Approximate Fair Loaded Dice
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1 Numerics

We have shown that there do not exist m n-sided die with all sums equally likely. The main question now is: how close can we get to having equal sums?

First we need a concrete measurement of being close to "equally likely". Perhaps the most natural metric is the variance of our sums. Let $k = mn - m + 1$. If the probabilities of the sums occuring are $x_1, \ldots, x_k$, the sums are close to equally likely when the variance

$$\frac{\left(x_1 - \frac{1}{k}\right)^2 + \cdots + \left(x_k - \frac{1}{k}\right)^2}{k}$$

is close to 0.

Another natural metric is the sum of squares of differences. That is, the smaller

$$\sum_{1 \le i < j \le k} (x_i - x_j)^2$$

is, the more equally likely the sums are. Since $x_1 + \cdots + x_k = 1$, we have

$$\begin{aligned} \frac{\left(x_1 - \frac{1}{k}\right)^2 + \cdots + \left(x_k - \frac{1}{k}\right)^2}{k} &= \frac{x_1^2 - \frac{2x_1}{k} + \frac{1}{k^2} + \cdots + x_k^2 - \frac{2x_k}{k} + \frac{1}{k^2}}{k} \\ &= \frac{x_1^2 + \cdots + x_k^2 - \frac{2}{k}(x_1 + \cdots + x_k) + \frac{k}{k^2}}{k} \\ &= \frac{x_1^2 + \cdots + x_k^2 - \frac{2}{k}(1) + \frac{1}{k}}{k} \\ &= \frac{x_1^2 + \cdots + x_k^2}{k} - \frac{1}{k^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{1 \le i < j \le k} (x_i - x_j)^2 &= \sum_{1 \le i < j \le k} (x_i^2 - 2x_i x_j + x_j^2) \\ &= (k-1)(x_1^2 + \cdots + x_k^2) - 2 \sum_{1 \le i \not< j \le k} x_i x_j \\ &= (k-1)(x_1^2 + \cdots + x_k^2) - \left[(x_1 + \cdots + x_k)^2 - (x_1^2 + \cdots + x_k^2)\right] \\ &= k(x_1^2 + \cdots + x_k^2) - 1. \end{aligned}$$

Therefore, minimizing the variance or sum of squares of differences is equivalent to minimizing the sum of squares $x_1^2 + \cdots + x_k^2$ metric. Therefore, we focus on minimizing the sum of squares.

Using mathematica, we have computed the dice which minimize the sum of squares of sums for small pairs of m and n ($(m,n) = (2,2),(2,3),(2,4),(3,2),(4,2)$). We have also approximately computed the dice for bigger pairs of m and n using java programming, and then extrapolated the fractions from the decimals. Our data strongly supports the following conjecture.

Conjecture 1.1. *The set of m n-sided dice with the minimum variance (or sum of squares) is*

	$P(1)$	$P(2)$	$\cdots$	$P(n-1)$	$P(n)$
dice 1	$\frac{m}{m\cdot(2n-2)-n+2}$	$\frac{2m-1}{m\cdot(2n-2)-n+2}$	$\cdots$	$\frac{2m-1}{m\cdot(2n-2)-n+2}$	$\frac{m}{m\cdot(2n-2)-n+2}$
dice 2	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
dice 3	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
$\vdots$			$\cdots$		
dice $m-1$	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
dice m	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$

The minimum sum of squares for a set of m n-sided dice is $\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m+(n-2)(2m-1))}$ (achieved by the above dice).

For these dice, the probability of rolling a sum of m is $\frac{m}{2^{m-1}(2m+(n-2)(2m-1))}$. The probability of rolling a sum of $m+1,\ldots,n+m-3$, or $n+m-2$ is each $\frac{(2m-1)}{2^{m-1}(2m+(n-2)(2m-1))}$. In general, the probability of rolling a sum of $in+m-i$ is $\frac{\binom{m}{i}m}{2^{m-1}(2m+(n-2)(2m-1)}$ and the probability of rolling a sum of $in+m-i+1,\ldots,(i+1)n+m-i-2$ is each $\frac{\binom{m-1}{i}(2m-1)}{2^{m-1}(2m+(n-2)(2m-1))}$. Therefore, the sum of squares for these dice is

