Three Proofs of the Hypergraph Ramsey Theorem (An Exposition)

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Abstract

Ramsey, Erdős-Rado, and Conlon-Fox-Sudakov have given proofs of the 3-hypergraph Ramsey Theorem with better and better upper bounds on the 3-hypergraph Ramsey numbers. Ramsey and Erdős-Rado also prove the a-hypergraph Ramsey Theorem. Conlon-Fox-Sudakov note that their upper bounds on the 3-hypergraph Ramsey Numbers, together with a recurrence of Erdős-Rado (which was the key to the Erdős-Rado proof), yield improved bounds on the a-hypergraph Ramsey numbers. We present all of these proofs and state explicit bounds for the 2-color case and the c-color case. We give a more detailed analysis of the construction of Conlon-Fox-Sudakov and hence obtain a slightly better bound.

1 Introduction

The 3-hypergraph Ramsey numbers $R(3, k)$ were first shown to exist by Ramsey [8]. His upper bounds on them were enormous. Erdős-Rado [3] obtained much better bounds, namely $R(3, k) \leq$

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$2^{2^{4k}}$. Recently Conlon-Fox-Sudakov [2] have obtained $R(3,k) \le 2^{2^{(2+o(1))k}}$. We present all three proofs. For the Conlon-Fox-Sudakov proof we give a more detailed analysis that required a non-trivial lemma, and hence we obtain slightly better bounds. Before starting the second and third proofs we will discuss why they improve the prior ones.

We also present extensions of all three proofs to the a-hypergraph case. The first two are known proofs and bounds. The Erdős-Rado proof gives a recurrence to obtain a-hypergraph Ramsey Numbers from $(a-1)$-hypergraph Ramsey Numbers. As Conlon-Fox-Sudakov note, this recurrence together with their improved bound on $R(3,k)$, yield better upper bounds on the a-hypergraph Ramsey Numbers. Can the Conlon-Fox-Sudakov method itself be extended to a proof of the a-hypergraph Ramsey Theorem? It can; however (alas), this does not seem to lead to better upper bounds. We include this proof in the appendix in the hope that someone may improve either the construction or the analysis to obtain better bounds on the a-hypergraph Ramsey Numbers.

For all of the proofs, the extension to c colors is routine. We present the results as notes; however, we leave the proofs as easy exercises for the reader.

2 Notation and Ramsey's Theorem

Def 2.1 Let X be a set and $a \in \mathsf{N}$. Then $\binom{X}{a}$ is the set of all subsets of X of size a.

Def 2.2 Let $a, n \in \mathsf{N}$. The *complete a-hypergraph on n vertices*, denoted K_n^a, is the hypergraph with vertex set $V = [n]$ and edge set $E = \binom{[n]}{a}$

Notation 2.3 In this paper a *coloring of a graph or hypergraph* always means a coloring of the *edges*. We will abbreviate $COL(\{x_1, \ldots, x_a\})$ by $COL(x_1, \ldots, x_a)$. We will refer to a c-coloring of the edges of the complete hypergraph K_n^a as a c-coloring of $\binom{[n]}{a}$.

Def 2.4 Let $a \geq 1$. Let COL be a c-coloring of $\binom{[n]}{a}$. A set of vertices H is *a-homogeneous for COL* if every edge in $\binom{H}{a}$ is the same color. We will drop the *for COL* when it is understood. We will drop the a when it is understood.

Convention 2.5 When talking about 2-colorings will often denote the colors by RED and BLUE.

Note 2.6 In Definition 2.4 we allow $a = 1$. Note that a c-coloring of $\binom{[n]}{1}$ is just a coloring of the numbers in $[n]$. A homogenous subset H is a subset of points that are all colored the same. Note that in this case the edges are 1-subsets of the points and hence are identified with the points.

Def 2.7 Let $a, c, k \in \mathsf{N}$. Let $R(a, k, c)$ be the least n such that, for all c-colorings of $\binom{[n]}{a}$ there exists an a-homogeneous set $H \in \binom{[n]}{k}$. We denote $R(a, k, 2)$ by $R(a, k)$. We have not shown that $R(a, k, c)$ exists; however, we will.

We state Ramsey's theorem for 1-hypergraphs (which is trivial) and for 2-hypergraphs (just graphs). The 2-hypergraph case (and the a-hypergraph case) is due to Ramsey [8] (see also [4, 6, 7]). The bound we give on $R(2, k)$ seems to be folklore (see [6]).

Def 2.8 The expression $\omega(1)$ means a function that goes to infinity monotonically. For example, $\lfloor \lg \lg n \rfloor$.

The following are well known.

Theorem 2.9 *Let $k \in \mathsf{N}$ and $c \geq 2$.*

1. $R(1, k) = 2k - 1$.
2. $R(1, k, c) = ck - c + 1$.
3. $R(2, k) \leq \binom{2k-2}{k-1} \leq 2^{2k - 0.5 \lg(k-1) - \Omega(1)}$.

4. $R(2,k,c) \le \frac{(c(k-1))!}{(k-1)!)^c} \le c^{ck-0.5\log_c(k-1)+O(c)}$.

5. *For all n, for every 2-coloring of $\binom{[n]}{2}$, there exists a 2-homogenous set H of size at least $\frac{1}{2}\lg n + \omega(1)$. (This follows from Part 3 easily. In fact, all you need is $R(2,k) \le 2^{2k-\Omega(1)}$.)*

Note 2.10 Theorem 2.9.2 has an elementary proof. A more sophisticated proof, by David Conlon [1] yields $R(2,k) \le k^{-E\frac{\log k}{\log\log k}}\binom{2k}{k}$, where E is some constant. A simple probabilistic argument shows that $R(2,k) \ge (1+o(1))\frac{1}{e\sqrt{2}}k2^{k/2}$. A more sophisticated argument by Spencer [9] (see [6]), that uses the Lovasz Local Lemma, shows $R(2,k) \ge (1+o(1))\frac{\sqrt{2}}{e}k2^{k/2}$.

We state Ramsey's theorem on a-hypergraphs [8] (see also [6, 7]).

Theorem 2.11 *Let $a,k,c \in \mathsf{N}$. For all $k \in \mathsf{N}$, $R(a,k,c)$ exists.*

3 Summary of Results

We will need both the tower function and Knuth's arrow notation to state the results.

Notation 3.1

$$c \uparrow^a k = \begin{cases} ck & \text{if } a = 0, \\ c^k, & \text{if } a = 1, \\ 1, & \text{if } k = 0, \\ c \uparrow^{a-1} (c \uparrow^a (k-1)), & \text{otherwise.} \end{cases}$$

Def 3.2 We define TOW which takes on a variable number of arguments.

1. $\mathrm{TOW}_c(b) = c^b$.

2. $\mathrm{TOW}_c(b_1,\ldots,b_L) = c^{b_1 \mathrm{TOW}_c(b_2,\ldots,b_L)}$.

When c is not stated it is assumed to be 2.

Example 3.3

1. $\mathrm{TOW}(2k) = 2^{2k}$.

2. $\mathrm{TOW}(1, 4k) = 2^{2^{4k}}$.

3. $\mathrm{TOW}(1) = 2$, $\mathrm{TOW}(1, 1) = 2^2$, $\mathrm{TOW}(1, 1, 1) = 2^{2^2}$.

The list below contains both who proved what bounds and the results we will prove in this paper.

1. Ramsey's proof [8] yields:

   (a) $R(3, k) \le 2 \uparrow^2 (2k - 1) = \mathrm{TOW}(1, \ldots, 1)$ where the number of 1's is $2k - 1$.

   (b) $R(a, k) \le 2 \uparrow^{a-1} (2k - 1)$.

2. The Erdős-Rado [3] proof yields:

   (a) $R(3, k) \le 2^{2^{4k-\lg(k-2)}}$.

   (b) $R(a, k) \le 2^{\binom{R(a-1,k-1)+1}{a-1}} + a - 2$.

   (c) Using the recurrence they obtain the following: For all $a \ge 4$, $R(a, k) \le \mathrm{TOW}(1, a - 1, a - 2, \ldots, 3, 4k - \lg(k - a + 1) - 4(a - 3)))$.

3. The Conlon-Fox-Sudakov [2] proof yields:

   (a) $R(3, k) \le 2^{B(k-1)^{1/2} 2^{2k}}$ where $B = \left(\frac{e}{\sqrt{2\pi}}\right)^3 \sim 1.28$.

   (b) If you combine this with the recurrence obtained by Erdős-Rado then one obtains:

      i. $R(3, k) \le \mathrm{TOW}(B(k - 1)^{1/2}, 2^{2k})$.

      ii. $R(4, k) \le \mathrm{TOW}(1, 3B(k - 2)^{1/2}, 2^{2k-2})$.

      iii. $R(5, k) \le \mathrm{TOW}(1, 4, 3B(k - 3)^{1/2}, 2^{2k-4})$.

iv. For all $a \geq 6$, for almost all k,

$$R(a,k) \leq \mathrm{TOW}(1, a-1, a-2, \ldots, 4, 3B(k-a+2)^{1/2}, 2^{2k-2a+6}).$$

4. The Appendix contains an alternative proof of the a-hypergraph Ramsey Theorem based on the ideas of Conlon-Fox-Sudakov. Since it does not yield better bounds we do not state the bounds here.

