_{n,m}$ is 2-colorable by Lemma 6.1 |
| $n$ | $1, 2$ | $G_{n,m}$ is 2-colorable by Lemma 6.1 |
| 5 | $3, 4$ | $G_{n,m}$ is 2-colorable by Lemma 6.1 |
| 5 | 5 | $G_{5,5} \in \mathrm{OBS}_2$ |
| 6 | $3, 4$ | $G_{n,m}$ is 2-colorable by Lemma 6.1 |
| 6 | $5, 6$ | $G_{n,m}$ is not 2-minimal since $G_{5,5}$ not 2-colorable |
| 7 | 3 | $G_{7,3} \in \mathrm{OBS}_2$ |
| $n \ge 7$ | $4, \ldots, n$ | $G_{n,m}$ is not 2-minimal since $G_{7,3}$ not 2-colorable |

∎

The following chart indicates exactly which grids are 2-colorable. The entry for $(n, m)$ is $C$ if $G_{n,m}$ is 2-colorable, and $N$ if $G_{n,m}$ is not 2-colorable.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| 3 | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ |
| 4 | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ |
| 5 | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ |
| 6 | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ |
| 7 | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 8 | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |

Table 2: Table of 3-colorable grids ($C$) and non 3-colorable grids $N$

# 7 Which Grids Can be Properly 3-Colored?

**Lemma 7.1**

1. *$G_{19,4}$ and $G_{4,19}$ are not 3-colorable.*
2. *$G_{16,5}$ and $G_{5,16}$ are not 3-colorable.*
3. *$G_{13,7}$ and $G_{7,13}$ are not 3-colorable.*
4. *$G_{12,10}$ and $G_{10,12}$ are not 3-colorable.*
5. *$G_{11,11}$ is not 3-colorable.*
6. *$G_{19,3}$ and $G_{3,19}$ are 3-colorable (this is trivial).*
7. *$G_{18,4}$ and $G_{4,18}$ are 3-colorable.*

8. $G_{15,6}$ *and* $G_{6,15}$ *are 3-colorable.*

9. $G_{12,9}$ *and* $G_{9,12}$ *are 3-colorable.*

**Proof:** We just consider the grids $G_{n,m}$ were $n \geq m$.
1, 2, 3, 4, 5)

In the following table we show that several grids are not 3-colorable. For each $(n, m)$ we plan to use either Corollary 2.8 or 2.9. In the table we give, for each $(n, m)$, the value of $\lceil \frac{nm}{3} \rceil$, the $q, r$ such that $\lceil \frac{nm}{3} \rceil = qn + r$ with $0 \leq r \leq n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

| $m$ | $n$ | $\lceil \frac{nm}{3} \rceil$ | $q$ | $r$ | Use | Prem | Arith |
|---|---|---|---|---|---|---|---|
| 4 | 19 | 26 | 1 | 7 | Cor 2.8 | $\frac{m(m-1)}{2} < r \leq n$ | $6 < 7 \leq 19$ |
| 5 | 16 | 27 | 1 | 11 | Cor 2.8 | $\frac{m(m-1)}{2} < r \leq n$ | $10 < 11 \leq 16$ |
| 7 | 13 | 31 | 2 | 5 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ $11 < 13$ | |
| 10 | 12 | 40 | 3 | 4 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ $11 < 12$ | |
| 11 | 11 | 41 | 3 | 8 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ $10\frac{1}{3} < 11$ | |

Table 3: Uncolorability values for $c = 3$

6) $G_{19,3}$ is clearly 3-colorable.

7) $G_{18,4}$ is 3-colorable by Theorem 4.4 with $c = 3$.

8) $G_{15,6}$ is 3-colorable by Corollary 4.8 with $c = 3$.

9) $G_{12,9}$ is 3-colorable by Corollary 4.14 with $p = 3$. ∎

**Lemma 7.2** $G_{10,10}$ *is 3-colorable.*

**Proof:** This is the 3-coloring:
∎

**Note 7.3** The coloring in Lemma 7.2 we found by first finding a size 34 rectangle free subset of $G_{10,10}$ (by hand). We used that for one of the colors. The final 3-coloring was then found with a simple computer program. It is an open problem to find a general theorem that has a corollary that $G_{10,10}$ is 3-colorable.

**Lemma 7.4** *If* $A \subseteq G_{11,10}$ *and* $A$ *is rectangle-free then* $|A| \leq 36 = \lceil \frac{11 \cdot 10}{3} \rceil - 1$. *Hence* $G_{11,10}$ *is not 3-colorable.*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $R$ | $R$ | $R$ | $R$ | $B$ | $B$ | $G$ | $G$ | $B$ | $G$ |
| $R$ | $B$ | $B$ | $G$ | $R$ | $R$ | $R$ | $G$ | $G$ | $B$ |
| $G$ | $R$ | $B$ | $G$ | $R$ | $B$ | $B$ | $R$ | $R$ | $G$ |
| $G$ | $B$ | $R$ | $B$ | $B$ | $R$ | $G$ | $R$ | $G$ | $R$ |
| $R$ | $B$ | $G$ | $G$ | $G$ | $B$ | $G$ | $B$ | $R$ | $R$ |
| $G$ | $R$ | $B$ | $B$ | $G$ | $G$ | $R$ | $B$ | $B$ | $R$ |
| $B$ | $G$ | $R$ | $B$ | $G$ | $B$ | $R$ | $G$ | $R$ | $B$ |
| $B$ | $B$ | $G$ | $R$ | $R$ | $G$ | $B$ | $G$ | $B$ | $R$ |
| $G$ | $G$ | $G$ | $R$ | $B$ | $R$ | $B$ | $B$ | $R$ | $B$ |
| $B$ | $G$ | $B$ | $R$ | $B$ | $G$ | $R$ | $R$ | $G$ | $G$ |

Table 4: Proper 3-coloring of $G_{10,10}$.