$$\frac{m^2+\binom{m}{1}m^2+\cdots+\binom{m}{m}m^2+(n-2)(2m-1)^2+(n-2)\binom{m-1}{1}^2(2m-1)^2+\cdots+(n-2)\binom{m-1}{m-1}^2(2m-1)^2}{2^{2m-2}(2m+(n-2)(2m-1))^2}$$
$$=\frac{m^2\left(\binom{m}{0}^2+\cdots+\binom{m}{m}^2\right)+(n-2)(2m-1)^2\left(\binom{m-1}{0}^2+\cdots+\binom{m-1}{m-1}^2\right)}{2^{2m-2}(2m+(n-2)(2m-1))^2}$$
$$=\frac{m^2\binom{2m}{m}+(n-2)(2m-1)^2\frac{2m-2}{m-1}}{2^{2m-2}(2m+(n-2)(2m-1))^2}$$
$$=\frac{m^2\frac{(2m)!}{(m!)^2}+(n-2)(2m-1)^2\frac{(2m-2)!}{((m-1)!)^2}}{2^{2m-2}(2m+(n-2)(2m-1))^2}$$
$$=\frac{(2m+(n-2)(2m-1))\frac{(2m-1)!}{((m-1)!)^2}}{2^{2m-2}(2m+(n-2)(2m-1))^2}$$
$$=\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m+(n-2)(2m-1))}$$

Let $\mu=\frac{1}{mn-m+1}$ be the mean. The coefficient of variance is

$$\frac{\sqrt{\frac{x_1^2+\cdots+x_k^2}{k}-\frac{1}{k^2}}}{\mu}=\frac{\sqrt{\mu\cdot\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m+(n-2)(2m-1))}-\mu^2}}{\mu}=\sqrt{\frac{(2m-1)!(mn-m+1)}{2^{2m-2}((m-1)!)^2(2m+(n-2)(2m-1))}-1}.$$

For fixed m, we can see that the coefficient of variance is decreasing in terms of n and that when n goes to ∞, the coefficient of variance approaches

$$\sqrt{-1 + \frac{m}{2^{2m-2}} \binom{2m-2}{m-1}}.$$

Note that this is positive for any $m > 2$ and is 0 when $m = 2$. We can also see that for fixed n, the coefficient of variance is increasing in m and approaches ∞ as $m \to \infty$ (although rather slowly). Therefore, for bigger m or n, the sums become less equally likely. Also, we can only get equally likely sums when $m = 1$ or when $m = 2$ and $n \to \infty$. Our computations for the coefficient of variance for a few pairs of m and n support our observations:

Coefficient of Variance Data

	m number of dice				
n sides	2	3	4	5	6
2	0.3536	0.5000	0.6060	0.6903	0.7610
3	0.2673	0.4395	0.5590	0.6517	0.7281
4	0.2236	0.4146	0.5409	0.6374	0.7161
5	0.1961	0.4009	0.5313	0.6299	0.7099
6	0.1768	0.3922	0.5254	0.6253	0.7061
7	0.1622	0.3863	0.5213	0.6222	0.7035
8	0.1508	0.3819	0.5184	0.6199	0.7017
9	0.1414	0.3785	0.5162	0.6182	0.7003
10	0.1336	0.3759	0.5144	0.6169	0.6992
100	0.0410	0.3557	0.5013	0.6070	0.6912
1000	0.0129	0.3538	0.5001	0.6061	0.6904
Limit	0	0.3536	0.5000	0.6060	0.6903

Another way to measure how equally likely the sums are is to minimize the range of our sums. That is, the smaller $\max_{1 \le i < j \le k} |x_i - x_j|$ is, the closer the sums are to uniformly likely. Again, we were able to find the dice which minimize the range for small pairs of m and n with mathematica, and we were able to use java programming to approximately verify this for bigger values of m and n. Our data strongly supports the following conjecture.

Conjecture 1.2. *The set of m n-sided dice with the minimum range is*

	$P(1)$	$P(2)$	$\cdots$	$P(n-1)$	$P(n)$
dice 1	$\frac{\lfloor \frac{m+1}{2} \rfloor}{2\lfloor \frac{m+1}{2} \rfloor + m(n-2)}$	$\frac{m}{2\lfloor \frac{m+1}{2} \rfloor + m(n-2)}$	$\cdots$	$\frac{m}{2\lfloor \frac{m+1}{2} \rfloor + m(n-2)}$	$\frac{\lfloor \frac{m+1}{2} \rfloor}{2\lfloor \frac{m+1}{2} \rfloor + m(n-2)}$
dice 2	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
dice 3	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
$\vdots$			$\cdots$		
dice $m-1$	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$
dice m	$\frac{1}{2}$	0	$\cdots$	0	$\frac{1}{2}$