Notation 3.4 PHP stands for Pigeon Hole Principle.

We will need the following lemma whose easy proof we leave to the reader.

Lemma 3.5 *For all $b, b_1, \ldots, b_L \in \mathsf{N}$ the following hold.*

1. $\mathrm{TOW}(b_1, \ldots, b_i, b_{i+1}, b_{i+2} \ldots, b_L) \leq \mathrm{TOW}(b_1, \ldots, 1, b_{i+1} + \lg(b_i), b_{i+2}, \ldots, b_L)$.
2. $\mathrm{TOW}(b_1, \ldots, b_L)^b = \mathrm{TOW}(bb_1, b_2, \ldots, b_L)$.
3. $(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L) \leq \mathrm{TOW}(b_1, b_2, \ldots, b_L + \delta)$.
4. $(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L)^b \leq \mathrm{TOW}(bb_1, b_2, \ldots, b_L + \delta)$. *(This follows from 1 and 2.)*
5. $2^{\mathrm{TOW}(b_1, \ldots, b_L)} = \mathrm{TOW}(1, b_1, \ldots, b_L)$.
6. $2^{(1+\delta)\mathrm{TOW}(b_1, \ldots, b_L)^b} \leq \mathrm{TOW}(1, bb_1, b_2, \ldots, b_L + \delta)$. *(This follows from 4 and 5.)*
7. $\lg^{(c)}(\mathrm{TOW}(1, \ldots, 1)) = 1$ *(there are c 1's).*

4 Ramsey's Proof

Theorem 4.1 *For almost k $R(3,k) \leq 2 \uparrow^2 (2k-1) = \mathrm{TOW}(1, \ldots, 1)$ where there are $2k-1$ 1's.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{2k-1}.$$

Here is the basic idea: Let $x_1 = 1$. This induces the following coloring of $\binom{[n]-\{1\}}{2}$:

$$COL^*(x, y) = COL(x_1, x, y).$$

By Theorem 2.9 there exists a 2-homogeneous set for COL^* of size $\frac{1}{2}\lg n + \omega(1)$. Keep that 2-homogeneous set and ignore the remaining points. Let x_2 be the least vertex that has been kept (bigger than x_1). Repeat the process.

We describe the construction formally.

CONSTRUCTION

$V_0 = [n]$

Assume $1 \le i \le 2k-1$ and that V_{i-1}, $x_1, x_2, \ldots, x_{i-1}$, $c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$\begin{aligned}
x_i = &\quad \text{the least number in } V_{i-1} \\
V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)} \\
COL^*(x, y) = &\quad COL(x_i, x, y) \text{ for all } \{x, y\} \in \tbinom{V_i}{2} \\
V_i = &\quad \text{the largest 2-homogeneous set for } COL^* \\
c_i = &\quad \text{the color of } V_i
\end{aligned}$$

KEY: for all $y, z \in V_i$, $COL(x_i, y, z) = c_i$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction can be carried out for $2k-1$ stages. For now assume the construction ends.

We have vertices

$$x_1, x_2, \ldots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \ldots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \ldots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

We take this color to be RED. We show that

$$H = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

is 3-homogenous for COL. For notational convenience we show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) =$ RED. The proof for any 3-set of H is similar. By the definition of c_{i_1} $(\forall A \in \binom{V_{i_1}-\{x_{i_1}\}}{2}))[COL(A \cup \{x_{i_1})\}) = c_i]$ In particular

$$COL(x_{i_1}, x_{i_2}, x_{i_3}) = c_{i_1} = \text{RED}.$$

We now see how large n must be so that the construction can be carried out. By Theorem 2.9, if k is large, at every iteration V_i gets reduced by a logarithm, cut in half, and then an $\omega(1)$ is added. Using this it is easy to show that, for almost all k,

$$|V_j| \geq \frac{1}{2}(\lg^{(j)} n) + \omega(1).$$

We want to run this iteration $2k-1$ times Hence we need

$$|V_{2k-1}| \geq \frac{1}{2}\log_2^{(2k-1)} n + \omega(1) \geq 1.$$

We can take $n = \mathrm{TOW}(1,\ldots,1)$ where 1 appears $2k-1$ times, and use Lemma 3.5. ∎

Note 4.2 The proof of Theorem 4.1 generalizes to c-colors to yield

$$R(3,k,c) \leq c \uparrow^2 (ck-c+1) = \mathrm{TOW}_c(1,\ldots,1)$$

where the number of 1's is $ck-c+1$.

We now prove Ramsey's Theorem for a-hypergraphs.

Theorem 4.3 *For all $a \geq 1$, for all $k \geq 1$, $R(a,k) \leq 2 \uparrow^{a-1} (2k-1)$.*

Proof:

We prove this by induction on a. Note that when we have the theorem for a we have it for a and for *all* $k \geq 1$.

Base Case: If $a = 1$ then, for all $k \geq 1$, $R(1,k) = 2k-1 \leq 2 \uparrow^0 (2k-1) = 4k-2$.

Induction Step: We assume that, for all k, $R(a-1,k) \leq 2 \uparrow^{a-2} (2k-1)$.

Let $k \geq 1$. Let n be a number to be determined later. Let COL be a 2-coloring of $\binom{[n]}{a}$. We show that there is an a-homogenous set for COL of size k.

CONSTRUCTION

$V_0 =]n]$.

Assume $1 \leq i \leq 2k-1$ and that $V_{i-1}, x_1, x_2, \ldots, x_{i-1}, c_1, \ldots, c_{i-1}$ are all defined. We define x_i, COL^*, V_i, and c_i:

$$
\begin{aligned}
x_i = &\quad \text{the least number in } V_{i-1} \\
V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)} \\
COL^*(A) = &\quad COL(x_i \cup A) \text{ for all } A \in \binom{V_i}{a-1}) \\
V_i = &\quad \text{the largest } a-1\text{-homogeneous set for } COL^* \\
c_i = &\quad \text{the color of } V_i
\end{aligned}
$$

KEY: For all $1 \le i \le 2k-1$, $(\forall A \in \binom{V_i}{a-1})[COL(A \cup x_i) = c_i]$

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction can be carried out for $2k-1$ stages. For now assume the construction ends.

We have vertices

$$x_1, x_2, \ldots, x_{2k-1}$$

and associated colors

$$c_1, c_2, \ldots, c_{2k-1}.$$

There are only two colors, hence, by PHP, there exists $i_1, \ldots, i_k$ such that $i_1 < \cdots < i_k$ and

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

We take this color to be RED. We show that

$$H = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}.$$

is a-homogenous for COL. For notational convenience we show that $COL(x_{i_1}, \ldots, x_{i_a}) = \text{RED}$. The proof for any a-set of H is similar. By the definition of c_{i_1} $(\forall A \in \binom{V_{i_1}}{a-1})[COL(A \cup x_{i_1}) = c_i]$ In particular

$$COL(x_{i_1}, \ldots, x_{i_a}) = c_{i_1} = \text{RED}.$$

We show that if $n = 2 \uparrow^{a-1} (2k-1)$ then the construction can be carried out for $2k-1$ stages.

Claim 1: For all $0 \le i \le 2k-1$, $|V_i| \ge 2 \uparrow^{a-1} (2k-(i+1))$.

Proof of Claim 1: We prove this claim by induction on i. For the base case note that

$$|V_0| = n = 2 \uparrow^{a-1} (2k-1).$$

Assume $|V_{i-1}| \ge 2 \uparrow^{a-1} (2k-i)$. By the definition of the uparrow function and by the inductive hypothesis of the theorem,

$$|V_{i-1}| \ge 2 \uparrow^{a-1} (2k-i) = 2 \uparrow^{a-2} (2 \uparrow^{a-1} (2k-(i+1))) \ge R(a-1, 2 \uparrow^{a-1} (2k-(i+1))).$$

By the construction V_i is the result of applying the $(a-1)$-ary Ramsey Theorem to a 2-coloring of $\binom{V_{i-1}}{a}$. Hence $|V_i| \ge 2 \uparrow^{a-1} (2k-(i+1))$.

End of Proof of Claim 1

By Claim 1 if $n = 2 \uparrow^{a-1} (2k-1)$ then the construction can be carried out for $2k-1$ stages. Hence $R(a,k) \le 2 \uparrow^{a-1} (2k-1)$. ∎

The proof of Theorem 4.1 is actually an ω^2-induction that is similar in structure to the original proof of van der Warden's theorem [5, 6, 10].

Note 4.4 The proof of Theorem 4.3 generalizes to c colors yielding

$$R(a,k,c) \le c \uparrow^{a-1} (ck-c+1).$$

5 The Erdős-Rado Proof

Why does Ramsey's proof yield such large upper bounds? Recall that in Ramsey's proof we do the following:

- Color a *node* by using Ramsey's theorem (on graphs). This cuts the number of nodes down by a log (from m to $\Theta(\log m)$). This is done $2k-1$ times.
- After the nodes are colored we use PHP once. This will cut the number of nodes in half.

