Ramsey Theory and the History of Pre-Christian England:

An Example of Interdisciplinary Research

An Exposition by William Gasarch

1 Introduction

The application of mathematics to the natural sciences, and the natural sciences as a source of interesting problems in mathematics, is a well known phenomenon. Eugene Wigner thought the application of math to the natural sciences was *unreasonably effective* [Wig60]. What about the application of mathematics to the social sciences? Herbert Scarf's application of the Brouwer fixed point theorem to economics [Sca83] is one of many examples of applying (arguably) pure mathematics to economics. Samuel Harrington's attempt to use mathematics in political science caused much controversy (see [Sim88, Kob88, SK88]).

What about sociology or history? The application of statistics to these fields is well known. Dorwin Cartwright and Frank Harary [CH77], and others, appear to have used graph theory to model social relationships; however, on closer inspection they just used the *language of graph theory*. While this is all well and good, the question arises: has *pure mathematics* ever been used in a serious way on a problem of sociology or history? In Charles Percy Snow's article, *The Two Cultures* [Sno59], he speculates that there is a cultural divide between the sciences and the humanities which may make such collaborations difficult. This points to the lack of interaction being a sociological problem in itself; however, we are not going to go there. While there may really be such connections between sociology and pure mathematics, they will be hard to find.

There was *an* application of Ramsey theory to sociology in the 1950's. In Jacob Fox's Lecture Notes in Combinatorics [Fox] he tells the the following well known story:

> In the 1950's, a Hungarian sociologist Sandor Szalai studied friendship relationships between children. He observed that in any group of around 20 children he was

able to find four children who were mutual friends, or four children such that no two of them were friends. Before drawing any sociological conclusions, Szalai consulted three eminent mathematicians in Hungary at that time: Paul Erdős, Paul Turán, and Vera Sós. A brief discussion revealed that indeed this is a mathematical phenomenon rather than a sociological one. For any symmetric relation R on at least 18 elements, there is a subset S of 4 elements such that R contains either all pairs in S or none of them. This fact is a special case of Ramsey's theorem, proved in 1930, the foundation of *Ramsey theory* which developed later into a rich area of combinatorics.

This could be called an anti-application since, in the end, there was no interesting sociological phenomena. And, while this story is amusing (and surprisingly true), sociology did not become a source of questions for mathematics.

Recently there *has* been a case where Ramsey theory has greatly simplified a topic in history, history has been the source of interesting new problems in Ramsey theory, and history has helped solve an open problem in Ramsey theory. This paper is an exposition of these events.

2 Pre-Christian History of England

Sir Woodsor Kneading is a scholar of pre-Christian English history. He is particularly interested in when wars (technically skirmishes) broke out. He noticed the following:

1. In 577 BC there were five lords in what is now West Essex, all with armies. Some of the pairs were allied and some were enemies, yet there was no war. Then a sixth Lord settled into that region and within a year war broke out.

2. In 552 BC there were five lords in what is now South Wales, all with armies. Similarly, once a sixth Lord settled, there was war.

3. He noticed this happening 42 times total.

He then looked for this pattern and discovered the following:

> Between the years 600 BC and 400 BC, whenever there were six lords in proximity war broke out, with one exception. That one exception was truly exceptional— that was when all six lords had an alliance with each other. The question arises: Why do six lords almost always mean war?
>
> I believe I have an answer. I noticed that either (1) three, four, or five of them formed an alliance and, thinking themselves quite powerful, merged armies and attacked the other lords, or (2) there were three or more of them who were pairwise enemies, and in that case war broke out among these factions, or (3) (and this is rare) all six formed an alliance and there was peace. Note that the wars in cases (1) and (2) were very different from each other, but they were still wars.

Kneading goes on:

> I conjectured the following: whenever there are six lords, not all in alliance, there must either be (1) three, four, or five who are all allied with each other, or (2) three, four, five, or six who are pairwise enemies. I hired a computer science undergraduate student H. K. Donnut to look into this. After a month he proved my conjecture by a clever computer search. We published a joint paper [KD11b].

After the year 400 BC there were cases of six lords in a region and no war. Why was this? Kneading speculated that around that time weapons became more high tech so wars became more costly. This speculation was verified by Moss Chill Beaches [Bea13].

Kneading then noticed that, between the years of 400 BC and 200 BC, whenever there were 18 lords in the same region there was a war.

> Between the years of 400 BC and 200 BC, whenever there were 18 lords in proximity either (1) between four and seventeen of them formed an alliance and, thinking

themselves quite powerful, merged armies and attacked the other lords, or (2) there were four or more of them who were pairwise enemies, and in that case war broke out among these factions. If ever there was a time when all 18 had alliances between them then peace could have broken out but, alas, this never happened.

Once again Kneading hired Donnut to look into this. After two months Donnut proved, to quote the title of their paper [KD11a], *18 lords means war!*.

3 Ramsey Theory

We state Ramsey's Theorem for graphs in a way that it can be applied to history easily.

1. K_n is the complete graph on n vertices. We will think of the vertices as being lords.
2. We will be 2-coloring the edges of K_n. If two lords are enemies then we draw a RED line between them (RED for blood). If two lords have an alliance then we draw a $BLUE$ line between them ($BLUE$ for friendship).
3. Let COL be a 2-coloring of the edges of K_n. A subset of vertices U of size k is called a *k-alliance set* if all of the lords in U have an alliance with each other. A subset of vertices U of size k is called a *k-enemy set* if all of the lords in U are pairwise enemies.