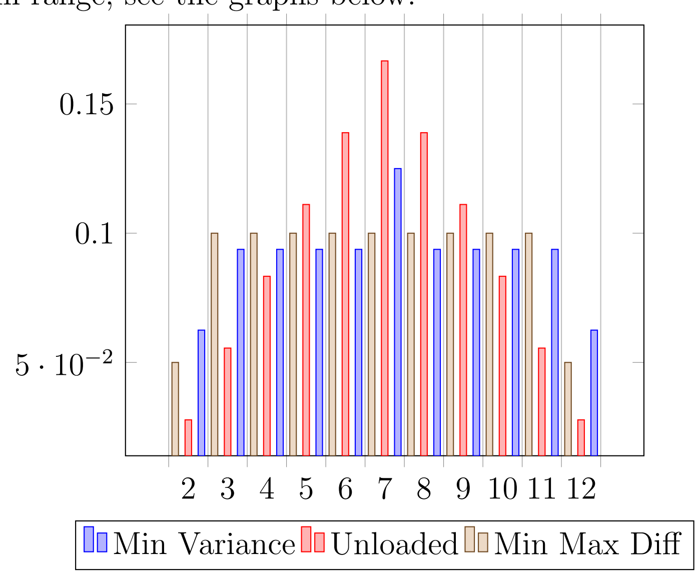
The minimum range for a set of m n-sided dice is

$$\frac{\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}m - \lfloor\frac{m+1}{2}\rfloor}{2^{m-1}\left(2\lfloor\frac{m+1}{2}\rfloor + m(n-2)\right)}.$$

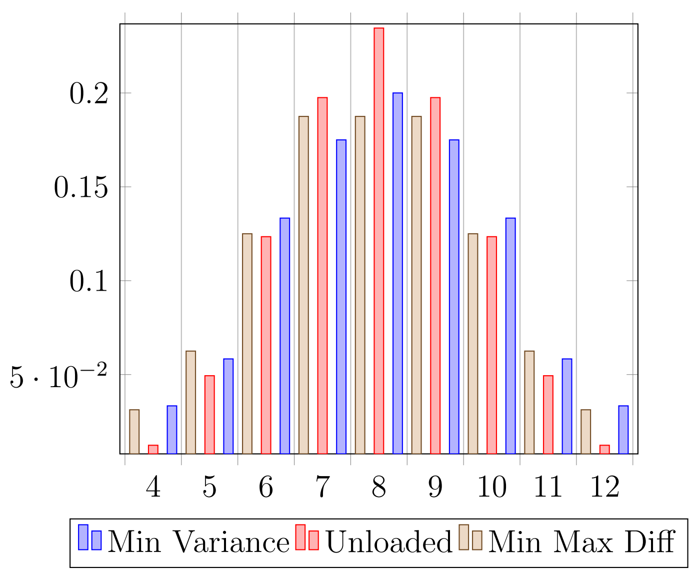
We can see that the probability of rolling a sum of $m + in - i$ is $\frac{\binom{m}{i}\lfloor\frac{m+1}{2}\rfloor}{2^{m-1}\left(2\lfloor\frac{m+1}{2}\rfloor+m(n-2)\right)}$ and the probability of rolling a sum of $m + in - i + 1, \ldots, m + (i+1)n - i - 2$ is each $\frac{\binom{m-1}{i}m}{2^{m-1}\left(2\lfloor\frac{m+1}{2}\rfloor+m(n-2)\right)}$. It is not hard to see that the range of these sums is

$$\frac{\binom{m-1}{\lfloor\frac{m-1}{2}\rfloor}m - \lfloor\frac{m+1}{2}\rfloor}{2^{m-1}\left(2\lfloor\frac{m+1}{2}\rfloor + m(n-2)\right)}.$$

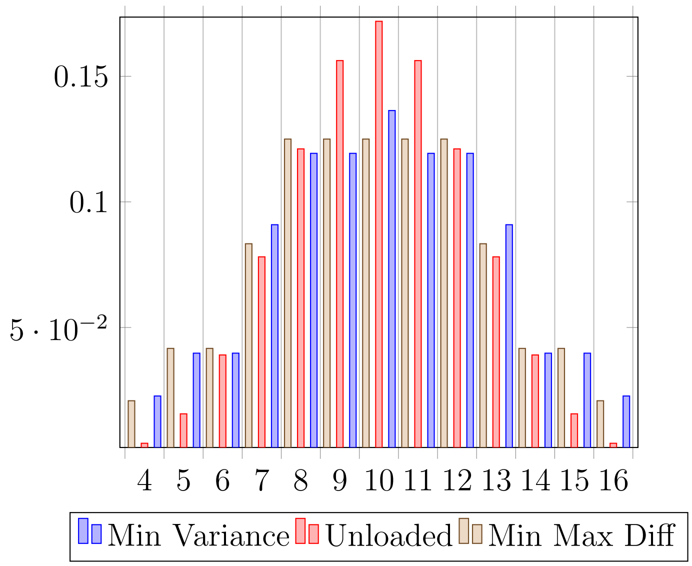
For visual comparison of the distribution of sums for our dice for minimum variance and minimum range, see the graphs below.