The key to the large bounds is the number of times we use Ramsey's theorem. The key insight of the proof by Erdős and Rado [3] is that they use PHP many times but Ramsey's theorem only once. In summary they do the following:

- Color an *edge* by using PHP. This cuts the number of nodes in half. This is done $R(2, k-1)+1$ times.
- After *all* the edges of a complete graph are colored we use Ramsey's theorem. This will cut the number of nodes down by a log.

We now proceed formally.

Theorem 5.1 *For almost all k, $R(3,k) \le 2^{2^{4k-\lg(k-2)}}$.*

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$. We define a sequence of vertices,

$$x_1, x_2, \ldots, x_{R(2,k-1)+1}.$$

Recall the definition of a 1-homogeneous set for a coloring of singletons from the note following Definition 2.4. We will use it here.

Here is the intuition: Let $x_1 = 1$. Let $x_2 = 2$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a 1-homogeneous for COL^* of size at least $\frac{n-2}{2}$. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let x_3 be the least vertex left (bigger than x_2).

The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$(\forall y \in V_1 - \{x_3\})[COL_1^*(y) = COL(x_1, x_3, y)]$$

$$(\forall y \in V_1 - \{x_3\})[COL_2^*(y) = COL(x_2, x_3, y)]$$

Let V_2 be a 1-homogeneous for COL_1^* of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Restrict COL_2^* to elements of V_2, though still call it COL_2^*. We reuse the variable name V_2 to be a 1-homogeneous for COL_2^* of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. Let x_4 be the least element of V_2. Repeat the process.

We describe the construction formally.

CONSTRUCTION

$$\begin{aligned} x_1 = {} & 1 \\ V_1 = {} & [n] - \{x_1\} \end{aligned}$$

Let $2 \le i \le R(2, k-1) + 1$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1, \ldots, x_{i-1}\}}{2} \to \{\text{RED}, \text{BLUE}\}$ are defined.

$x_i =$ the least element of V_{i-1}

$V_i =$ $V_{i-1} - \{x_i\}$ (We will change this set without changing its name).

We define $COL^{**}(x_1, x_i)$, $COL^{**}(x_2, x_i)$, …, $COL^{**}(x_{i-1}, x_i)$. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 1$ to $i - 1$

1. $COL^* : V_i \rightarrow \{\text{RED}, \text{BLUE}\}$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.
2. Let V_i be redefined as the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.
3. $COL^{**}(x_j, x_i)$ is the color of V_i.

KEY: For all $1 \leq i_1 < i_2 \leq i$, for all $y \in V_i$, $COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2})$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the the construction can be carried out for $R(2, k-1) + 1$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \ldots, x_{R(2,k-1)+1}\}$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the definition of $R(2, k-1) + 1$ there exists a set

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$

such that the first $k - 1$ elements of it are a 2-homogenous set for COL^{**}. Let the color of this 2-homogenous set be RED. We show that H (including x_{i_k}) is a 3-homogenous set for COL. For notational convenience we show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$. The proof for any 3-set of H is similar.

By the definition of COL^{**} for all $y \in V_{i_2}$, $COL(x_{i_1}, x_{i_2}, y) = COL^{**}(x_{i_1}, x_{i_2}) = \text{RED}$. In particular $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$.

We now see how large n must be so that the construction be carried out. Note that in stage i $|V_i|$ be decreases by at most half, i times. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^i}$.

Therefore

$$|V_i| \geq \frac{|V_1|}{2^{1+2+\cdots+(i-1)}} \geq \frac{n-1}{2^{(i-1)^2}}.$$

We want $|V_{R(2,k-1)+1}| \geq 1$. It suffice so take $n = 2^{R(2,k-1)^2} + 1$.

By Theorem 2.9

$$R(2,k-1)^2 + 1 \leq (2^{2k-0.5\lg(k-2)})^2 \leq 2^{4k-\lg(k-2)}.$$

Hence

$$R(3,k) \leq 2^{2^{4k-\lg(k-2)}}.$$

∎

Note 5.2 A slightly better upper bound for $R(3,k)$ can be obtained by using Conlon's upper bound on $R(2,k)$ given in Note 2.10.

Note 5.3 The proof of Theorem 5.1 generalizes to c-colors yielding

$$R(3,k,c) \leq c^{c^{2ck-\log_c(k-2)+O(c)}}.$$

We state Ramsey's theorem on a-hypergraphs [8] (see also [6, 7]).

Theorem 5.4

1. *For all $a \geq 2$, for all k,* $R(a,k) \leq 2^{\binom{R(a-1,k-1)+1}{a-1}} + a - 2.$

2. $R(3,k) \leq \mathrm{TOW}(1, 4k - \lg(k-2)).$

3. *For all* $a \geq 4$*, for almost all* k*,*

$$R(a,k) \leq \mathrm{TOW}(1, a-1, a-2, \ldots, 3, 4k - \lg(k-a+1) - 4(a-3)).$$

Proof:

1) Assume that $R(a-1, k-1)$ exists and $a \geq 2$.

CONSTRUCTION

$$\begin{aligned} x_1 = &\quad 1 \\ \vdots = &\quad \vdots \\ x_{a-2} = &\quad a-2 \\ V_{a-2} = &\quad [n] - \{x_1, \ldots, x_{a-2}\}. \text{ We start indexing here for convenience.} \end{aligned}$$

Let $a-1 \leq i \leq R(a-1, k-1)+1$. Assume that $x_1, \ldots, x_{i-1}, V_{i-1}$, and $COL^{**} : \binom{\{x_1, \ldots, x_{i-1}\}}{a-1} \to \{\mathrm{RED}, \mathrm{BLUE}\}$ are defined.

$$\begin{aligned} x_i = &\quad \text{the least element of } V_{i-1} \\ V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name).} \end{aligned}$$

We define $COL^{**}(A \cup \{x_i\})$ for every $A \in \binom{\{x_1, \ldots, x_{i-1}\}}{a-1}$. We will also define smaller and smaller sets V_i.

For $A \in \binom{\{x_1, \ldots, x_{i-1}\}}{a-1}$

1. $COL^* : V_i \to \{\mathrm{RED}, \mathrm{BLUE}\}$ is defined by $COL^*(y) = COL(A \cup \{y\})$.

2. Let V_i be redefined as the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.

3. $COL^{**}(A \cup \{x_i\})$ is the color of V_i.

KEY: For all $l \leq i_1 < \cdots < i_a \leq i$, $COL(x_{i_1}, \ldots, x_{i_a}) = COL^{**}(x_{i_1}, \ldots, x_{i_{a-1}})$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the the construction can be carried out for $R(a-1,k-1)+1$ stages. For now assume the construction ends.

We have vertices

$$X = \{x_1, x_2, \ldots, x_{R(a-1,k-1)+1}\}$$

and a 2-coloring COL^{**} of $\binom{X}{2}$. By the definition of $R(a-1,k-1)+1$ there exists a set

$$H = \{x_{i_1}, \ldots, x_{i_k}\}.$$

such that the first $k-1$ elements of it are a $(a-1)$-homogenous set for COL^{**}. Let the color of this $(a-1)$-homogenous set be RED. We show that H (including x_{i_k}) is a a-homogenous set for COL. For notational convenience we show that $COL(x_{i_1}, \ldots, x_{i_a}) =$ RED. The proof for any a-set of H is similar, including the case where the last vertex is x_{i_k}.

By the definition of COL^{**} for all $y \in V_{i_2}$, $COL(x_{i_1}, \ldots, x_{i_{a-1}}, y) = COL^{**}(x_{i_1}, \ldots, x_{i_{a-1}}) =$ RED. In particular $COL(x_{i_1}, \ldots, x_{i_a}) =$ RED.

We now see how large n must be so that the construction can be carried out. Note that during stage i there will be $\binom{i}{a-2}$ times where $|V_i|$ decreases by at most half. Hence $|V_{i+1}| \geq \frac{|V_i|}{2^{\binom{i}{a-2}}}$.

Therefore

$$|V_i| \geq \frac{|V_{a-2}|}{2^{\binom{a-2}{a-2}+\binom{a-1}{a-2}+\binom{a}{a-2}+\cdots+\binom{i-1}{a-2}}} = \frac{n-a+2}{2^{\binom{i}{a-1}}}.$$

We want $|V_{R(a-1,k-1)+1}| \geq 1$.

Hence we need

$$|V_{R(a-1,k-1)+1}| \geq \frac{n-a+2}{2^{\binom{R(a-1,k-1)+1}{a-1}}} \geq 1$$

$$n - a + 2 \geq 2^{\binom{R(a-1,k-1)+1}{a-1}}$$

Hence

$$n \geq 2^{\binom{R(a-1,k-1)+1}{a-1}} + a - 2.$$

Therefore

$$R(a,k) \leq 2^{\binom{R(a-1,k-1)+1}{a-1}} + a - 2.$$

2) This is a restatement of Theorem 5.1.