**Proof:**

We divide the proof into cases. Every case will either conclude that $|A| \le 36$ or $A$ cannot exist.

For $1 \le j \le 10$ let $x_j$ be the number of elements of $A$ in column $j$. We assume

$$x_1 \ge \cdots \ge x_{10}.$$

1. $5 \le x_1 \le 11$.

   By Lemma 3.7 with $x = 5$, $n = 11$, $m = 10$ we have

   $$|A| \le x+m-1+\text{maxrf}(n-x, m-1) \le 5+10-1+\text{maxrf}(11-5, 10-1) \le 14+\text{maxrf}(6, 9).$$

   By Lemma 12.1 we have $\text{maxrf}(6, 9) = 21$. Hence

   $$|A| \le 14 + 21 = 35 \le 36.$$

2. $x_1 \le 3$. Then $|A| \le 3 \times 11 = 33 < 36$. Note that we now know that $x_1 = 4$.

3. There exists $k$, $1 \le k \le 6$, such that $x_1 = \cdots = x_k = 4$ and $x_{k+1} \le 3$. Then

   $$|A| = \sum_{j=1}^{10} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{10} x_j) \le 4k + 3(10-k) = 30 + k$$

   Since $k \le 6$ this quantity is $\le 30 + 6 = 36$. Hence $|A| \le 36$.

4. $x_1 = \cdots = x_7 = 4$ and, for all, $1 \le j_1 < j_2 < j_3 \le 7$,

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 0.$$

Let $G'$ be the grid restricted to the first 7 columns. Let $B$ be $A$ restricted to $G'$. Since every column of $G'$ has 4 elements of $B$, $|B| = 7 \times 4 = 28$. Since every row of $G'$ has $\le 2$ elements of $B$, $|B| \le 2 \times 11 = 22$. Therefore $A$ does not exist.

5. $x_1 = \cdots = x_7 = 4$ and there exists $1 \le j_1 < j_2 < j_3 \le 7$ such that

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3}| = 1,$$

but for all $1 \le j_1 < j_2 < j_3 < j_4 \le 7$

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 0.$$

By renumbering we can assume that

$$|C_1 \cap C_2 \cap C_3| = 1$$

and that the intersection is in row 11. Let $G'$ be the grid restricted to the first 7 columns. Let $B$ be $A$ restricted to $G'$. The following picture portrays what is in the first 3 columns of $G'$.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | $R$ | | | | | | |
| 2 | $R$ | | | | | | |
| 3 | $R$ | | | | | | |
| 4 | | $R$ | | | | | |
| 5 | | $R$ | | | | | |
| 6 | | $R$ | | | | | |
| 7 | | | $R$ | | | | |
| 8 | | | $R$ | | | | |
| 9 | | | $R$ | | | | |
| 10 | | | | | | | |
| 11 | $R$ | $R$ | $R$ | | | | |

Since there are no $1 \le j_1 < j_2 < j_3 < j_4 \le 7$ with

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1,$$

there will be no other elements of $A$ in row 11 of $G'$.

Consider $C_4$. Since $|C_4| = 4$, $|C_1 \cap C_4| \ge 1$, $|C_2 \cap C_4| \ge 1$, $|C_3 \cap C_4| \ge 1$, and $11 \notin C_4$ we must have (1) $|C_1 \cap C_4| = 1$, $|C_2 \cap C_4| = 1$, $|C_3 \cap C_4| = 1$, and (2) $10 \in C_4$.

By the same reasoning on $C_5, C_6, C_7$ we have that, for all $5 \le i \le 7$, (1) $|C_1 \cap C_i| = 1$, $|C_2 \cap C_i| = 1$, $|C_4 \cap C_i| = 1$, and (1) $10 \in C_i$.

The following picture portrays all that we know so far.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | $R$ | | | | | | |
| 2 | $R$ | | | | | | |
| 3 | $R$ | | | | | | |
| 4 | | $R$ | | | | | |
| 5 | | $R$ | | | | | |
| 6 | | $R$ | | | | | |
| 7 | | | $R$ | | | | |
| 8 | | | $R$ | | | | |
| 9 | | | $R$ | | | | |
| 10 | | | | $R$ | $R$ | $R$ | $R$ |
| 11 | $R$ | $R$ | $R$ | | | | |

Note that $C_4 \cap C_5 \cap C_6 \cap C_7 \neq \emptyset$. This contradicts the premise of this case.

Note that $C_4 \cap C_5 \cap C_6 \cap C_7 \neq \emptyset$. This contradicts the premise of this case.

Without loss of generality we can assume $C_4 = \{1, 4, 7, 10\}$. Consider $C_5$. In addition to the above constraints on $C_5$ we also have that $1, 4, 7 \notin C_5$. Hence, without loss of generality, we can assume that $C_5 = \{2, 5, 8, 10\}$. In order to meet our requirements, we need $C_6 = \{3, 6, 9, 10\}$, but now there is no way to color $C_7$ without getting a rectangle. So $A$ cannot exist.

6. $x_1 = \cdots = x_7 = 4$ and there exists $1 \le j_1 < j_2 < j_3 < j_4 \le 7$ such that

$$|C_{j_1} \cap C_{j_2} \cap C_{j_3} \cap C_{j_4}| = 1.$$

By renumbering we can assume that

$$|C_1 \cap C_2 \cap C_3 \cap C_4| = 1.$$

By Lemma 3.5.2 with $k = 4$ and $x_1 = x_2 = x_3 = x_4 = 4$:

$$16 = x_1 + x_2 + x_3 + x_4 \le 11 + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 11 + 6 - 4 + 1 = 14.$$

Hence $A$ does not exist.