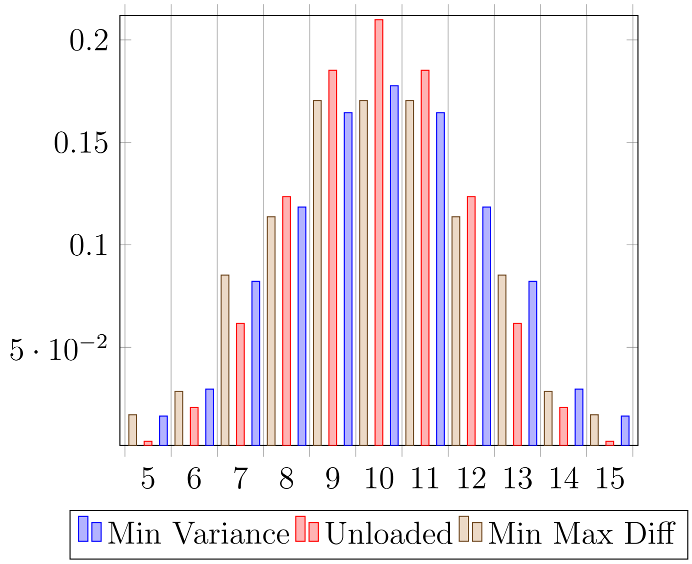
2d6



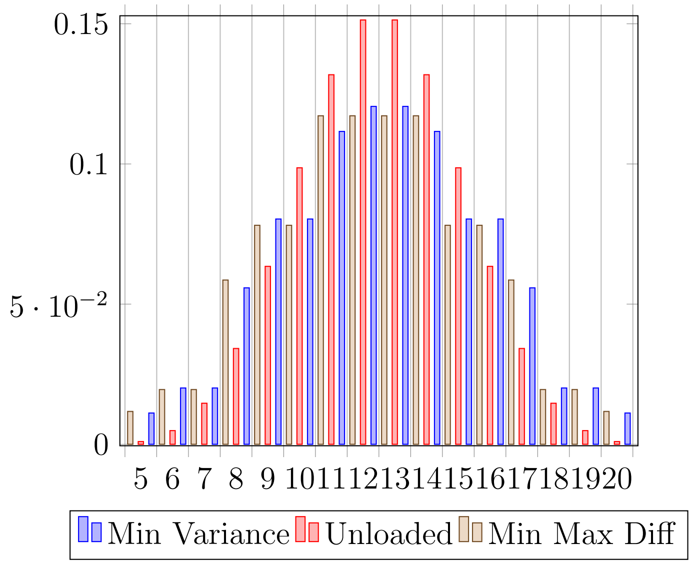
4d3


0.15
0.1
$5 \cdot 10^{-2}$
4 5 6 7 8 9 10 11 12 13 14 15 16
Min Variance
Unloaded
Min Max Diff

4d4


0.2
0.15
0.1
$5 \cdot 10^{-2}$
5 6 7 8 9 10 11 12 13 14 15
Min Variance
Unloaded
Min Max Diff

5d3


0.15
0.1
$5 \cdot 10^{-2}$
0
5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Min Variance
Unloaded
Min Max Diff

5d4

2 Proof of two sides case of conjecture

First we prove the $n = 2$ special case of Conjecture 1.1.

Theorem 2.1. *The following m two-sided dice (the $n = 2$ case of Conjecture 1.1)*

	$P(1)$	$P(2)$
dice 1	$\frac{1}{2}$	$\frac{1}{2}$
dice 2	$\frac{1}{2}$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$
dice m	$\frac{1}{2}$	$\frac{1}{2}$

minimizes the sum of squares (or variance).