3) We use Lemma 3.5 throughout this proof implicitly. We will also use a weak form of the recurrence from Part 1, namely:

$$R(a,k) \leq 2^{R(a-1,k-1)^{a-1}}.$$

We prove the bound on $R(a,k)$ for $a \geq 4$ by induction on a.

Base Case: $a = 4$**:** By Part 2, $R(3,k) \leq \mathrm{TOW}(1, 4k - \lg(k-2))$. Hence

$$R(4,k) \leq 2^{R(3,k-1)^2} \leq \mathrm{TOW}(1,3,4k - \log(k-3) - 4) = \mathrm{TOW}(1,3,4k - \log(k-3) - 4 \times (4-3)).$$

Induction Step: We assume

$$\begin{aligned} R(a-1,k-1) \leq & \quad \mathrm{TOW}(1, a-2, \ldots, 3, 4(k-1) - \lg((k-1) - (a-1) + 1) - 4(a-4)) \\ = & \quad \mathrm{TOW}(1, a-2, \ldots, 3, 4k - 4 - \lg(k-a+1) - 4(a-3)). \end{aligned}$$

Hence

$$R(a,k) \le 2^{R(a-1,k-1)^{a-1}} \le \mathrm{TOW}(1, a-1, a-2, \ldots, 3, 4k - \lg(k-a+1) - 4(a-3)).$$

∎

Corollary 5.5 *For all $a \ge 3$, for almost all k, $R(a,k) \le \mathrm{TOW}(1,1,\ldots,1,4k)$ where there are $a-2$ 1's. (This is often called 2 to the 2 to the 2 ..., $a-2$ times and then a $4k$ at the top.)*

Note 5.6 The proof of Theorem 5.4 easily generalizes to yield the following.

1. For all $a \ge 2$, for all k, $R(a,k,c) \le c^{\binom{R(a-1,k-1,c)+1}{a-1}} + a - 2$.
2. $R(3,k,c) \le \mathrm{TOW}_c(1, 2ck - \log_c(k-2) + O(c))$.
3. For all $a \ge 4$, for almost all k,

$$R(a,k,c) \le \mathrm{TOW}_c(1, a-1, a-2, \ldots, 3, 2ck - \log_c(k-a+1) + O(c)).$$

6 The Conlon-Fox-Sudakov Proof

Recall the following high level description of the Erdős-Rado proof:

- Color an edge by using PHP. This cuts the number of nodes in half. This is done $R(2,k-1)+1$ times.
- After *all* the edges of a complete graph are colored we use Ramsey's theorem. This will cut the number of nodes down by a log.

Every time we colored an edge we cut the number of vertices in half. Could we color fewer edges? Consider the following scenario:

$COL^{**}(x_1, x_2) =$ RED and $COL^{**}(x_1, x_3) =$ BLUE. Intuitively the edge from x_2 to x_3 might not be that useful to us. *Therefore we will not color that edge!*

Two questions come to mind:

Question: How will we determine which edges are potentially useful?

Answer: We will associate to each x_i a 2-colored 1-hypergraph G_i that keeps track of which edges $(x_{i'}, x_i)$ are colored, and if so what they are colored. For example, if $COL^{**}(x_7, x_9) =$ RED then $(7, \text{RED}) \in G_9$. (We use the terminology *2-colored 1-hypergraphs* and the notation G_i so that when we extend this to the a-hypergraph Ramsey Theorem, in the appendix, the similarity will be clear.)

We will have $x_1 = 1$ and $G_1 = \emptyset$. Say we already have

$$x_1, \ldots, x_i$$

$$G_1, \ldots, G_i.$$

Assume $i' < i$. Assume that for each of $COL^{**}(x_1, x_i), \ldots, COL^{**}(x_{i'-1}, x_i)$ we have either defined it or intentionally chose to not define it. We are wondering if we should define $COL^{**}(x_{i'}, x_i)$. At this point the vertices of G_i are a subsets of $\{1, \ldots, i' - 1\}$. If G_i is equal (not just isomorphic) to $G_{i'}$ (as colored 1-hypergraphs) then we will define $COL^{**}(x_{i'}, x_i)$ and add i' to G_i with that color. If G_i is not equal to $G_{i'}$ then we will not define $COL^{**}(x_{i'}, x_i)$.

Question: Since we only color some of the edges how will we use Ramsey's theorem?

Answer: We will not. Instead we go until one of the 1-hypergraphs has k monochromatic points. Hence we will be using the 1-ary Ramsey Theorem. (When we prove the a-hypergraph Ramsey theorem we will use the $(a-2)$-hypergraph Ramsey Theorem.)

We need a lemma that will help us in both the case of $c = 2$ and the case of general c.

Lemma 6.1 *Let $S \subseteq [c]^*$ be such that no string in S has $\geq k-1$ of any $i \in [c]$. Then the following hold:*

1.
$$\sum_{\sigma \in S} |\sigma| \leq k^{3/2-c/2} c^{c(k-1)+2} \left(\frac{e}{\sqrt{2\pi}}\right)^{c+1}$$

2. *If $c = 2$ then the summation is bounded above by $\left(\frac{e}{\sqrt{2\pi}}\right)^3 k^{1/2} 2^{2k}$.*

Proof:

Let

$$A = \sum \{|\sigma| \ : \ \sigma \in [c]^*, \sigma \text{ contains at most } k-1 \text{ of any element}\}.$$

Grouping by the number of appearances of each element of $[c]$, we get

$$A = \sum_{j_1=0}^{k-1} \cdots \sum_{j_c=0}^{k-1} (j_1 + \ldots + j_c) \frac{(j_1 + \ldots + j_c)!}{j_1! \cdots j_c!}.$$

We may split up the innermost sum to get c different sums, each containing a single j_i in the summand. Since each of these sums is equal, we get

$$A = c \cdot \sum_{j_1=0}^{k-1} j_1 \cdot \sum_{j_2=0}^{k-1} \cdots \sum_{j_c=0}^{k-1} \frac{(j_1 + \ldots + j_c)!}{j_1! \cdots j_c!}.$$

We split this up into the part which depends on j_c, and the part which doesn't:

$$A = c \cdot \sum_{j_1=0}^{k-1} j_1 \cdot \sum_{j_2=0}^{k-1} \cdots \sum_{j_c=0}^{k-1} \frac{(j_1 + \ldots + j_{c-1})!}{j_1! \cdots j_{c-1}!} \binom{j_1 + \ldots + j_c}{j_c}. \tag{1}$$

Claim For all ℓ, with $0 \leq \ell \leq c-1$,

$$A \leq c \cdot \sum_{j_1=0}^{k-1} \frac{j_1}{\prod_{i=0}^{\ell-1}(j_1 + ik)} \cdot \sum_{j_2=0}^{k-1} \cdots \sum_{j_{c-\ell}=0}^{k-1} B_\ell \cdot \binom{j_1 + \ldots + j_{c-\ell} + \ell k}{j_{c-\ell}},$$

where

$$B_\ell = \frac{(j_1 + \ldots + j_{c-\ell-1} + \ell k)!}{j_1! \cdots j_{c-\ell-1}!(k-1)!^\ell}$$

does not depend on $j_{c-\ell}$. Note that, in the case $\ell = c-1$, all the inner sums are gone, so we are left with

$$A \leq c \cdot \sum_{j_1=0}^{k-1} \frac{j_1}{\prod_{i=0}^{c-2}(j_1 + ik)} \cdot B_{c-1} \cdot \binom{j_1 + (c-1)k}{j_1}.$$

Proof of Claim

We will prove this by induction on ℓ. The base case is Equation 1.

For the inductive step, we need only to look at the innermost sum, whose value we call S.

$$\begin{aligned} S &= \sum_{j_{c-\ell}=0}^{k-1} B_\ell \cdot \binom{j_1 + \ldots + j_{c-\ell} + \ell k}{j_{c-\ell}} \\ &= B_\ell \cdot \sum_{j_{c-\ell}=0}^{k-1} \binom{j_1 + \ldots + j_{c-\ell} + \ell k}{j_{c-\ell}} \\ &= B_\ell \cdot \binom{j_1 + \ldots + j_{c-\ell-1} + \ell k + k}{k-1}. \end{aligned}$$

Here we used Pascal's 2^{nd} Identity:

$$\sum_{b=0}^{n} \binom{a+b}{b} = \binom{a+n+1}{n}.$$

with $a = j_1 + \ldots + j_{c-\ell-1} + \ell k$.