Note 3.1 Kneading and Donnut called these colored graphs *Alliance-Enemy Descriptors* or AED's.

The following theorem was essentially rediscovered in the papers by Woodsor and Donnut [KD11b, KD11a].

Theorem 3.2

1. *For any 2-coloring of K_6 there is either a 3-enemies set or a 3-alliance set. (Note that the only case in [KD11b] where war did not occur was when there was a 6-alliance set, which was rare in the time period considered.)*

2. *For any 2-coloring of K_{18} there is either a 4-enemies set or a 4-alliance set. (In [KD11a] they note that if every there was an 18-alliance then war would not have broken out, but this never happened in the time period considered.)*

Had Kneading known this ahead of time he would have saved time. If Donnut had known these theorems he would not have gotten paid for three months of work, nor would he have gotten to be a co-author on the two papers.

4 Kneading's Book

Kneading and Donnut kept on working on history viewed in terms of AED's. They ended up rediscovering several known small Ramsey Numbers including bipartite ones (useful if the lords live on different sides of a river). Kneading wrote an entire book [Kne13] on this topic without knowing any Ramsey theory. (Donnut declined to be a co-author as he thought of himself as merely being a coder whereas Kneading had done the intellectual heavy lifting.)

This book was a breakthrough in the study of war and peace. It was largely because of this book that Kneading was elected a foreign member of the (American) National Academy of Sciences [Gra].

We present the general Ramsey Theorem for 2-colorings of graphs and then a passage from Kneading's book. We believe the passage can be construed as stating Ramsey's Theorem.

Theorem 4.1 *For all k there exists n such that for all 2-colorings of the edges of K_n there is either a k-enemies set or a k-alliance set. (It is known that $2^{k/2} \leq n \leq 2^{2k}$.)*

The Passage:

> As wars get more and more costly the number of lords (or in the modern world countries) that need to either align or be pairwise enemies to guarantee war will increase. Is there a limit? I doubt it. If, for example, a 17-enemies set or 17-alliance

set is needed to start a war then I expect there is some number of lords, perhaps quite large, so that if that number of lords were in close proximity, a war would break out.

5 Simplification Due to Ramsey's Theorem

Alma Grand-Rho was visiting Cambridge, where Kneading teaches, to give a talk on her ideas for a simplification of D.H.J. Polymath's proof of the Density Hales-Jewitt Theorem [Pol12]. She lost her way and found herself in the History department. The department had one of those bookshelves where books of faculty are displayed. Fortunately it was not behind any glass so she just happened to look at Kneading's book. The title *Alliances, Enemies, and War in Pre-Christian England* intrigued her since she had minored in history and did a senior thesis on the English kings. While browsing the book she found the following passage:

> In the year 100 AD, in West Wales, there were 48 lords in proximity but no war broke out. In this time period whenever five lords formed either a 5-alliance or a 5-enemies set there was a war. I realized that either (1) there was a 5-alliance or 5-enemies set yet no war broke out, which would be interesting and perhaps cause me to change my timeline as to when 48 was enough, or (2) there was no 5-alliance or 5-enemies set, which would be less interesting. I had Donnut enter the data and run his algorithm to see if there were any 5-alliance or 5-enemies sets. There were not. Oh well.

She then thought: *OMG!*[1] *He just proved* $R(5) = 49$. What does this mean? It was known (by mathematicians) that every AED of size 49 has either a 5-alliance or a 5-enemies set, but it was not known for 48. Now it is known that 49 is optimal. Later on, Kneading and Grand-Rho met and got out a paper [GK14] that essentially summarized his book succinctly using Ramsey theory.

[1] OMG is Text-speak for Oh My God.

Kneading says, without embarrassment, “My paper with Alma says cleanly in 30 pages what I said clumsily in 300 pages.”

Their paper was a lucid explanation of concepts in both Ramsey theory and history. For this paper they were awarded the Steele Prize for Mathematical Exposition [Wri]. There work will soon be popularized in the recreational math columns of Tim Andrer Gran and Ana Writset

6 Open Problems

Kneading’s model of alliance and enemies might be too simple to grapple with today’s countries and other groups. At the conference celebrating noted Ramsey theorists Tee A. Cornet’s 100th birthday Grand-Rho and Kneading[GK15] presented the following refinements.

1. Some pairs of lords are enemies and some are allied. This seems too black-and-white. There could be Fifty Shades of Grey [Jam11]. This may lead to applications of, and open problems in, multicolor Ramsey theory.

2. There may be alliances between three, four, or even more lords. However, there is no such thing as a set of three lords that are enemies except to say it is pairwise. This may lead to a new type of Ramsey theory where you color (say) edges RED or $BLUE$, but hyperedges are either colored $BLUE$ or not colored at all.

3. You can combine the two above to get a new type of multicolor, hypergraph Ramsey theory.

4. Kneading assumed that alliances and enemies are symmetric. There can be cases where A likes B but B does not like A. Hence one can look at asymmetric Ramsey theory. Again, one may combine this with the above to get asymmetric, multicolored, hypergraph Ramsey theory.

In summary, the story I tell above is strong evidence of the value of interdisciplinary research. Ramsey theory simplified some work in history, and history is now the impetus for new work in

Ramsey theory. And, of course, most surprisingly, history solved an open problem in mathematics.

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