Proof. Let our m dice be $p_1(x) = a_1 + (1 - a_1)x, \dots, p_m(x) = a_m + (1 - a_m)x$. It suffices to show that $a_1 = \cdots = a_m = \frac{1}{2}$ is the global minimum for the sum of squares S. Note that the sum of squares S can be represented by

$$S = \frac{1}{2\pi i}\int_{|x|=1} \frac{(a_1 + (1-a_1)x)\cdots(a_n + (1-a_m)x)(a_1 + \frac{1-a_1}{x})\cdots(a_n + \frac{1-a_m}{x})}{x}dx.$$

We will first show that $a_1 = \cdots = a_m = \frac{1}{2}$ is the only critical point of S. Note that

$$\begin{aligned}\frac{\partial S}{\partial a_1} &= \frac{1}{2\pi i}\int_{|x|=1} \frac{\left[4a_1 - 2 + (1-2a_1)\left(\frac{1}{x} + x\right)\right]\prod_{j=2}^{n}\left[(a_j + (1-a_j)x)(a_j + \frac{1-a_j}{x})\right]}{x}dx \\ &= \frac{2a_1 - 1}{2\pi i}\int_{|x|=1} \frac{(2 - \frac{1}{x} - x)\prod_{j=2}^{n}\left[(a_j + (1-a_j)x)(a_j + \frac{1-a_j}{x})\right]}{x}dx\end{aligned}$$

This partial vanishes when $a_1 = \frac{1}{2}$. Similarly $\frac{\partial S}{\partial a_j}$ vanishes when $a_j = \frac{1}{2}$. Therefore, $a_1 = \cdots = a_m = \frac{1}{2}$ is a critical point.

Now we show that there are no other critical points. We show that $\frac{\partial S}{\partial a_1} \neq 0$ when $a_1 \neq \frac{1}{2}$. Note that

$$\begin{aligned}\frac{1}{2\pi i}\int_{|x|=1} \frac{\prod_{j=2}^{n}\left[(a_j + (1-a_j)x)(a_j + \frac{1-a_j}{x})\right]}{x}dx &= \frac{1}{2\pi i}\int_{|x|=1} \frac{\prod_{j=2}^{n}\left[(a_j + (1-a_j)x)(a_j + (1-a_j)\overline{x})\right]}{x}dx \\ &= \frac{1}{2\pi i}\int_{|x|=1} \frac{\prod_{j=2}^{n}\left|(a_j + (1-a_j)x)\right|^2}{x}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\prod_{j=2}^{n}\left|(a_j + (1-a_j)e^{i\theta})\right|^2}{e^{i\theta}}ie^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \prod_{j=2}^{n}\left|(a_j + (1-a_j)e^{i\theta})\right|^2 d\theta,\end{aligned}$$

and that

$$\frac{1}{2\pi i}\int_{|x|=1}\frac{\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]}{x^2}dx,\ \frac{1}{2\pi i}\int_{|x|=1}\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]dx$$

are the linear and constant coefficients of $\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]$. Therefore both are real. Thus we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{|x|=1}\frac{\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]}{x^2}dx &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|x|=1}\frac{\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]}{x^2}dx\right)\\ &= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\prod_{j=2}^{n}(a_j+(1-a_j)e^{i\theta})\right|^2}{e^{i\theta}}d\theta\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\prod_{j=2}^{n}(a_j+(1-a_j)e^{i\theta})\right|^2\cos\theta d\theta\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2\pi i}\int_{|x|=1}\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]dx &= \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|x|=1}\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]dx\right)\\ &= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|\prod_{j=2}^{n}(a_j+(1-a_j)e^{i\theta})\right|^2 e^{i\theta}d\theta\right)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\left|\prod_{j=2}^{n}(a_j+(1-a_j)e^{i\theta})\right|^2\cos\theta d\theta.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\partial S}{\partial a_1} &= (2a_1-1)\frac{1}{2\pi i}\int_{|x|=1}\frac{(2-\frac{1}{x}-x)\prod_{j=2}^{n}\left[(a_j+(1-a_j)x)(a_j+\frac{1-a_j}{x})\right]}{x}dx\\ &= 2(2a_1-1)\frac{1}{2\pi}\int_0^{2\pi}\left|\prod_{j=2}^{n}(a_j+(1-a_j)e^{i\theta})\right|^2(1-\cos\theta)d\theta\end{aligned}$$

Note that the integrand is positive when $\theta=\frac{\pi}{2}$. Since the integrand is continuous and nonnegative, the integral is strictly positive. Therefore, $\frac{\partial S}{\partial a_1}\neq 0$ unless $a_1=\frac{1}{2}$. Thus $a_i=\frac{1}{2}$ for all i is the only critical point.