Writing our answer out in terms of factorials, we get

$$\begin{aligned}
S &= \frac{(j_1+\ldots+j_{c-\ell-1}+\ell k)!}{j_1!\cdots j_{c-\ell-1}!(k-1)!^{\ell}} \cdot \frac{(j_1+\ldots+j_{c-\ell-1}+(\ell+1)k)!}{(k-1)!(j_1+\ldots+j_{c-\ell-1}+\ell k+1)!} \\
&= \frac{(j_1+\ldots+j_{c-\ell-1}+\ell k)!}{(j_1+\ldots+j_{c-\ell-1}+\ell k+1)!} \cdot \frac{(j_1+\ldots+j_{c-\ell-1}+(\ell+1)k)!}{j_1!\cdots j_{c-\ell-1}!(k-1)!^{\ell+1}} \\
&= \left(\frac{1}{j_1+\ldots+j_{c-\ell-1}+\ell k+1}\right) \cdot \frac{(j_1+\ldots+j_{c-\ell-1}+(\ell+1)k)!}{j_1!\cdots j_{c-\ell-1}!(k-1)!^{\ell+1}} \\
&\le \left(\frac{1}{j_1+\ell k}\right) \cdot \frac{(j_1+\ldots+j_{c-\ell-1}+(\ell+1)k)!}{j_1!\cdots j_{c-\ell-1}!(k-1)!^{\ell+1}} \\
&= \left(\frac{1}{j_1+\ell k}\right)\left(\frac{(j_1+\ldots+j_{c-\ell-2}+(\ell+1)k)!}{j_1!\cdots j_{c-\ell-2}!(k-1)!^{\ell+1}}\right)\binom{j_1+\ldots+j_{c-\ell-1}+(\ell+1)k}{j_{c-\ell-1}}. \\
&= \left(\frac{1}{j_1+\ell k}\right) B_{\ell+1}\binom{j_1+\ldots+j_{c-\ell-1}+(\ell+1)k}{j_{c-\ell-1}}.
\end{aligned}$$

Reinserting this value S back into the formula for A, and factoring the fraction $\frac{1}{j_1+\ell k}$ to the outermost sum, we get the desired result.

The induction stops when we hit the outermost sum, where the format of the summand changes.

End of Proof of Claim

Using this claim, with $\ell = c-1$, we get the bound

$$A \le c \cdot \sum_{j_1=0}^{k-1} \frac{j_1}{\prod_{i=0}^{c-2}(j_1+ik)} \cdot B_{c-1} \cdot \binom{j_1+(c-1)k}{j_1}.$$

Note the first fraction: the j_1 in the numerator cancels with the $i=0$ term of the denominator. As for the rest of the terms, they reach their maxima when $j_1 = 0$.

Renaming j_1 to be n and filling in the value of B_{c-1}, we get

$$
\begin{aligned}
A &\leq c \cdot \sum_{n=0}^{k-1} \frac{n}{\prod_{i=0}^{c-2}(n+ik)} \cdot \frac{((c-1)k)!}{(k-1)!^{c-1}} \cdot \binom{n+(c-1)k}{n} \\
&\leq \frac{c}{\prod_{i=1}^{c-2} ik} \cdot \frac{((c-1)k)!}{(k-1)!^{c-1}} \sum_{n=0}^{k-1} \binom{n+(c-1)k}{n} \\
&= \frac{c}{(c-2)!k^{c-2}} \cdot \frac{((c-1)k)!}{(k-1)!^{c-1}} \sum_{n=0}^{k-1} \binom{n+(c-1)k}{n} \\
&= \frac{c}{(c-2)!k^{c-2}} \cdot \frac{((c-1)k)!}{(k-1)!^{c-1}} \cdot \binom{ck}{k-1} \\
&= \frac{c}{(c-2)!k^{c-2}} \cdot \frac{((c-1)k)!}{(k-1)!^{c-1}} \cdot \frac{(ck)!}{(k-1)!((c-1)k+1)!} \\
&\leq \frac{c}{(c-2)!k^{c-2}} \cdot \frac{((c-1)k)!}{((c-1)k+1)!} \cdot \frac{(ck)!}{(k-1)!^{c}} \\
&= \frac{c}{(c-2)!k^{c-2}} \cdot \frac{1}{(c-1)k+1} \cdot \frac{(ck)!}{(k-1)!^{c}} \\
&\leq \frac{c^2 k}{c!} \cdot \frac{(ck)!}{k!^{c}}
\end{aligned}
$$

Now we use the bounds associated with Stirling's approximation:

$$
\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n
$$

$$
\begin{aligned}
A &\le \frac{c^2 k}{c!} \cdot \frac{e(ck)^{1/2}(ck)^{ck}}{(2\pi k)^{c/2} k^{ck}} \\
&\le \frac{e}{c!} \cdot c^{ck+5/2} k^{3/2-c/2} (2\pi)^{-c/2} \\
&\le \frac{e}{\sqrt{2\pi c}(c/e)^c} \cdot c^{ck+5/2} k^{3/2-c/2} (2\pi)^{-c/2} \\
&\le c^{c(k-1)+2} k^{3/2-c/2} \left(\frac{e}{\sqrt{2\pi}} \right)^{c+1}.
\end{aligned}
$$

$\blacksquare$

The following proof is by Conlon-Fox-Sudakov [2]; however, we do a more careful analysis with the aide of Lemma 6.1.2.

Theorem 6.2 *For all k, $R(3,k) \le 2^{B(k-1)^{1/2}2^{2k}}$ where $B = (\frac{e}{\sqrt{2\pi}})^3 \sim 1.28$.*

Proof: Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{3}$.

We define a finite sequence of vertices $x_1, x_2, \ldots, x_L$ where we will bound L later. For every $1 \le i \le L$ we will also define G_i, a 2-colored 1-hypergraph. We will represent G_i as a subset of $\mathsf{N} \times \{\mathrm{RED}, \mathrm{BLUE}\}$. For example, G_i could be $\{(1, \mathrm{RED}), (4, \mathrm{BLUE}), (5, \mathrm{RED})\}$. The notation $G_i = G_i \cup \{(12, \mathrm{RED})\}$ means that we add the edge $\{12\}$ to G_i and color it RED. When we refer to the vertices of the G_i 1-hypergraph we will often refer to them as *1-edges* since (1) in a 1-hypergraph, vertices are edges, and (2) the proof will generalize to a-hypergraphs more easily. We use the term 1-edges so the reader will remember they are vertices also.

The construction will stop when one of the G_i has a 1-homogenous set of size $k-1$ (more commonly called a set of $k-1$ monochromatic points). We will later show that this must happen.

Recall the definition of a 1-homogeneous set relative to a coloring of a 1-hypergraph from the note following Definition 2.4. We will use it here.

Here is the intuition: Let $x_1 = 1$ and $x_2 = 2$. Let $G_1 = \emptyset$. The vertices x_1, x_2 induces the following coloring of $\{3, \ldots, n\}$.

$$COL^*(y) = COL(x_1, x_2, y).$$

Let V_1 be a 1-homogeneous set of size at least $\frac{n-2}{2}$. We will only work within V_1 from now on. Let $COL^{**}(x_1, x_2)$ be the color of V_1. Let $G_2 = \{(1, COL^{**}(x_1, x_2)\}$.

Let x_3 be the least vertex in V_1. The number x_3 induces *two* colorings of $V_1 - \{x_3\}$:

$$COL^*_{1,3}(y) = COL(x_1, x_3, y)$$

$$COL^*_{2,3}(y) = COL(x_2, x_3, y)$$

Let V_2 be a 1-homogeneous for $COL^*_{1,3}$ of size $\frac{|V_1|-1}{2}$. Let $COL^{**}(x_1, x_3)$ be the color of V_2. We also set $G_3 = \{(1, COL^{**}(x_1, x_3))\}$, though we will may add to G_3 later. Restrict $COL^*_{2,3}$ to elements of V_2, though still call it $COL^*_{2,3}$. We will only work within V_2 from now on.

Will we color (x_2, x_3)? If $G_2 = G_3$ (that is, if they both colored 1 the same) then YES. If not then we won't. This is the KEY— every time we color an edge we divide V in half. We will not always color an edge- only the promising ones. Hence V will not decrease as quickly as was done in the proof of Theorem 5.1.

If $G_2 = G_3$ then we reuse the variable name V_2 to be a 1-homogeneous for $COL^*_{2,3}$ of size at least $\frac{|V_2|}{2}$. Let $COL^{**}(x_2, x_3)$ be the color of V_2. Add $(2, COL^{**}(x_2, x_3))$ to G_3.

If $G_2 \neq G_3$ then we do not color (x_2, x_3) and do not add anything to G_3.

In the actual construction we will not define COL^{**} since the information it contains will be stored in the 2-colored 1-hypergraphs G_i.

We describe the construction formally.

Def 6.3 Let G_{i_1}, G_{i_2} be 2-colored 1-hypergraphs. Let $j \in \mathsf{N}$.

1. G_{i_1} and G_{i_2} *agree on* j if, either (1) G_{i_1} and G_{i_2} both have 1-edge j and color it the same or, (2) neither G_{i_1} nor G_{i_2} has 1-edge j.

2. G_{i_1} and G_{i_2} *agree on* $\{1, \ldots, j\}$ if G_{i_1} and G_{i_2} agree on all of the 1-edges in the set $\{1, \ldots, j\}$.

3. G_{i_1} and G_{i_2} *disagree on* j if either (1) G_{i_1} and G_{i_2} both have 1-edge j and color it differently or (2) one of them has 1-edge j but the other one does not.