∎

**Theorem 7.5**

$$\mathrm{OBS}_3 = \{G_{19,4}, G_{16,5}, G_{13,7}, G_{11,10}, G_{10,11}, G_{7,13}, G_{5,16}, G_{4,19}\}.$$

**Proof:**

For each $G_{n,m}$ listed above, (1) $G_{n,m}$ is not 3-colorable by Lemma 7.1 or 7.4, (2) both $G_{n-1,m}$ and $G_{n,m-1}$ are 3-colorable by Lemma 7.1 or 7.2 . Hence all of the grids listed are in $\mathrm{OBS}_3$. We need to show that no other grids are in $\mathrm{OBS}_3$. This is a straightforward use of Lemmas 7.1, 7.2, and 7.4. The proof is similar to how Theorem 6.2 was proven. We leave the details to the reader. ∎

The following chart indicates exactly which grids are 3-colorable. The entry for $(n, m)$ is $C$ if $G_{n,m}$ is 3-colorable, and $N$ if $G_{n,m}$ is not 3-colorable.

| | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ |
| 4 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ |
| 5 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 6 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 7 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 8 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 9 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 10 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 11 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 12 | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 13 | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 14 | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 15 | $C$ | $C$ | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 16 | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 17 | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 18 | $C$ | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 19 | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |
| 20 | $C$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ | $N$ |

Table 5: Summary of information on which grids are 3-colorable.

# 8 Which Grids Can be Properly 4-Colored?

In the first section we give absolute results about which grids are 4 colorable. In the second section we give results that assume a conjecture.

## 8.1 Absolute Results

**Theorem 8.1**

1. *$G_{41,5}$ and $G_{5,41}$ are not 4-colorable.*
2. *$G_{31,6}$ and $G_{6,31}$ are not 4-colorable.*
3. *$G_{29,7}$ and $G_{7,29}$ are not 4-colorable.*
4. *$G_{25,9}$ and $G_{9,25}$ are not 4-colorable.*
5. *$G_{23,10}$ and $G_{10,23}$ are not 4-colorable.*
6. *$G_{22,11}$ and $G_{11,22}$ are not 4-colorable.*
7. *$G_{21,13}$ and $G_{13,21}$ are not 4-colorable.*
8. *$G_{20,17}$ and $G_{17,20}$ are not 4-colorable.*
9. *$G_{19,18}$ and $G_{18,19}$ are not 4-colorable.*
10. *$G_{41,4}$ and $G_{4,41}$ are 4-colorable (this is trivial).*
11. *$G_{40,5}$ and $G_{5,40}$ are 4-colorable.*
12. *$G_{30,6}$ and $G_{6,30}$ are 4-colorable.*
13. *$G_{28,8}$ and $G_{8,28}$ are 4-colorable.*
14. *$G_{20,16}$ and $G_{16,20}$ are 4-colorable.*

**Proof:**

We only consider grids $G_{n.m}$ where $n \ge m$.

1,2,3,4,5,6,7,8,9)

In the following table we show that several grids are not 4-colorable. For each $(n, m)$ we plan to use either Corollary 2.8 or 2.9. In the table we give, for each $(n, m)$, the value of $\lceil \frac{nm}{4} \rceil$, the $q, r$ such that $\lceil \frac{nm}{4} \rceil = qn + r$ with $0 \le r \le n - 1$, which corollary we use (Use), the premise of the corollary (Prem), and the arithmetic showing the premise is true (arith).

10) $G_{41,4}$ is clearly 4-colorable.

11) $G_{40,5}$ is 4-colorable by Theorem 4.4 with $c = 4$.

12) $G_{30,6}$ is 4-colorable by Theorem 4.5 with $c = 4$ and $c' = 2$.

13) $G_{28,8}$ is 4-colorable by Theorem 4.12 with $p = 2$, $d = 3$, and $s = 1$.

14) $G_{20,16}$ is 4-colorable by Theorem 4.12 with $p = 2$, $d = 2$, and $s = 2$

▮

| $m$ | $n$ | $\lceil \frac{nm}{4} \rceil$ | $q$ | $r$ | Use | Prem | Arith |
|---|---|---|---|---|---|---|---|
| 5 | 41 | 52 | 1 | 11 | Cor 2.8 | $\frac{m(m-1)}{2} < r \le n$ | $10 < 11 \le 41$ |
| 6 | 31 | 47 | 1 | 16 | Cor 2.8 | $\frac{m(m-1)}{2} < r \le n$ | $15 < 16 \le 31$ |
| 7 | 29 | 51 | 1 | 22 | Cor 2.8 | $\frac{m(m-1)}{2} < r \le n$ | $21 < 22 \le 29$ |
| 9 | 25 | 57 | 2 | 7 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $22 < 25$ |
| 10 | 23 | 58 | 2 | 12 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $21 < 23$ |
| 11 | 22 | 61 | 2 | 17 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $21 < 22$ |
| 13 | 21 | 69 | 3 | 6 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $20 < 21$ |
| 17 | 20 | 85 | 4 | 5 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $19\frac{1}{3} < 20$ |
| 18 | 19 | 86 | 4 | 10 | Cor 2.9 | $\frac{m(m-1)-2qr}{q(q-1)} < n$ | $18\frac{5}{6} < 19$ |

Table 6: Uncolorability values for $c = 4$

**Lemma 8.2** *If $A \subseteq G_{19,17}$ and $A$ is rectangle-free then $|A| \le 80 = \lceil \frac{19 \cdot 17}{4} \rceil - 1$. Hence $G_{19,17}$ is not 4-colorable.*

**Proof:** We divide the proof into cases. Every case will either conclude that $|A| \le 80$ or $A$ cannot exist.

For $1 \le j \le 17$ let $x_j$ be the number of elements of $A$ in column $j$. We assume

$$x_1 \ge \cdots \ge x_{17}.$$

1. $6 \le x_1 \le 19$.

