Generating Functions vs Elementary Methods: Loaded Dice

by

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0.1 Point

Generating functions are a powerful tool to prove theorems. However, there are times when a more elementary proof is available. We look at several loaded dice questions and answer them both with generating function proofs and elementary proofs. Which technique is better is a matter of opinion.

0.2 Loaded Dice

When you role a pair of fair 6-sided dice the probability of getting a 2 is 1/36 and the probability of getting a 7 is 1/6. Note that they are not equal! Can you load dice to get fair sums? The answer is no, as was shown by [Honsberger (1978)] using elementary methods and [Hofri (1995)] using generating functions. [Chen *et al.* (1997)], [Gasarch and Kruskal (1999)], and [Morrison (2014)] have looked at generalizations of this question. We look at this question for two 6-sided dice and M d-sided dice.

Definition 0.1. A *d-sided die* is a tuple of d real numbers $(p_1, \ldots, p_d)$ such that $0 \le p_i \le 1$ and $\sum_{i=1}^{d} p_i = 1$.

0.3 Elementary and Gen Function Proofs for Two 6-Sided Dice

Lemma 0.1. *For all real $x > 0$, $x + \frac{1}{x} \ge 2$.*

Proof. Since $x > 0$ $\sqrt{x}, \frac{1}{\sqrt{x}} \in \mathbb{R}^+$. Hence the following algebra makes sense.

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \ge 0$$

Hence

$$x - 2\sqrt{x}\frac{1}{\sqrt{x}} + \frac{1}{x} \geq 0$$

$$x + \frac{1}{x} \geq 2$$

□

Theorem 0.1. *There is no way to load two six-sided dice to obtain fair sums.*

Proof. Assume, by way of contradiction, that $(p_1, \ldots, p_6)$ and $(q_1, \ldots, q_6)$ are a loaded pair of dice that yields fair sums. Each sum has probability $1/11$ of occurring. From this premise we give two proofs.

Elementary Proof

If $q_1 = 0$ then $\text{Prob}(2) = 0 \neq \frac{1}{11}$, hence $q_1 \neq 0$. Similar for q_6. Hence we can divide by either of them.

$\text{Prob}(2) = \text{Prob}(12) = p_1q_1 = p_6q_6$.

$$\begin{aligned}
\text{Prob}(7) &= p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 \\
&\geq p_1q_6 + p_6q_1 = (p_1q_1)\tfrac{q_6}{q_1} + (p_6q_6)\tfrac{q_1}{q_6} \\
&= \tfrac{1}{11}\left(\tfrac{q_6}{q_1} + \tfrac{q_1}{q_6}\right) \\
&\geq \tfrac{2}{11} \text{ This step uses Lemma 0.1.}
\end{aligned}$$

This is a contradiction.

Generating Function Proof

Consider the function

$$(p_1x+p_2x^2+p_3x^3+p_4x^4+p_5x^5+p_6x^6)(q_1x+q_2x^2+q_3x^3+q_4x^4+q_5x^5+q_6x^6)$$

Note that the coefficient of x^n is $\text{Prob}(n) = \frac{1}{11}$. Hence

$$(p_1x+p_2x^2+p_3x^3+p_4x^4+p_5x^5+p_6x^6)(q_1x+q_2x^2+q_3x^3+q_4x^4+q_5x^5+q_6x^6)$$

$$= \frac{1}{11}(x^2 + \cdots + x^{12})$$

so

$$(p_1+p_2x+p_3x^2+p_4x^3+p_5x^4+p_6x^5)(q_1+q_2x+q_3x^2+q_4x^3+q_5x^4+q_6x^5)$$

$$=\frac{1}{11}(1+\cdots+x^{10})$$

The left hand side has at least two real roots (counting multiplicities) since it is the product of two odd degree polynomials over the reals. Note that $(x^{11}-1)=(x-1)(x^{10}+\cdots+1)$ so the right hand side has as its roots 10 of the 11 11th roots of unity. Since neither 1 nor -1 are roots of the right hand side, the right hand side has no real roots. That is a contradiction. □

0.4 Generalizing Both Proofs to Two d-sided Dice

Which proof is better? One way to judge is to see if the proof easily generalizes to the case of two d-sided dice. We leave it to the reader to prove the following:

- The elementary proof easily generalizes to show that, for all $d \geq 2$, there is no way to load two d-sided dice to obtain fair sums.
- The generating function proof easily generalizes to show that, for all $d \geq 2$, d even, there is no way to load two d-sided dice to obtain fair sums.