CONSTRUCTION

$$\begin{aligned}
x_1 = &\quad 1 \\
x_2 = &\quad 2 \\
G_1 = &\quad \emptyset \\
V_1 = &\quad [n] - \{x_1, x_2\} \\
COL^*(y) = &\quad COL(x_1, x_2, y) \text{ for all } y \in V_1 \\
V_2 = &\quad \text{the largest 1-homogeneous set for } COL^* \\
G_2 = &\quad \{(1, \text{ the color of } V_2)\}
\end{aligned}$$

KEY: for all $y \in V_2$, $COL(x_1, x_2, y)$ is the color of 1 in G_2.

Let $i \geq 2$, and assume that V_{i-1}, $x_1, \ldots, x_{i-1}$, $G_1, \ldots, G_{i-1}$ are defined. If G_{i-1} has a 1-homogenous set of size $k-1$ then stop (yes, $k-1$- this is not a typo). Otherwise proceed.

$$\begin{aligned}
G_i = &\quad \emptyset \text{ (This will change.)} \\
x_i = &\quad \text{the least element of } V_{i-1} \\
V_i = &\quad V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)}
\end{aligned}$$

We will add some colored 1-edges to G_i. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

For $j = 0$ to $i - 1$

1. If $G_j = G_i$ then proceed, else go to the next value of j. (Note that we are asking if $G_j = G_i$ at a time when G_i's vertex set is a subset of $\{1, \ldots, j-1\}$.)

2. $COL^* : V_i \to \{\text{RED}, \text{BLUE}\}$ is defined by $COL^*(y) = COL(x_j, x_i, y)$.

3. V_i is the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.

4. $G_i = G_i \cup \{(j, \text{ color of } V_i)\}$

KEY: Let $1 \le i_1 < i_2 \le i$ such that i_1 is a 1-edge of G_{i_2}. Let c_{i_1} be such that $(i_1, c_{i_1}) \in G_{i_2}$. For all $y \in V_i$, $COL(x_{i_1}, x_{i_2}, y) = c_{i_1}$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction ends. For now assume the construction ends.

When the construction ends we have a G_L that has a 1-homogenous set of size $k-1$. We assume the color is RED. Let $\{i_1 < i_2 < \cdots < i_{k-1}\}$ be the 1-homogenous set. Define $i_k = L$. We show that

$$H = \{x_{i_1}, \ldots, x_{i_k}\}$$

is a 3-homogenous set with respect to the original coloring COL. For notational convenience we show that $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$. The proof for any 3-set of H is similar, even for the case where the last point is x_L.

Look at G_{i_2}. Since i_2 is a 1-edge in G_L we know that G_{i_2} and G_L agree on all 1-edges in $\{1, \ldots, i_2 - 1\}$. Since $(i_1, \text{RED}) \in G_L$ and $i_1 \le i_2 - 1$, $(i_1, \text{RED}) \in G_{i_2}$. Hence, for all $y \in V_{i_2}$, $COL(x_{i_1}, x_{i_2}, y) = \text{RED}$. In particular $COL(x_{i_1}, x_{i_2}, x_{i_3}) = \text{RED}$.

We now establish bounds on n.

Def 6.4 Let $G = V$ be a 2-colored 1-hypergraph on vertex set $V = \{L_1 < \cdots < L_m\}$ and edge set E. Define $\mathrm{squash}(G)$ to be $G' = (V', E')$, the following 2-colored 1-hypergraph:

- The vertex sets $V' = \{1, \ldots, m\}$.
- For each edge $\{L_i\}$ in E the edge $\{i\}$ is in E'.
- The color of $\{i\}$ in G' is the color of $\{L_i\}$ in G.

Claim 1: For all $2 \le i_1 < i_2$, $\mathrm{squash}(G_{i_1}) \ne \mathrm{squash}(G_{i_2})$.

Proof of Claim 1: Assume, by way of contradiction, that $i_1 < i_2$ and $\mathrm{squash}(G_{i_1}) = \mathrm{squash}(G_{i_2})$. Let G_{i_1} have vertex set U_1. Let f_1 be the isomorphism that maps U_1 to the vertex set of $\mathrm{squash}(G_{i_1})$. Note that f_1 is order preserving. If f_1 is applied to a number not in U_1 then the result is undefined. Let U_2 and f_2 be defined similarly for G_{i_2}.

We will prove that, for all $1 \le j \le i_1 - 1$, (1) f_1 and f_2 agree on $\{1, \ldots, j\}$, (2) G_{i_1} and G_{i_2} agree on $\{1, \ldots, j\}$. The proof will be by induction on j.

Base Case: $j = 1$. Since $2 \le i_1, i_2$, the edge $E = \{1\}$ is in both G_{i_1} and G_{i_2}, hence $f_1(1) = f_2(1)$. If the color of E is different in G_{i_1} and G_{i_2} then $\mathrm{squash}(G_{i_1}) \ne \mathrm{squash}(G_{i_2})$. Hence the color of E is the same in both graphs. Hence G_{i_1} and G_{i_2} agree on $\{1\}$.

Induction Step: Assume that G_{i_1} and G_{i_2} agree on $\{1, 2, \ldots, j-1\}$. Assume that f_1 and f_2 agree on $\{1, \ldots, j-1\}$. We use these assumptions without stating them. Look at what happens when G_{i_1} (G_{i_2}) has to decide what to do with j.

If G_j and G_{i_1} agree on $\{1, \ldots, j-1\}$ then, since $j < i_1$, G_j also agrees with G_{i_2} on $\{1, \ldots, j-1\}$. Hence edge $E = \{j\}$ will be put into both G_{i_1} and G_{i_2}. Hence j will be a vertex in both G_{i_1} and G_{i_2} so $f_1(j) = f_2(j)$. Since f_1 and f_2 agree on $\{1, \ldots, j\}$ and $\mathrm{squash}(G_{i_1}) = \mathrm{squash}(G_{i_2})$, E must be the same color in G_{i_1} and G_{i_2}. Hence G_{i_1} and G_{i_2} agree on $\{1, \ldots, j\}$.

If G_j does not agree with G_{i_1} on $\{1, \ldots, j-1\}$ then there must be an edge $E \in \{1, \ldots, j-1\}$ such that G_j and G_{i_1} disagree on E. Hence G_j and G_{i_2} disagree on E. Thus j will not be made a vertex of G_{i_1} or G_{i_2} ever. Hence both $f_1(j)$ and $f_2(j)$ are undefined. The edge E is not added to G_{i_1} or G_{i_2} in stage j. Since G_{i_1} and G_{i_2} agree on $\{1, \ldots, j-1\}$ they agree on $\{1, \ldots, j\}$.

We now know that G_{i_1} and G_{i_2} agree on $\{1, \ldots, i_1 - 1\}$. Note that G_{i_1} only has vertices in $\{1, \ldots, i_1 - 1\}$. Look at stage i_1 in the construction of G_{i_2}. Since G_{i_1} agrees with G_{i_2} on $\{1, \ldots, i_1 - 1\}$ i_1 is an vertex in G_{i_2}. At that point G_{i_2} will have more vertices then G_{i_1} hence $\text{squash}(G_{i_1}) \neq \text{squash}(G_{i_2})$. This is a contradiction.

End of Proof of Claim 1

We now bound L, the length of the sequence. The sequence $G_1, G_2, \ldots,$ will end when some G_i has $2k - 3$ points in it (so at least $k - 1$ must be the same color) or earlier. For all i, map G_i to $\text{squash}(G_i)$. This mapping is 1-1 by Claim 1. Hence the length of the sequence is bounded by the number of 2-colored 1-hypergraphs on *an initial segment of* $\{1, \ldots, 2k-3\}$ so $L \leq 2^0 + \cdots + 2^{2k-3} \leq 2^{2k-2} - 1$. We have shown the construction terminates.

Strangely enough, this is not quite what we care about when we are bounding n. We care about the number of *edges* in all of the G_i's since each edge at most halves the number of vertices.

By Lemma 6.1, the number of edges in all of the G_i is bounded by $B(k-1)^{1/2}2^{2k}$ where $B = \left(\frac{e}{\sqrt{2\pi}}\right)^3$. Hence the number of times $|V|$ is cut in at most half is bounded by that same quantity. Hence it suffices to take $n = 2^{B(k-1)^{1/2}2^{2k}}$.

▮

Note 6.5 For $c \geq 2$ let $B_c = \left(\frac{e}{\sqrt{2\pi}}\right)^{c+1}$. The proof of Theorem 6.2 generalize to c colors yielding $R(3, k, c) \leq c^{B_c(k-1)^{1/2}c^{ck}}$.

Theorem 6.6 *Throughout this theorem* $B = (\frac{e}{\sqrt{2\pi}})^3 \sim 1.28$.

1. $R(3, k) \leq \text{TOW}(B(k-1)^{1/2}, 2^{2k})$.

2. $R(4, k) \leq \text{TOW}(1, 3B(k-2)^{1/2}, 2^{2k-2})$.

3. $R(5, k) \leq \text{TOW}(1, 4, 3B(k-3)^{1/2}, 2^{2k-4})$.