   By Lemma 3.7 with $x = 6$, $n = 19$, $m = 17$,

   $$|A| \le x+m-1+\text{maxrf}(n-x, m-1) \le 6+17-1+\text{maxrf}(19-6, 17-1) = 22+\text{maxrf}(13, 16).$$

   Assume, by way of contradiction, that $|A| \ge 81$. Then $\text{maxrf}(13, 16) \ge 59$ By Lemma 2.7 with $n = 16$, $m = 13$, $a = 59$, $q = 3$, $r = 11$

   $$16 \le \left\lfloor \frac{13 \times 12 - 2 \times 3 \times 11}{3 \times 2} \right\rfloor = 15.$$

   This is a contradiction.

2. There exists $k$, $1 \le k \le 12$, such that $x_1 = \cdots = x_k = 5$ and $x_{k+1} \le 4$. Then

   $$|A| = \sum_{j=1}^{17} x_j = (\sum_{j=1}^{k} x_j) + (\sum_{j=k+1}^{17} x_j) \le 5k + 4(17-k) = 68 + k.$$

   Since $k \le 12$ this quantity is $\le 68 + 12 = 80$. Hence $|A| \le 80$.

3. $x_1 = x_2 = \cdots = x_{13} = 5$. Look at the grid restricted to the first 13 columns. Let $B$ be $A$ restricted to that grid. Note that $B$ is a rectangle-free subset of $G_{19,13}$ of size 65. By Lemma 2.7 with $n = 19$, $m = 13$, $a = 65$, $q = 3$, and $r = 8$ we have

$$19 \leq \left\lfloor \frac{13 \times 12 - 2 \times 8 \times 3}{3 \times 2} \right\rfloor = 18.$$

This is a contradiction, hence $A$ cannot exist.

▮

**Lemma 8.3** *$G_{24,9}$ is 4-colorable.*

**Proof:** We show that $G_{9,6}$ is strongly $(4,1)$-colorable and then apply Lemma 4.3 with $c = 4$ and $c' = 1$.

The following is a strong 4-coloring of $G_{9,6}$.

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $Y$ | $R$ | $R$ | $Y$ | $R$ | $R$ |
| 2 | $Y$ | $B$ | $B$ | $R$ | $Y$ | $B$ |
| 3 | $Y$ | $G$ | $G$ | $B$ | $B$ | $Y$ |
| 4 | $R$ | $Y$ | $G$ | $Y$ | $G$ | $B$ |
| 5 | $B$ | $Y$ | $R$ | $B$ | $Y$ | $G$ |
| 6 | $G$ | $Y$ | $B$ | $G$ | $R$ | $Y$ |
| 7 | $G$ | $B$ | $Y$ | $Y$ | $B$ | $G$ |
| 8 | $R$ | $G$ | $Y$ | $G$ | $Y$ | $R$ |
| 9 | $B$ | $R$ | $Y$ | $R$ | $G$ | $Y$ |

Table 7: Strong 4-coloring of $G_{9,6}$.

▮

**Lemma 8.4** *$G_{21,11}$ is 4-colorable*

**Proof:** Brad Lorsen has obtained the following 4-coloring of $G_{21,11}$.
▮

**Lemma 8.5** *$G_{22,10}$ is 4-colorable*

**Proof:** Brad Lorsen has obtained the following 4-coloring of $G_{22,10}$.
▮

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $G$ | $B$ | $B$ | $G$ | $R$ | $Y$ | $R$ | $G$ | $Y$ | $B$ | $Y$ |
| 2 | $B$ | $G$ | $G$ | $Y$ | $B$ | $G$ | $Y$ | $R$ | $R$ | $B$ | $R$ |
| 3 | $R$ | $R$ | $B$ | $Y$ | $B$ | $Y$ | $B$ | $Y$ | $G$ | $G$ | $R$ |
| 4 | $Y$ | $R$ | $Y$ | $G$ | $B$ | $B$ | $R$ | $Y$ | $R$ | $G$ | $B$ |
| 5 | $R$ | $Y$ | $G$ | $B$ | $B$ | $Y$ | $Y$ | $B$ | $R$ | $G$ | $G$ |
| 6 | $B$ | $R$ | $Y$ | $R$ | $G$ | $Y$ | $B$ | $R$ | $G$ | $Y$ | $B$ |
| 7 | $Y$ | $G$ | $B$ | $R$ | $G$ | $B$ | $R$ | $G$ | $Y$ | $Y$ | $R$ |
| 8 | $Y$ | $Y$ | $G$ | $B$ | $R$ | $B$ | $G$ | $R$ | $G$ | $B$ | $Y$ |
| 9 | $R$ | $B$ | $R$ | $B$ | $G$ | $G$ | $R$ | $Y$ | $Y$ | $G$ | $B$ |
| 10 | $R$ | $Y$ | $Y$ | $R$ | $G$ | $R$ | $B$ | $B$ | $Y$ | $B$ | $G$ |
| 11 | $B$ | $Y$ | $R$ | $R$ | $Y$ | $B$ | $G$ | $G$ | $R$ | $Y$ | $G$ |
| 12 | $R$ | $B$ | $Y$ | $Y$ | $Y$ | $B$ | $B$ | $R$ | $G$ | $R$ | $G$ |
| 13 | $G$ | $G$ | $B$ | $B$ | $R$ | $R$ | $Y$ | $Y$ | $R$ | $Y$ | $G$ |
| 14 | $G$ | $B$ | $R$ | $Y$ | $B$ | $G$ | $G$ | $R$ | $B$ | $Y$ | $Y$ |
| 15 | $G$ | $Y$ | $G$ | $Y$ | $G$ | $R$ | $R$ | $R$ | $B$ | $B$ | $B$ |
| 16 | $B$ | $B$ | $R$ | $G$ | $Y$ | $G$ | $Y$ | $B$ | $Y$ | $R$ | $G$ |
| 17 | $Y$ | $G$ | $B$ | $G$ | $Y$ | $Y$ | $R$ | $B$ | $G$ | $R$ | $B$ |
| 18 | $B$ | $Y$ | $B$ | $G$ | $G$ | $R$ | $G$ | $Y$ | $B$ | $R$ | $R$ |
| 19 | $Y$ | $G$ | $R$ | $Y$ | $R$ | $B$ | $G$ | $B$ | $B$ | $G$ | $R$ |
| 20 | $B$ | $R$ | $Y$ | $B$ | $R$ | $G$ | $Y$ | $G$ | $G$ | $R$ | $Y$ |
| 21 | $G$ | $R$ | $R$ | $B$ | $Y$ | $R$ | $B$ | $Y$ | $B$ | $G$ | $Y$ |