Now we are ready to prove our theorem by induction. The base case of $m = 1$ is trivial. Suppose our theorem is true for some $m - 1 \geq 1$. For m dice, since $a_i = \frac{1}{2}$ for all i is the only critical point, we know that the global minimum occurs when $a_i = 0$ or $a_i = 1$ for some i (the boundary) or when $a_i = \frac{1}{2}$ for all i. By our inductive hypothesis, our first case is minimized when $a_i = 0$, $a_j = \frac{1}{2}$ for all $j \neq i$. As we computed earlier, the sum of squares for our first case is

$$\frac{(2m-3)!}{2^{2m-4}((m-2)!)^2((2m-2)}$$

and the sum of squares for our second case is

$$\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m)}.$$

Since

$$\frac{\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m)}}{\frac{(2m-3)!}{2^{2m-4}((m-2)!)^2((2m-2)}} = \frac{(2m-1)(2m-2)}{2^2(m-1)^2\left(\frac{2m}{2m-2}\right)} = \frac{2m-1}{2m} < 1$$

we see that the global minimum for the m dice two sides case is $a_j = \frac{1}{2}$ for all j. Our induction is complete. □

3 Local Minimum Proof for two dice

Next we partially prove a different special case of Conjecture 1.1. Specifically, we prove that the $m = 2$ case given in Conjecture 1.1 is a local minimizer for the sum of squares.

Theorem 3.1. *The following two n-sided dice (which is the $m = 2$ case of Conjecture 1.1)*

	$P(1)$	$P(2)$	$\cdots$	$P(n-1)$	$P(n)$
dice 1	$\frac{2}{3n-2}$	$\frac{3}{3n-2}$	$\cdots$	$\frac{3}{3n-2}$	$\frac{2}{3n-2}$
dice 2	$\frac{1}{2}$	*0*	$\cdots$	*0*	$\frac{1}{2}$

are a local minima for the sum of squares (and variance).

Proof. Our two dice are $p_1 + p_2x + \cdots + p_nx^{n-1}$ and $q_1 + q_2x + \cdots + q_nx^{n-1}$ and we will show that $p_1 = p_n = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, q_1 = q_n = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0$ are local minimizers of

$$S = (p_1q_1)^2 + (p_1q_2 + p_2q_1)^2 + \cdots + (p_nq_n)^2$$

subject to the constraints

$$p_1 + \cdots + p_n = 1$$

$$q_1 + \cdots + q_n = 1$$

$$p_1, p_2, \ldots, p_n \geq 0$$

$$q_1, q_2, \ldots, q_n \geq 0.$$

First note that

$$S = \frac{1}{2\pi i}\int_{|x|=1} \frac{P(x)Q(x)P(\frac{1}{x})Q(\frac{1}{x})}{x}dx$$

and we have the Lagrangian

$$L = S+\lambda(p_1+\cdots+p_n-1)+\mu(q_1+\cdots+q_n-1)+\lambda_1(-p_1)+\cdots+\lambda_n(-p_n)+\mu_1(-q_1)+\cdots+\mu_n(-q_n)$$

where λ and μ are Lagrange multipliers and $\lambda_1, \ldots, \lambda_n, \mu_1 \geq 0, \ldots, \mu_n \geq 0$ are Karush-Kuhn-Tucker multipliers. We will first show that $p_1 = p_n = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, q_1 = q_n = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0$, $\lambda = -\frac{3}{3n-2}$, $\mu = -\left(\frac{2}{3n-2}\right)^2 [3 + \left(\frac{3}{2}\right)^2 (n-2)]$, $\lambda_1 = \cdots = \lambda_n = 0$, $\mu_1 = \mu_n = 0, \mu_2 = \cdots = \mu_{n-1} = \frac{3}{(3n-2)^2}$ is a solution of the Karush Kuhn Tucker equations

$$\frac{\partial S}{\partial p_\ell} + \lambda - \lambda_\ell = 0$$

$$\frac{\partial S}{\partial q_\ell} + \mu - \mu_\ell = 0$$

$$\lambda_1(-p_1) + \cdots + \lambda_n(-p_n) + \mu_1(-q_1) + \cdots + \mu_n(-q_n) = 0$$

$$p_1 + p_2 + \cdots + p_n - 1 = 0$$

$$q_1 + q_2 + \cdots + q_n - 1 = 0$$

$$-p_1 \leq 0, \ldots, -p_n \leq 0$$

$$-q_1 \leq 0, \ldots, -q_n \leq 0.$$

First observe that

$$\begin{aligned}
\frac{\partial S}{\partial p_\ell} &= \frac{1}{2\pi i}\int_{|x|=1} \left[P\left(\frac{1}{x}\right)Q(x)Q\left(\frac{1}{x}\right)x^{\ell-2} + P(x)Q(x)Q\left(\frac{1}{x}\right)\frac{1}{x^\ell}\right]dx \\
&= \frac{1}{4}\left(\frac{6}{3n-2}\right) + \frac{1}{4}\left(\frac{6}{3n-2}\right) \\
&= \frac{3}{3n-2} = \lambda_\ell - \lambda
\end{aligned}$$