4. *For all $a \geq 6$, for almost all k,*

$$R(a,k) \leq \mathrm{TOW}(1, a-1, a-2, \ldots, 4, 3B(k-a+2)^{1/2}, 2^{2k-2a+6})$$

Proof:

Part 1 is a restatement of Theorem 6.2.

From Theorem 5.4 we have $R(a,k) \leq 2^{R(a-1,k-1)^{a-1}}$. We apply this recurrence to Part 1 to get Part 2, and to Part 2 to get Part 3. We then use it to get Part 4 by induction.

∎

Note 6.7 For $c \geq 2$ let $B_c = \left(\frac{e}{\sqrt{2\pi}}\right)^{c+1}$. The proof of Theorem 6.6 generalize to c colors yielding the following.

1. $R(3,k,c) \leq \mathrm{TOW}_c(B_c(k-1)^{1/2}, c^{ck})$.

2. $R(4,k,c) \leq \mathrm{TOW}_c(1, 3B(k-2)^{1/2}, c^{ck-c})$.

3. $R(5,k,c) \leq \mathrm{TOW}_c(1, 4, 3B(k-3)^{1/2}, c^{ck-2c})$.

4. For all $a \geq 6$, for almost all k,

$$R(a,k,c) \leq \mathrm{TOW}_c(1, a-1, a-2, \ldots, 4, 3B(k-a+2)^{1/2}, c^{ck-ac+3c})$$

7 Open Problems

The best known lower bounds are attributed to Erdős and Hajnal in [6]. They are as follows:

1. $R(3,k) \geq 2^{\Omega(k^2)}$ by a simple probabilistic argument.

2. $R(a,k) \geq \mathrm{TOW}(1, \ldots, 1, \Omega(k^2))$ ($a-1$ 1's) by the lower bound on $R(3,k)$ and the stepping up lemma.

For 4 colors the situation is very different. Erdős and Hajnal showed that

$$R(3, k, 4) \geq 2^{2^{\Omega(k)}}.$$

Obtaining matching upper and lower bounds for the hypergraph Ramsey Numbers seems to be a hard open problem. We suspect that a bound of the form $R(a, k) \leq 2^{2^{k+o(k)}}$ can be obtained.

8 Acknowledgments

We would like to thank David Conlon whose talk on this topic at RATLOCC 2011 inspired this paper. We would also like to thank David Conlon (again), Jacob Fox and Benny Sudakov for their paper [2] which contains the new proof of the 3-hypergraph Ramsey Theorem. We also thank Jessica Shi and Sam Zbarsky who helped us clarify some of the results.

A Extending Conlon-Fox-Sudakov to a-Hypergraph Ramsey

In this appendix we extend the Conlon-Fox-Sudakov proof to prove the a-hypergraph Ramsey Theorem. Unfortunately it does not yield better bounds on $R(a, k, 2)$. We include it in the hope that in the future someone may modify the construction, or our analysis of it, to yield better bounds.

In order to prove an upper bound on $R(a, k))$ (and $R(a, k, c)$) we need a lemma similar to Lemma 6.1. The lemma below gives a crude estimate. It is possible that a more careful bound would lead to a better analysis of the construction and hence to a better bound on the hypergraph Ramsey numbers.

Lemma A.1 *Let S be the subset of c-colored complete $(a-2)$-hypergraphs whose vertex sets are an initial segments of* N *and that have no $(a-2)$-homogenous set of size $k-1$. Then*

$$\sum_{(V,E,COL) \in S} |E| \leq R(a-2, k-1, c)^{a-1} c^{R(a-2,k-1,c)^{a-2}}.$$

Proof:

The largest size of V such that a c-colored $(a-2)$-hypergraph (V,E) has no $(a-2)$-homogenous set of size $k-1$ is bounded above by $R(a-2,k-1,c)$. Hence we want to bound.

$$\sum_{i=1}^{R(a-2,k-1,c)} \sum_{(V,E,COL)\in S, |V|=i} |E| \le \sum_{i=1}^{R(a-2,k-1,c)} \sum_{(V,E,COL)\in S, |V|=i} i^{a-2}.$$

The number of c-colored $(a-2)$-hypergraphs on i vertices is bounded above by $c^{i^{a-2}}$. Hence we can bound the above sum by

$$\begin{aligned}\sum_{i=1}^{R(a-2,k-1,c)} c^{i^{a-2}} i^{a-2} &\le R(a-2,k-1,c)2^{R(a-2,k-1,c)^{a-2}}R(a-2,k-1,c)^{a-2}\\ &\le R(a-2,k-1,c)^{a-1}2^{R(a-2,k-1,c)^{a-2}}\end{aligned}$$

▮

Theorem A.2 *For all $a \ge 3$, for all $k \ge 3$*

$$R(a,k) \le 2^{R(a-2,k-1)^{a-1}2^{R(a-2,k-1)^{a-2}}}.$$

Proof:

Let n be a number to be determined. Let COL be a 2-coloring of $\binom{[n]}{a}$.

We define a finite sequence of vertices $x_1, x_2, \ldots, x_L$ where we will bound L later. For every $1 \le i \le L$ we will also define G_i, a 2-colored $(a-2)$-hypergraph. We will represent G_i as a subset of $\binom{\mathsf{N}}{a-2} \times \{\text{RED}, \text{BLUE}\}$. For example, if $a=6$, G_i could be

$$\{(\{1,2,4,5\},\text{RED}), (\{1,3,4,9\},\text{BLUE}), (\{4,5,6,10\},\text{RED})\}.$$

The notation $G_i = G_i \cup \{\{12,13,19,99\},\text{RED}))\}$ means that we add the edge $\{12,13,19,99\}$

to G_i and color it RED in G_i.

The construction will stop when one of the G_i has a $(a-2)$-homogenous set of size $k-1$. We will later show that this must happen.

Def A.3 Let G_{i_1}, G_{i_2} be 2-colored $(a-2)$-hypergraphs. Let $J \in \binom{\mathsf{N}}{a-2}$.

1. G_{i_1} and G_{i_2} *agree on* J if either (1) G_{i_1} and G_{i_2} both have edge J and color it the same or (2) neither G_{i_1} nor G_{i_2} has edge J.

2. G_{i_1} and G_{i_2} *agree on* $\{1, \ldots, j\}$ if G_{i_1} and G_{i_2} agree on all of the edges in $\binom{[j]}{a-2}$.

3. G_{i_1} and G_{i_2} *disagree on* J if either (1) G_{i_1} and G_{i_2} both have edge J and color it differently or (2) one of them has edge J but the other one does not.

CONSTRUCTION

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 2 \\
\vdots &= \vdots \\
x_{a-1} &= a-1 \\
G_1 &= \emptyset \\
G_2 &= \emptyset \\
\vdots & \ \ \vdots \\
G_{a-2} &= \emptyset \\
V_{a-2} &= [n] - \{x_1, \ldots, x_{a-1}\}. \text{ We start indexing here for convenience.} \\
COL^*(y) &= COL(x_1, x_2, \ldots, x_{a-1}, y) \text{ for all } y \in V_{a-2} \\
V_{a-1} &= \text{ the largest } (a-2)\text{-homogeneous set for } COL^* \\
G_{a-1} &= (\{1, \ldots, a-2\}, \text{ the color of } V_{a-1})
\end{aligned}
$$

The G_i's will be 2-colored $(a-2)$-hypergraphs.

KEY: for all $y \in V_{a-1}$, $COL(x_1, \ldots, x_{a-1}, y)$ is the color of $\{1, \ldots, a-2\}$ in G_{a-1}.

Let $i \geq a-1$, and assume that V_{i-1}, $x_1, \ldots, x_{i-1}$, and $G_1, \ldots, G_{i-1}$ are defined. If G_{i-1} has an $(a-2)$-homogenous set of size $k-1$ then stop (yes $k-1$- this is not a typo). Otherwise proceed.

$$\begin{aligned} G_i &= \emptyset \text{ (This will change.)} \\ x_i &= \text{ the least element of } V_{i-1} \\ V_i &= V_{i-1} - \{x_i\} \text{ (We will change this set without changing its name.)} \end{aligned}$$

We will add colored $(a-2)$-edges to G_i. We will also define smaller and smaller sets V_i. We will keep the variable name V_i throughout.

In the next step we will, for all $J \in \binom{[i-1]}{a-2}$, consider adding J to G_i. The order in which we consider the J matters. Assume the order first considers each edge whose maximum entry is $a-2$, then each edges with maximum entry is $a-1$, etc, until the maximum entry is $i-1$.
For $J \in \binom{[i-1]}{a-2}$

1. If for every $j \in J$, G_j and G_i agree on $\{1, \ldots, j-1\}$ then proceed, otherwise go to the next J. (Note that when edge J is being considered all of the edges J' with $\max(J') < \max(J)$ have already been decided upon. Hence if J becomes an edge of G_i then it will always be the case that, for every $j \in J$, G_j and G_i agree on $\{1, \ldots, j-1\}$.

2. $COL^* : V_i \rightarrow \{\text{RED}, \text{BLUE}\}$ is defined by $COL^*(y) = COL(J \cup \{x_i, y\})$.