Table 8: A 4-coloring of $G_{21,11}$ due to Brad Lorsen

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $Y$ | $G$ | $R$ | $R$ | $G$ | $G$ | $Y$ | $Y$ | $B$ | $B$ |
| 2 | $G$ | $Y$ | $B$ | $G$ | $B$ | $B$ | $Y$ | $R$ | $Y$ | $R$ |
| 3 | $B$ | $G$ | $B$ | $R$ | $Y$ | $Y$ | $G$ | $R$ | $Y$ | $B$ |
| 4 | $Y$ | $Y$ | $G$ | $G$ | $R$ | $R$ | $B$ | $B$ | $G$ | $Y$ |
| 5 | $Y$ | $B$ | $Y$ | $Y$ | $G$ | $R$ | $R$ | $G$ | $G$ | $R$ |
| 6 | $Y$ | $B$ | $R$ | $B$ | $R$ | $Y$ | $G$ | $R$ | $G$ | $G$ |
| 7 | $G$ | $Y$ | $G$ | $Y$ | $B$ | $Y$ | $R$ | $B$ | $R$ | $G$ |
| 8 | $Y$ | $R$ | $R$ | $B$ | $Y$ | $B$ | $G$ | $G$ | $B$ | $R$ |
| 9 | $Y$ | $B$ | $B$ | $R$ | $R$ | $G$ | $R$ | $G$ | $Y$ | $G$ |
| 10 | $R$ | $R$ | $B$ | $B$ | $Y$ | $G$ | $R$ | $B$ | $G$ | $Y$ |
| 11 | $R$ | $G$ | $G$ | $Y$ | $R$ | $B$ | $B$ | $G$ | $Y$ | $R$ |
| 12 | $R$ | $B$ | $R$ | $G$ | $G$ | $Y$ | $Y$ | $B$ | $B$ | $G$ |
| 13 | $B$ | $R$ | $G$ | $B$ | $G$ | $R$ | $B$ | $R$ | $Y$ | $Y$ |
| 14 | $G$ | $G$ | $Y$ | $B$ | $B$ | $Y$ | $R$ | $R$ | $G$ | $B$ |
| 15 | $R$ | $G$ | $Y$ | $R$ | $B$ | $R$ | $B$ | $Y$ | $Y$ | $G$ |
| 16 | $B$ | $B$ | $Y$ | $G$ | $Y$ | $B$ | $Y$ | $G$ | $R$ | $R$ |
| 17 | $G$ | $Y$ | $B$ | $R$ | $Y$ | $G$ | $B$ | $Y$ | $B$ | $R$ |
| 18 | $R$ | $B$ | $G$ | $Y$ | $B$ | $G$ | $Y$ | $R$ | $R$ | $Y$ |
| 19 | $G$ | $B$ | $R$ | $Y$ | $Y$ | $R$ | $B$ | $G$ | $R$ | $B$ |
| 20 | $B$ | $R$ | $Y$ | $G$ | $R$ | $G$ | $G$ | $B$ | $R$ | $Y$ |
| 21 | $B$ | $R$ | $G$ | $R$ | $B$ | $Y$ | $G$ | $Y$ | $B$ | $Y$ |
| 22 | $G$ | $Y$ | $Y$ | $R$ | $G$ | $B$ | $G$ | $B$ | $R$ | $B$ |

Table 9: A 4-coloring of $G_{22,10}$ due to Brad Lorsen

**Theorem 8.6**

1. *The following grids are in* $\mathrm{OBS}_4$*:*

   $G_{41,5}$, $G_{31,6}$, $G_{29,7}$, $G_{25,9}$, $G_{23,10}$, $G_{22,11}$, $G_{11,22}$, $G_{10,23}$, $G_{9,25}$, $G_{7,29}$, $G_{6,31}$, $G_{5,41}$.

2. *For each of the following grids it is not known if it is 4-colorable. These are the only such.* $G_{17,17}$, $G_{17,18}$, $G_{18,17}$, $G_{18,18}$. $G_{21,12}$, $G_{12,21}$.

3. *Exactly one of the following grids is in* $\mathrm{OBS}_4$*:* $G_{21,12}$ $G_{21,13}$.

4. *Exactly one of the following grids is in* $\mathrm{OBS}_4$*:* $G_{19,17}$, $G_{18,17}$, $G_{17,17}$.

5. *If* $G_{19,17} \in \mathrm{OBS}_4$ *then it is possible that* $G_{18,18} \in \mathrm{OBS}_4$.