Since the elementary proof generalizes easily for all d, and the generating function proof only for d even, this is a win for the elementary proof. But is there *some* generating function proof that works for all d? There is! However, rather than show you that we'll show you both an elementary and a generating function proof for the case of M d-sided dice.

0.5 Elementary and Gen Function Proof for M d-sided Dice

Definition 0.2. A polynomial $f(x)=a_{d-1}x^{d-1}+\cdots+a_0$ is *palindromic* if, $a_{d-1}=a_0$, $a_{d-2}=a_1$, etc.

Lemma 0.2. *Let $f(x)$ be a polynomial such that the sum of the coefficients is nonzero. If for every root r of $f(x)$, $\frac{1}{r}$ is also a root with the same multiplicity, then f is palindromic.*

Proof. Let $f(x) = p_{d-1}x^{d-1} + \cdots + p_1 x + p_0$. Let r be some root. Then $1/r$ is also a root. Hence

$$p_{d-1}\frac{1}{r^{d-1}} + \cdots + p_1\frac{1}{r} + p_0 = 0.$$

$$p_{d-1} + p_{d-2}r + \cdots + p_0 r^{d-1} = 0$$

Hence r is a root of

$$g(x) = p_{d-1} + p_{d-2}x + \cdots + p_0 x^{d-1}.$$

Therefore

$$f(x) = p_{d-1}x^{d-1} + \cdots + p_1 x + p_0$$

and

$$g(x) = p_0 x^{d-1} + p_1 x^{d-2} + \cdots + p_{d-1}$$

have the same roots (counting multiplicity). Hence $f(x)$ is a multiple of $g(x)$, say $f(x) = Ag(x)$. Note that $f(1)$ and $g(1)$ are both the sum of the coefficients, hence they are equal and nonzero. Therefore $A = f(1)/g(1) = 1$, so $f(x) = g(x)$. Thus we have f is palindromic. □

Definition 0.3. If $a+bi \in \mathbb{C}$ then $\overline{a+bi} = a - bi$ is the conjugate of $a+bi$. It is well known that if a $f \in \mathbb{R}[x]$ and $f(r) = 0$ then $f(\overline{r}) = 0$.

The following is well known and easy to prove. It is the $n = 2$ case of the AM-GM inequality.

Lemma 0.3. *For all $x, y \geq 0$, $\frac{x+y}{2} \geq \sqrt{xy}$.*

Theorem 0.2. *Let $M \geq 2$ and $d \geq 2$. There is no way to load M d-sided dice to produce fair sums.*

Proof. Assume, by way of contradiction, that $(p_{11}, \ldots, p_{1d})$, $(p_{21}, \ldots, p_{2d})$, $\ldots$ $(p_{M1}, \ldots, p_{Md})$, are a set of loaded M d-sided dice that yield fair sums. There are M d-sided dice so the possible rolls range from M 1's (total M) to M d's (total Md). Hence there are $Md - M + 1$ rolls. The probability of any numbers between M and Md is $\frac{1}{Md-M+1}$.

Notation 0.1. The probability that dice $M/2$ rolls an i we write as $p_{(M/2),i}$ rather than the rather ambiguous $p_{(M/2)i}$. We will also add the comma in other places where the meaning is unclear.

Elementary Proof

The probability of rolling an M is the probability that all of the die roll a 1. Hence

$$\frac{1}{Md-M+1} = \text{Prob}(M) = p_{11}\cdots p_{M1}$$

which yields

$$p_{11} = \frac{1}{Md-M+1}\frac{1}{p_{21}\cdots p_{M_1}}$$

The probability of rolling an Md is the probability that all of the die roll a d. Hence

$$\text{Prob}(Md) = p_{1d}\cdots p_{Md} = \frac{1}{Md-M+1}.$$

which yields

$$p_{1d} = \frac{1}{(Md-M+1)p_{2d}\cdots p_{Md}}$$

The proof from this point needs two cases.