3. V_i is the largest 1-homogeneous set for COL^*. Note that $|V_i|$ decreases by at most half.

4. $G_i = G_i \cup \{(J, \text{ the color of } V_i\ )\}$

KEY: Let $A \in \binom{[i-1]}{a-2}$ and $b > \max(A)$ such that A is an $(a-2)$-edge of G_i. Let c_A be such that $(A, c_A) \in G_i$. For all $y \in V_i$, $COL(A \cup \{x_b, y\}) = c_A$.

END OF CONSTRUCTION

When we derive upper bounds on n we will show that the construction ends. For now assume the construction ends.

When the construction ends we have a G_L that has a $(a-2)$-homogenous set of size $k-1$. We assume the color is RED. Let $\{i_1 < i_2 < \cdots < i_{k-1}\}$ be the $(a-2)$-homogenous set. Define $i_k = L$. We show that

$$H = \{x_{i_1}, \ldots, x_{i_k}\}$$

is a a-homogenous set with respect to the original coloring COL. For notational convenience we show that $COL(x_{i_1}, \ldots, , x_{i_a}) = \text{RED}$. The proof for any a-set of H is similar, even for the case where the last point is x_L.

Look at $G_{i_{a-1}}$. Since i_{a-1} is a vertex in G_L we know that $G_{i_{a-1}}$ and G_L agree on $\{1, \ldots, i_{a-1}-1\}$. Since $(i_{a-1}, \text{RED}) \in G_L$ and $i_1, \ldots, i_{a-2} \leq i_2 - 1$, $(\{i_1, \ldots, i_{a-2}\}, \text{RED}) \in G_{i_{a-1}}$. Hence, for all $y \in V_{i_{a-1}}$, $COL(x_{i_1}, \ldots, x_{i_{a-2}}, x_{i_{a-1}}, y) = \text{RED}$. In particular $COL(x_{i_1}, \ldots, x_{i_a}) = \text{RED}$.

We now establish bounds on n.

Def A.4 Let G be a 2-colored $(a-2)$-hypergraph on vertex set $V = \{L_1 < \cdots < L_m\}$ and edge set E. Define $\text{squash}(G)$ to be $G' = (V', E')$, the following 2-colored $(a-2)$-hypergraph:

- The vertex sets $V' = \{1, \ldots, m\}$.
- For each edges $\{L_{i_1}, \ldots, L_{i_{a-2}}\}$ in E the edge $\{i_1, \ldots, i_{a-2}\}$ is in E'.
- The color of $\{i_1, \ldots, i_{a-2}\}$ in G'. is the color of $\{L_{i_1}, \ldots, L_{i_{a-2}}\}$ in G.

Claim 1: For all $a-1 \leq i_1 < i_2$, $\text{squash}(G_{i_1}) \neq \text{squash}(G_{i_2})$.

Proof of Claim 1: Assume, by way of contradiction, that $a-1 \leq i_1 < i_2$ and $\text{squash}(G_{i_1}) = \text{squash}(G_{i_2})$. Let G_{i_1} have vertex set U_1 and let f_1 be the isomorphism that maps U_1 to the vertex set of $\text{squash}(G_{i_1})$. Note f_1 is order preserving and, if f_1 is applied to a number not in U_1, then the result is undefined. Define U_2 and f_2 for G_{i_2} similarly.

We will prove that, for all $1 \le j \le i_1 - 1$, (1) f_1 and f_2 agree on $\{1, \ldots, j\}$, (2) G_{i_1} and G_{i_2} agree on $\{1, \ldots, j\}$. The proof will be by induction on j.

Base Case: $j \in \{1, 2, \ldots, a-2\}$. Since $a - 1 \le i_1, i_2$ the edge $E = \{1, 2, \ldots, a-2\}$ is in both G_{i_1} and G_{i_2}; therefore, $f_1(1) = f_2(1)$, …, $f_1(a-2) = f_2(a-2)$. If the color of E is different in G_{i_1} and G_{i_2} then $\mathrm{squash}(G_{i_1}) \ne \mathrm{squash}(G_{i_2})$. Hence the color of E is the same in both graphs. Thus we have that G_{i_1} and G_{i_2} agree on $\{1, \ldots, a-2\}$.

Induction Step: Assume that G_{i_1} and G_{i_2} agree on $\{1, 2, \ldots, j-1\}$. Assume that f_1 and f_2 agree on $\{1, \ldots, j-1\}$. We use these assumptions without stating them throughout. Look at what happens when G_{i_1} (G_{i_2}) has to decide what to do with j.

If G_j and G_{i_1} agree on $\{1, \ldots, j-1\}$ then, since $j < i_1$, G_j also agrees with G_{i_2} on $\{1, \ldots, j-1\}$. Hence the edge $\{1, 2, \ldots, a-3, j\}$ will be put into both G_{i_1} and G_{i_2}. Hence j will be a vertex in both G_{i_1} and G_{i_2} so $f_1(j) = f_2(j)$. Let $E \in \binom{[j]}{a-2}$ such that $j \in E$. If for every vertex j' of E, $G_{j'}$ and G_{i_1} agree on $\{1, \ldots, j'-1\}$ then, since $j' < i_1$, $G_{j'}$ also agrees with G_{i_2} on $\{1, \ldots, j'-1\}$. Hence E will be in both G_{i_1} and G_{i_2}. Since f_1 and f_2 agree on $\{1, \ldots, j\}$ and $\mathrm{squash}(G_{i_1}) = \mathrm{squash}(G_{i_2})$, E must be the same color in G_{i_1} and G_{i_2}. Hence every edge put into G_{i_1} in stage j is also in G_{i_2} and with the same color. By a similar argument we can show that every edge put into G_{i_2} in stage j is also in G_{i_1} and with the same color. Hence G_{i_1} and G_{i_2} agree on $\{1, \ldots, j\}$.

If G_j does not agree with G_{i_1} on $\{1, \ldots, j-1\}$ then there must be an edge $E \in \binom{[j-1]}{a-2}$ such that G_j and G_{i_1} disagree on E. Hence G_j and G_{i_2} disagree on E. Thus j will not be made a vertex of G_{i_1} or G_{i_2} ever. Hence both $f_1(j)$ and $f_2(j)$ are undefined. No new edges are added to G_{i_1} or G_{i_2} in stage j hence, since G_{i_1} and G_{i_2} agree on $\{1, \ldots, j-1\}$ they agree on $\{1, \ldots, j\}$.

We now know that G_{i_1} and G_{i_2} agree on $\{1, \ldots, i_1 - 1\}$. Note that G_{i_1} only has vertices in $\{1, \ldots, i_1 - 1\}$. Look at stage i_1 in the construction of G_{i_2}. Since G_{i_1} agrees with G_{i_2} on $\{1, \ldots, i_1 - 1\}$, i_1 will be a vertex of G_{i_1}. At that point G_{i_2} will have more vertices then G_{i_1} hence $\mathrm{squash}(G_{i_1}) \ne \mathrm{squash}(G_{i_2})$. This is a contradiction.

End of Proof of Claim 1

Claim 2: All of the G_i are complete $(a-2)$-hypergraphs.

Proof of Claim 2:

Let $i_1 < i_2 < \cdots < i_{a-2}$ be vertices of G_i. We will show that $\{i_1, \ldots, i_{a-2}\}$ is an edge in G_i.

For all $1 \le j \le a-2$, since i_j is a vertex of G_i we know that G_{i_j} and G_i agree on $\{1, \ldots, i_j - 1\}$. Hence, in stage i, this will be noted and $\{i_1, \ldots, i_{a-2}\}$ will be added to G_i.

End of Proof of Claim 2

We now bound L, the length of the sequence. The sequence $G_1, G_2, \ldots,$ will end when some G_i has $R(a-2, k-1)$ vertices (since by the definition of $R(a-2, k-1)$ there will be a homogenous set of size $(k-1)$ or earlier. For all $i \ge a-2$ map G_i to $\mathrm{squash}(G_i)$. This mapping is 1-1 by Claim 1. Hence the length of the sequence is bounded by the $a-3$ plus the number of 2-colored $(a-2)$-hypergraphs on *an initial segment of* $\{1, \ldots, R(a-2, k-1)\}$, so $L \le a-3+2^0+\cdots+2^{R(a-2,k-1)} \le 2^{R(a-2,k-1)+1} + a - 4$. We have shown the construction terminates.

Strangely enough, this is not quite what we care about when we are bounding n. We care about the number of *edges* in all of the G_i's since each edge at most halves the number of vertices.

By Lemma A.1 the number of edges in all of the G_i's is bounded above by

$$R(a-2, k-1)^{a-1} 2^{R(a-2,k-1)^{a-2}}.$$

Hence the number of times that the number of vertices are decreased by at most half is bounded by this same quantity. Therefore it suffices to take $n = 2^{R(a-2,k-1)^{a-1} 2^{R(a-2,k-1)^{a-2}}}$. Hence

$$R(a,k) \le 2^{R(a-2,k-1)^{a-1} 2^{R(a-2,k-1)^{a-2}}}.$$

■

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