**Proof:** This is easily proven from Theorem 8.1, Lemmas 8.2, 8.3, 8.4, and 8.5. For a visual aid see the following chart where we put a $C$ in the $(n,m)$ spot if $G_{n,m}$ is Colorable, an $N$ if it is not colorable, and a $U$ if it is not known. ▮

## 8.2 Assuming the Rectangle-Free Conjecture

We have the following conjecture which, if true, yields more 4-colorings and allows us to state exactly what $\mathrm{OBS}_4$ is.

**Rectangle-Free Conjecture (RFC): Let $n, m, c \geq 2$. If there exists a rectangle-free subset of $G_{n,m}$ of size $\lceil nm/c \rceil$ then $G_{n,m}$ is $c$-colorable.**

**Lemma 8.7** *There exists a rectangle-free subset of* $G_{21,12}$ *of size* $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$*. Hence, if RFC is true, there is a 4-coloring of* $G_{21,12}$ *and* $G_{12,21}$.

**Proof:**

We show there exists a rectangle free subset of $G_{21,12}$ of size 63 by showing it to you:

▮

**Lemma 8.8** *There exists a rectangle-free subset of* $G_{18,18}$ *of size* $81 = \left\lceil \frac{18 \cdot 18}{4} \right\rceil$*. Hence, if RFC is true, there is a 4-coloring of* $G_{18,18}$.

**Proof:**

Here is the rectangle-free set.

▮

**Note 8.9** If the $5^{\text{th}}$ row and the $2^{\text{nd}}$ column were removed then this would be a rectangle free set of $G_{17,17}$ of size 74. Note that $\left\lceil \frac{17 \times 17}{4} \right\rceil = 73$. Hence if we had a weaker version of RFC then we would have that $G_{17,17}$ is 4-colorable.

| | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C |
| 9 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C |
| 10 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C |
| 11 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C |
| 12 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | U |
| 13 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | N |
| 14 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | N |
| 15 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | N |
| 16 | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | C | N |
| 17 | C | C | C | C | C | C | C | C | C | C | C | C | C | U | U | N | N | N |
| 18 | C | C | C | C | C | C | C | C | C | C | C | C | C | U | U | N | N | N |
| 19 | C | C | C | C | C | C | C | C | C | C | C | C | C | N | N | N | N | N |
| 20 | C | C | C | C | C | C | C | C | C | C | C | C | C | N | N | N | N | N |
| 21 | C | C | C | C | C | C | C | C | U | N | N | N | N | N | N | N | N | N |
| 22 | C | C | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N |
| 23 | C | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N |
| 24 | C | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N |
| 25 | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 26 | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 27 | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 28 | C | C | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 29 | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 30 | C | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 31 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 32 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 33 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 34 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 35 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 36 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 37 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 38 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 39 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 40 | C | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |
| 41 | C | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N | N |

Table 10: Summary of what is known about 4-coloring of grids.

| | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $R$ | $R$ | | | | | | | | | | |
| 2 | $R$ | | $R$ | | | | | | | | | |
| 3 | | $R$ | $R$ | | | | | | | | | |
| 4 | | | $R$ | $R$ | $R$ | | | | | | | |
| 5 | | $R$ | | $R$ | | $R$ | | | | | | |
| 6 | $R$ | | | | $R$ | $R$ | | | | | | |
| 7 | | | | | | $R$ | $R$ | $R$ | | | | |
| 8 | | | | | $R$ | | $R$ | | $R$ | | | |
| 9 | | | | $R$ | | | | $R$ | $R$ | | | |
| 10 | | | | | | $R$ | | | | $R$ | $R$ | |
| 11 | | | | | $R$ | | | | | $R$ | | $R$ |
| 12 | | | | $R$ | | | | | | | $R$ | $R$ |
| 13 | | | $R$ | | | $R$ | | | $R$ | | | $R$ |
| 14 | | $R$ | | | $R$ | | | $R$ | | | $R$ | |
| 15 | $R$ | | | $R$ | | | $R$ | | | $R$ | | |
| 16 | | | $R$ | | | | $R$ | | | | $R$ | |
| 17 | | $R$ | | | | | | | $R$ | $R$ | | |
| 18 | $R$ | | | | | | | $R$ | | | | $R$ |
| 19 | | | $R$ | | | | | $R$ | | $R$ | | |
| 20 | | $R$ | | | | | $R$ | | | | | $R$ |
| 21 | $R$ | | | | | | | | $R$ | | $R$ | |

Table 11: A Rectangle Free Subset of $G_{21,12}$ of size 63.

| | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | $R$ | | $R$ | | | | | | | | | | $R$ | | $R$ | $R$ | |
| 2 | $R$ | $R$ | | | | | | | | $R$ | $R$ | | $R$ | | | | | |
| 3 | $R$ | | | | | | | | $R$ | | | | | | $R$ | $R$ | | $R$ |
| 4 | | | | | | $R$ | | | $R$ | | | $R$ | $R$ | $R$ | | | | |
| 5 | | $R$ | $R$ | | | $R$ | | | | | | | | | | | | $R$ |
| 6 | $R$ | | | $R$ | | $R$ | $R$ | | | | | | | | | | | |
| 7 | | | | | | | $R$ | $R$ | | $R$ | | | | $R$ | | | | $R$ |
| 8 | | | $R$ | | | | $R$ | | $R$ | | $R$ | | | | | | $R$ | |
| 9 | | $R$ | | | $R$ | | $R$ | | | | | $R$ | | | $R$ | | | |
| 10 | | | | $R$ | | | | | | | $R$ | $R$ | | | | | | $R$ |
| 11 | $R$ | | $R$ | | $R$ | | | | | | | | | $R$ | | | | |
| 12 | | | $R$ | $R$ | | | | $R$ | | | | | $R$ | | $R$ | | | |
| 13 | | | | | $R$ | $R$ | | $R$ | | | $R$ | | | | | $R$ | | |
| 14 | $R$ | | | | | | | $R$ | | | | $R$ | | | | | $R$ | |
| 15 | | | | $R$ | $R$ | | | | $R$ | $R$ | | | | | | | | |
| 16 | | | | | | $R$ | | | | $R$ | | | | | $R$ | | $R$ | |
| 17 | | | $R$ | | | | | | | $R$ | | $R$ | | | | $R$ | | |
| 18 | | | | | $R$ | | | | | | | | $R$ | | | | $R$ | $R$ |

Table 12: A Rectangle Free Subset of $G_{18,18}$ of size 81.