Case 1 M **even** We will use that $\frac{M}{2}$ is an integer.

The probability of rolling an $(Md+M)/2$ is bounded below by the sum of the following (1) the prob that die 1 is a 1, dice $2,\ldots,M/2$ are 1's, and dice $(M/2)+1,\ldots,M$ are d's, and (2) the prob that die 1 is a d, dice $2,\ldots,M/2$ are d's and dice $(M/2)+1,\ldots,M$ are 1's. We write this as

$$\frac{1}{Md-M+1} = \text{Prob}\left(\frac{Md+M}{2}\right) \geq$$

$$p_{11}\cdots p_{(M/2),1}\times p_{(M/2+1),d}\cdots p_{Md} + p_{1d}\cdots p_{(M/2),d}\times p_{(M/2+1),1}\cdots p_{M1}.$$

(Note that if $d=1$ there would only be one term in the above summation. We leave it to the reader to determine where the proof fails.)

By using the equations for p_{11} and p_{1d} above we get

$$= \frac{p_{21}\cdots p_{(M/2),1}\times p_{(M/2+1),d}\cdots p_{Md}}{(Md-M+1)p_{21}\cdots p_{M1}} + \frac{p_{2d}\cdots p_{(M/2),d}\times p_{((M/2)+1),1}\cdots p_{M1}}{(Md-M+1)p_{2d}\cdots p_{Md}}$$

Note that lots of terms cancel and we can factor out the $\frac{1}{Md-M+1}$ to obtain

$$= \frac{1}{Md - M + 1}\left(\frac{p_{(M/2+1),d}\cdots p_{Md}}{p_{(M/2+1),1}\cdots p_{M1}} + \frac{p_{(M/2+1),1}\cdots p_{M1}}{p_{(M/2+1),d}\cdots p_{Md}}\right)$$

$$\geq \frac{2}{Md - M + 1} \text{ (This follows from Lemma 0.1.)}$$

Putting this all together we get

$$\frac{1}{Md - M + 1} > \frac{2}{Md - M + 1}$$

which is a contradiction.

Case 2 M **is odd** Using the same equations above we can come to a similar conclusion. We will assume that $p_{1,1} \geq p_{1,n}$ without loss of generality.

Knowing the probability of rolling $M \cdot d$ and using our assumption we arrive at the following inequality:

$$p_{1,1} \cdot p_{2,d} \cdots p_{M,d} \geq p_{1,d} \cdot p_{2,d} \cdots p_{M,d} = P(M \cdot d) = \frac{1}{Md - M + 1}$$

The probability of rolling $\left(\frac{M+1}{2} + \frac{M-1}{2} \cdot d\right)$ is at least the following and will also have an equal chance probability of $\frac{1}{Md-M+1}$

So

$$P\left(\frac{M+1}{2} + \frac{M-1}{2} \cdot d\right) \geq p_{1,1} \cdot p_{2,1} \cdots p_{\frac{M+1}{2},1} \cdot p_{\frac{M+3}{2},d} \cdots p_{M,d} + p_{1,1} \cdot p_{2,d} \cdots p_{\frac{M+1}{2},d} \cdot p_{\frac{M+3}{2},1} \cdots p_{M,1}$$

$$\frac{1}{Md - M + 1} \geq p_{1,1} \cdot p_{2,1} \cdots p_{\frac{M+1}{2},1} \cdot p_{\frac{M+3}{2},d} \cdots p_{M,d} + p_{1,1} \cdot p_{2,d} \cdots p_{\frac{M+1}{2},d} \cdot p_{\frac{M+3}{2},1} \cdots p_{M,1}$$