and that

$$\frac{\partial S}{\partial q_1} = \frac{1}{2\pi i}\int_{|x|=1}\left[P(x)P\left(\frac{1}{x}\right)Q\left(\frac{1}{x}\right)\frac{1}{x} + P(x)P\left(\frac{1}{x}\right)Q(x)\frac{1}{x}\right]dx$$
$$= \left(\frac{2}{3n-2}\right)^2\left[1^2 + \left(\frac{3}{2}\right)^2(n-2) + 1^2\right] + \left(\frac{2}{3n-2}\right)^2\left(\frac{1+1}{2}\right)$$
$$= \left(\frac{2}{3n-2}\right)^2\left[3 + \left(\frac{3}{2}\right)^2(n-2)\right]$$
$$= -\mu,$$

$$\frac{\partial S}{\partial q_1} = \frac{1}{2\pi i}\int_{|x|=1}\left[P(x)P\left(\frac{1}{x}\right)Q\left(\frac{1}{x}\right)x^{n-2} + P(x)P\left(\frac{1}{x}\right)Q(x)\frac{1}{x^n}\right]dx$$
$$= \left(\frac{2}{3n-2}\right)^2\left[1^2 + \left(\frac{3}{2}\right)^2(n-2) + 1^2\right] + \left(\frac{2}{3n-2}\right)^2\left(\frac{1+1}{2}\right)$$
$$= \left(\frac{2}{3n-2}\right)^2\left[3 + \left(\frac{3}{2}\right)^2(n-2)\right]$$
$$= -\mu,$$

and

$$\frac{\partial S}{\partial q_\ell} = \frac{1}{2\pi i}\int_{|x|=1}\left[P(x)P\left(\frac{1}{x}\right)Q\left(\frac{1}{x}\right)x^{\ell-1} + P(x)P\left(\frac{1}{x}\right)Q(x)\frac{1}{x^\ell}\right]dx$$
$$= \left(\frac{2}{3n-2}\right)^2\left[\frac{3}{2} + \left(\frac{3}{2}\right)^2(n-3) + \frac{3}{2}\right] + \left(\frac{2}{3n-2}\right)^2\left(\frac{3}{2}+\frac{3}{2}\right)$$
$$= \left(\frac{2}{3n-2}\right)^2\left[6 + \left(\frac{3}{2}\right)^2(n-3)\right]$$
$$= \left(\frac{2}{3n-2}\right)^2\left[3 + \left(\frac{3}{2}\right)^2(n-2) + \frac{3}{4}\right]$$
$$= -\mu + \mu_\ell.$$

Thus we have shown that our conjectured dice satisfy the Karush-Kuhn-Tucker condtions, which means that it is a critical point. Now we use the Hessian to show that it is indeed a strict local minimizer.

Since the constraint equations and constraint inequalities are all linear, we have $\mathrm{Hess}(L) = \mathrm{Hess}(S)$. We consider $y = (y_1 \cdots y_{2n})^T$ such that

$$\mathrm{grad}(p_1 + p_2 + \cdots + p_n - 1)y = 0,$$

$$\text{grad}(q_1 + q_2 + \cdots + q_n - 1)y = 0$$

and

$$\text{grad}(-q_j)y = 0$$

for $j = 2, \ldots, n-1$ because $\mu_2, \ldots, \mu_{n-1} > 0$. These equations imply $y_{n+2} = \cdots = y_{2n-1} = 0$, $y_1 + \cdots + y_n = 0$, and $y_{n+1} + y_{2n} = 0$.

$$\begin{aligned}
&y^T(\text{Hess}(L))y \\
&= y^T(\text{Hess}(S))y \\
&= \sum_{k,\ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \sum_{k,\ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_\ell} y_k y_{n+\ell} + \sum_{k,\ell=1}^{n} \frac{\partial^2 S}{\partial q_k \partial q_\ell} y_{n+k} y_{n+\ell} \\
&= \sum_{k,\ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_1} y_k y_{n+1} + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_n} y_k y_{2n} + \frac{\partial^2 S}{\partial q_1^2} y_{n+1}^2 + 2\frac{\partial^2 S}{\partial q_1 \partial q_n} y_{n+1} y_{2n} + \frac{\partial^2 S}{\partial q_n^2} y_n^2
\end{aligned}$$

where the last equality follows from the fact $y_{n+2} = \cdots = y_{2n-1} = 0$. We have