**Theorem 8.10** *Assume RFC is true. Then*

$$\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \bigcup$$

$$\{G_{17,19}, G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}$$

**Proof:**

In this proof we only consider $G_{n,m}$ where $n \geq m$. The grids

$$G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}$$

are in the obstruction set by Theorem 8.6.

$G_{21,13}$ is not 4-colorable by Theorem 8.1. $G_{20,13}$ is 4-colorable by Theorem 8.1 (since $G_{20,16}$ is 4-colorable). Assuming RFC $G_{21,12}$ is 4-colorable by Lemma 8.7. Hence, assuming RFC, $G_{21,13}$ is a minimal grid that is not 4-colorable, hence it is in $OBS_4$.

$G_{19,17}$ is not 4-colorable by Lemma 8.2. Assuming RFC $G_{18,17}$ is 4-colorable by Lemma 8.8 (since $G_{18,18}$ is 4-colorable). $G_{19,16}$ is 4-colorable by Theorem8.1 (since $G_{20,16}$ is 4-colorable). Hence, assuming RFC, $G_{19,17}$ is a minimal grid that is not 4-colorable, hence it is in $OBS_4$.

The proof that these are all of the elements in $OBS_4$ is similar to the proof of Theorem 6.2 and hence we omit it. ∎

# 9 Application to Bipartite Ramsey Numbers

We state the Bipartite Ramsey Theorem. See [6] for history, details, and proof.

**Def 9.1** A complete bipartite graph, $G = (V1 + V2, E)$, is a bipartite graph such that for any two vertices, $v_1 \in V_1$ and $v_2 \in V_2$, $(v_1, v_2)$ is an edge in $G$. The complete bipartite graph with partitions of size $|V_1| = a$ and $|V_2| = b$, is denoted $K_{a,b}$.

**Theorem 9.2** *For all $a, c$ there exists $n = BR(a, c)$ such that for all c-colorings of the edges of $K_{n,n}$ there will be a monochromatic $K_{a,a}$.*

The following theorem is easily seen to be equivalent to this.

**Theorem 9.3** *For all $a, c$ there exists $n = BR(a, c)$ so that for all c-colorings of $G_{n,n}$ there will be a monochromatic $a \times a$ submatrix.*

In this paper we are $c$-coloring $G_{n,m}$ and looking for a $2 \times 2$ monochromatic submatrix. We have the following theorems which, except where noted, seem to be new.

**Theorem 9.4**

1. $BR(2,2) = 5$. *(This was also shown in [11].)*

2. $BR(2,3) = 11$.

3. $17 \leq BR(2,4) \leq 19$.

4. $BR(2,c) \leq c^2 + c$.

5. *If $p$ is a prime and $s \in \mathsf{N}$, then $BR(2,p^s) > p^{2s}$.*

6. *For almost all $c$, $BR(2,c) \geq c^2 - 2c^{1.525} + c^{1.05}$.*

**Proof:**
1) By Lemma 6.1, $G_{5,5}$ is not 2-colorable and $G_{4,4}$ is 2-colorable.
2) By Lemma 7.4, $G_{11,11}$ is not 3-colorable. By Lemma 7.2 $G_{10,10}$ is 3-colorable.
3) By Lemma 8.2, $G_{19,19}$ is not 4-colorable. By Theorem 8.1 $G_{16,16}$ is 4-colorable.
4) By Theorem 5.1, $G_{c^2+c,c^2+c}$ is not $c$-colorable.
5) By Theorem 4.12, $G_{cr,cm}$ is $c$-colorable where $c = p^s$, $r = p^s$, and $m = \frac{p^{2s}-1}{p^s-1}$. Note that $m \geq p^s$. Hence $G_{p^{2s},p^{2s}}$ is $p^s$-colorable.
6) Baker, Harman, and Pintz [2] (see [9] for a survey) showed that for almost all $c$, there is a prime between $c$ and $c - c^{0.525}$. Let $p$ be that prime. By part 5 with $s = 1$, $BR(2,p) \geq p^2$. Hence

$$BR(2,c) \geq BR(2,p) \geq p^2 \geq (c - c^{0.525})^2 = c^2 - 2c^{1.525} + c^{1.05}.$$

∎

# 10 Open Questions

1. Find $\mathrm{OBS}_4$. We feel this is possible since we are so close. A clever computer program may be needed. The second author has offered a prize of \$289.00 for a proper 4-coloring of $G_{17,17}$.

2. Refine our tools so that our ugly proofs can be corollaries of our tools.

3. Find an algorithm that will, given $c$, find $\mathrm{OBS}_c$ or $|\mathrm{OBS}_c|$ quickly.

4. We know that $2\sqrt{c}(1 - o(1)) \leq |\mathrm{OBS}_c| \leq 2c^2$. Bring these bounds closer together.

5. Is the Rectangle-Free Conjecture True? If so then this may help us find $c$-colorings. If not then this may open up new techniques for proving that a grid is not $c$-colorable.