Using Lemma 0.3

$$\frac{1}{Md - M + 1} \geq 2 \cdot \sqrt{p_{1,1} \cdot p_{2,1} \cdots p_{\frac{M+1}{2},1} \cdot p_{\frac{M+3}{2},d} \cdots p_{M,d} \cdot p_{1,1} \cdot p_{2,d} \cdots p_{\frac{M+1}{2},d} \cdot p_{\frac{M+3}{2},1} \cdots p_{M,1}}$$

$$\frac{1}{Md - M + 1} \geq 2 \cdot \sqrt{\left(p_{1,1} \cdot p_{2,1} \cdots p_{\frac{M+1}{2},1} \cdot p_{\frac{M+3}{2},1} \cdots p_{M,1}\right) \cdot \left(\cdot p_{1,1} \cdot p_{2,d} \cdots p_{\frac{M+1}{2},d} \cdot p_{\frac{M+3}{2},d} \cdots p_{M,d}\right)}$$

$$\frac{1}{Md - M + 1} \geq 2 \cdot \sqrt{\left(\frac{1}{Md - M + 1}\right)^2}$$

$$\frac{1}{Md-M+1} \geq \frac{2}{Md-M+1}$$

We arrive at a contradiction $\therefore$ We are done with the case where M is odd.

Generating Function Proof

Let the generating function for the ith dice be $P_i(x) = p_{i1}x + p_{i2}x^2 + \cdots + p_{id}x^d$. If each sum occurs with equal probability, we have

$$P_1(x)\cdots P_M(x) = \frac{1}{Md-M+1}(x^M + \cdots + x^{Md}) \tag{0.1}$$

which is equivalent to

$$(p_{11} + \cdots + p_{1d}x^{d-1})\cdots(p_{M1}x + \cdots + p_{Md}x^{d-1}) = \frac{x^{Md-M+1}-1}{(Md-M+1)(x-1)}$$

Let $Q_i(x) = p_{i1} + p_{i2}x + \cdots + p_{id}x^{d-1}$. We rewrite the equation above as

$$Q_1(x)Q_2(x)\cdots Q_M(x) = \frac{x^{Md-M+1}-1}{(Md-M+1)(x-1)}$$

The roots of the RHS are all of the $(Md-M+1)$-roots of unity except 1. Thus, the roots of $Q_1(x)Q_2(x)\cdots Q_M(x)$ come in conjugate pairs, except possibly -1.

We consider two cases

Case 1: d is even. Since $Q_i(x)$ is of degree $d-1$ which is odd, $Q_i(x)$ has an odd number of roots. Since the nonreal roots of $Q_i(x)$ come in conjugate pairs, -1 must be a root of $Q_i(x)$. Hence, $Q_1(x)Q_2(x)\cdots Q_M(x)$ has root -1 with multiplicity $M > 1$. This is a contradiction.

Case 2: d is odd. $Md - M + 1$ is odd, so -1 is not a root. Thus, all roots of $Q_i(x)$ come in conjugate pairs. Note that r is a root of $Q_i(x)$ if and only if $\overline{r} = \frac{1}{r}$ is also a root of $Q_i(x)$. Hence, by Lemma 0.2, $Q_i(x)$ is palindromic. In particular $p_{i1} = p_{id}$.

By equating the the coefficients of x^{Md} and $x^{(M-1)d+1}$ in equation (0.1), we have

$$p_{1d}\cdots p_{Md} = \frac{1}{Md-M+1}$$

and

$$p_{11}p_{2d}\cdots p_{Md}+p_{1d}p_{21}p_{3d}\cdots p_{Md}+\text{other nonnegative terms} = \frac{1}{Md-M+1}.$$

By using that $p_{11} = p_{1d}$ and $p_{21} = p_{2d}$ we have

$$p_{1d}p_{2d}\cdots p_{Md}+p_{1d}p_{2d}p_{3d}\cdots p_{Md}+\text{other nonnegative terms} = \frac{1}{Md-M+1}.$$

Hence

$$\frac{2}{Md-M+1} + \text{other nonnegative terms} = \frac{1}{Md-M+1},$$

which is a contradiction. □

0.6 Many Dice

Using generating functions one can determine *exactly* when one can obtain fair sums. That is, given any set of types of dice (e.g., four 3-sided dice, six 10-sided dice, and four 11-sided dice), one can determine if they can be loaded to obtain fair sums. This was done by Gasarch and Kruskal [Gasarch and Kruskal (1999)] and later refined by Morrison [Morrison (2014)]. Gasarch and Kruskal proved that there exists a way to load a set of dice iff there is a way to load them so that every sum occurs exactly once. They also gave an algorithm to determine this. Morison gave a faster algorithm. We leave it as an exercise to load a 2-sided die and a 4-sided die to obtain fair sums.

Bibliography

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