$$\begin{aligned}
\frac{\partial^2 S}{\partial p_k \partial p_\ell} &= \frac{1}{2\pi i} \int_{|x|=1} \left[Q(x) Q\left(\frac{1}{x}\right) x^{k-\ell-1} + Q(x) Q\left(\frac{1}{x}\right) x^{\ell-1-k} \right] dx \\
&= \frac{1}{2\pi i} \int_{|x|=1} \frac{1}{4} \left(2 + x^{n-1} + \frac{1}{x^{n-1}} \right) \left(x^{k-\ell-1} + x^{\ell-1-k} \right) dx \\
&= \begin{cases} 1 & \text{if } k = \ell \\ \frac{1}{2} & \text{if } k = n,\ \ell = 1 \\ \frac{1}{2} & \text{if } k = 1,\ \ell = n \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

The matrix $\left(\frac{\partial^2 S}{\partial p_k \partial p_\ell}\right)$ is positive definite by checking principle minors.

We can also check that

$$\frac{\partial^2 S}{\partial p_k \partial q_1} = \frac{\partial^2 S}{\partial p_k \partial q_n}$$

at $p_1 = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, p_n = \frac{2}{3n-2}, q_1 = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0, q_n = \frac{1}{2}$. Hence,

$$y^T \text{Hess}(L) y = \sum_{k,\ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \frac{\partial^2 S}{\partial q_1^2} y_{n+1}^2 + 2\frac{\partial^2 S}{\partial q_1 \partial q_n} y_{n+1} y_{2n} + \frac{\partial^2 S}{\partial q_n^2} y_{2n}^2.$$

We also have

$$\frac{\partial^2 S}{\partial q_1^2} = \frac{1}{2\pi i}\int_{|x|=1}\left[P(x)P\left(\frac{1}{x}\right) + P(x)P\left(\frac{1}{x}\right)\right]\frac{1}{x}dx = 2\left(\frac{2}{3n-2}\right)^2\left[1^2 + \left(\frac{3}{2}\right)^2(n-2) + 1^2\right],$$

$$\frac{\partial^2 S}{\partial q_n^2} = \frac{1}{2\pi i}\int_{|x|=1}\left[P(x)P\left(\frac{1}{x}\right)\frac{x^{n-2}}{x^{n-1}} + P(x)P\left(\frac{1}{x}\right)\frac{x^{n-1}}{x^n}\right]dx = 2\left(\frac{2}{3n-2}\right)^2\left[1^2 + \left(\frac{3}{2}\right)^2(n-2) + 1^2\right],$$

$$\frac{\partial^2 S}{\partial q_1 \partial q_n} = \frac{1}{2\pi i}\int_{|x|=1}\left(P(x)P\left(\frac{1}{x}\right)\frac{1}{x^n} + P(x)P\left(\frac{1}{x}\right)x^{n-2}\right)dx = \left(\frac{2}{3n-2}\right)^2[1+1].$$

Finally,

$$\begin{aligned}
y^T\text{Hess}(L)y &= [y_1 \cdots y_n]\left(\frac{\partial^2 S}{\partial q_k \partial q_\ell}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \left(2\frac{\partial^2 S}{\partial q_1^2} - 2\frac{\partial^2 S}{\partial q_1 \partial q_n}\right)y_{n+1}^2 \\
&= [y_1 \cdots y_n]\left(\frac{\partial^2 S}{\partial q_k \partial q_\ell}\right)\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + 2\left(\frac{2}{3n-2}\right)^2\left[\left(\frac{3}{2}\right)^2(n-2) + 1\right]y_{n+1}^2.
\end{aligned}$$

Recall that $\left(\frac{\partial^2 S}{\partial q_k \partial q_\ell}\right)$ is positive definite. Thus, if $y_{n+1} \neq 0$, we have $y^T\text{Hess}(L)y \geq 2\left(\frac{2}{3n-2}\right)^2\left[\left(\frac{3}{2}\right)^2(n-2)+1\right]y_{n+1}^2 > 0$. If $y_{n+1} = 0$, then since $y_{n+1} + y_{2n} = 0$ and $(y_1 \cdots y_n, y_{n+1}, 0 \cdots 0, y_{2n})^T \neq \vec{0}$, we have $y_{2n} = 0$ and thus $(y_1 \cdots y_n)^T \neq 0$. Therefore, we have $y^T\text{Hess}(L)y = y^T\left(\frac{\partial^2 S}{\partial q_k \partial q_\ell}\right)y > 0$. Thus we have proved that our conjectured dice are indeed a strict local minimizer. □