# 11 Acknowledgments

We would like to thank Brad Lorsen for providing us with 4-colorings of $G_{21,11}$ and $G_{22,10}$.

We would like to thank Michelle Burke, Brett Jefferson, and Krystal Knight who worked with the second and third authors over the Summer of 2006 on this problem. As noted earlier, Brett Jefferson has his own paper on this subject [10].

We would also like to thank László Székely for pointing out the connection to bipartite Ramsey numbers, Larry Washington for providing information on number theory that was used in the proof of Theorem 9.4, and Russell Moriarty for proofreading and intelligent commentary.

# 12 Appendix: Exact Values of $\mathrm{maxrf}(n,m)$ for $0 \le n \le 6$, $m \le n$

**Lemma 12.1**

*0) For* $m \ge 0$, $\mathrm{maxrf}(0,m) = 0$.

*1) For* $m \ge 1$, $\mathrm{maxrf}(1,m) = m$.

*2) For* $m \ge 2$, $\mathrm{maxrf}(2,m) = m+1$.

*3) For* $m \ge 3$, $\mathrm{maxrf}(3,m) = m+3$.

*4)*

$$\mathrm{maxrf}(4,m) = \begin{cases} m+5 & \textit{if } 4 \le m \le 5 \\ m+6 & \textit{if } m \ge 6 \end{cases}$$

*5)*

$$\mathrm{maxrf}(5,m) = \begin{cases} 12 & \textit{if } m = 5 \\ m+8 & \textit{if } 6 \le m \le 7 \\ m+9 & \textit{if } 8 \le m \le 9 \\ m+10 & \textit{if } m \ge 10 \end{cases}$$

*6)*

$$\mathrm{maxrf}(6,m) = \begin{cases} 2m+4 & \textit{if } 6 \le m \le 7 \\ 19 & \textit{if } m = 8 \\ m+12 & \textit{if } 9 \le m \le 10 \\ m+13 & \textit{if } 11 \le m \le 12 \\ m+14 & \textit{if } 13 \le m \le 14 \\ m+15 & \textit{if } m \ge 15 \end{cases}$$

**Proof:**

Lemma 2.7 will provide all of the upper bounds. The lower bounds are obtained by actually exhibiting rectangle-free sets of the appropriate size. We do this for the case of maxrf$(6, m)$. Our technique applies to all of the other cases.

**Case 1:** maxrf$(6, m)$ **where** $6 \le m \le 7$ **and** $m = 8$**:** Fill the first four columns with 3 elements (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 12 are blocked. Hence we can fill the next $15 - 12 = 3$ columns with two elements each, and the remaining column (if $m = 8$) with 1 element. The picture below shows the result for maxrf$(6, 8) = 19$; however, if you just look at the first 6 (7) columns you get the result for maxrf$(6, 6)$ (maxrf$(6, 7)$).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $R$ | | $R$ | | $R$ | | | $R$ |
| $R$ | | | $R$ | | $R$ | | |
| $R$ | $R$ | | | | | $R$ | |
| | $R$ | $R$ | | | $R$ | | |
| | $R$ | | $R$ | $R$ | | | |
| | | $R$ | $R$ | | | $R$ | |

**Case 2:** maxrf$(6, m)$ **where** $9 \le m \le 10$**:** Fill the first three columns with 3 elements each (all pairs overlapping). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 9 are blocked. Hence we can fill the next $15 - 9 = 6$ columns with two elements each and the remaining column (if $m = 10$) with 1 element. The picture below shows the result for maxrf$(6, 10) = 22$; however, if you just look at the first 9 columns you get the result maxrf$(6, 9) = 21$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $R$ | | $R$ | | $R$ | | | | | |
| $R$ | | | | | $R$ | | $R$ | $R$ | |
| $R$ | $R$ | | | | | $R$ | | | |
| | $R$ | $R$ | | | $R$ | | | | |
| | $R$ | | $R$ | $R$ | | | $R$ | | |
| | | $R$ | $R$ | | | $R$ | | $R$ | $R$ |

**Case 3:** maxrf$(6, m)$ **where** $11 \le m \le 12$**:** Fill the first two columns with 3 elements each (they overlap). Each column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs, hence 6 are blocked. Hence we can fill the next $15 - 6 = 9$ columns with two elements each and the remaining column (if $m = 12$) with 1 element. The picture below shows the result for maxrf$(6, 12) = 25$; however, if you just look at the first 11 columns you get the result maxrf$(6, 11) = 24$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $R$ | | $R$ | | $R$ | | | | | | $R$ | |
| $R$ | | | | | $R$ | | $R$ | $R$ | | | |
| $R$ | $R$ | | | | | $R$ | | | | | |
| | $R$ | $R$ | | | $R$ | | | | $R$ | | |
| | $R$ | | $R$ | $R$ | | | $R$ | | | | |
| | | | $R$ | | | $R$ | | $R$ | $R$ | $R$ | $R$ |

**Case 4: maxrf$(6, m)$ where $13 \le m \le 14$:** Fill the first column with 3 elements. This column of 3 blocks exactly $\binom{3}{2} = 3$ of the possible $\binom{6}{3} = 15$ ordered pairs. Hence we can fill the next $15 - 3 = 12$ columns with two elements each and the remaining column (if $m = 14$) with 1 element. We omit the picture.

**Case 5: maxrf$(6, m)$ where $m \ge 15$:** Fill the first $\binom{6}{2} = 15$ columns with two elements each in a way so that each column has a distinct pair. Fill the remaining $m - 15$ columns with one element each. The result is a rectangle-free set of size $30 + m - 15 = m + 15$. ∎

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