Approximate Fair Loaded Dice
by
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1 Numerics

We have shown that there do not exist \( mn \)-sided die with all sums equally likely. The main question now is: how close can we get to having equal sums?

First we need a concrete measurement of being close to “equally likely”. Perhaps the most natural metric is the variance of our sums. Let \( k = mn - m + 1 \). If the probabilities of the sums occurring are \( x_1, \ldots, x_k \), the sums are close to equally likely when the variance

\[
\frac{(x_1 - \frac{1}{k})^2 + \cdots + (x_k - \frac{1}{k})^2}{k}
\]

is close to 0.

Another natural metric is the sum of squares of differences. That is, the smaller

\[
\sum_{1 \leq i < j \leq k} (x_i - x_j)^2
\]

is, the more equally likely the sums are. Since \( x_1 + \cdots + x_k = 1 \), we have

\[
\frac{(x_1 - \frac{1}{k})^2 + \cdots + (x_k - \frac{1}{k})^2}{k} = \frac{x_1^2 - \frac{2x_1}{k} + \frac{1}{k^2} + \cdots + x_k^2 - \frac{2x_k}{k} + \frac{1}{k^2}}{k} = \frac{x_1^2 + \cdots + x_k^2}{k} - \frac{2}{k} (x_1 + \cdots + x_k) + \frac{1}{k^2}
\]

and

\[
\sum_{1 \leq i < j \leq k} (x_i - x_j)^2 = \sum_{1 \leq i < j \leq k} (x_i^2 - 2x_i x_j + x_j^2)
\]

\[
= (k - 1)(x_1^2 + \cdots + x_k^2) - 2 \sum_{1 \leq i < j \leq k} x_i x_j
\]

\[
= (k - 1)(x_1^2 + \cdots + x_k^2) - [(x_1 + \cdots + x_k)^2 - (x_1^2 + \cdots + x_k^2)]
\]

\[
= k(x_1^2 + \cdots + x_k^2) - 1.
\]

Therefore, minimizing the variance or sum of squares of differences is equivalent to minimizing the sum of squares \( x_1^2 + \cdots + x_k^2 \) metric. Therefore, we focus on minimizing the sum of squares.
Using mathematica, we have computed the dice which minimize the sum of squares of sums for small pairs of \( m \) and \( n \) \((m, n) = (2, 2), (2, 3), (2, 4), (3, 2), (4, 2)\)). We have also approximately computed the dice for bigger pairs of \( m \) and \( n \) using java programming, and then extrapolated the fractions from the decimals. Our data strongly supports the following conjecture.

**Conjecture 1.1.** The set of \( m \) \( n \)-sided dice with the minimum variance (or sum of squares) is

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{dice} & P(1) & P(2) & \cdots & P(n - 1) & P(n) \\
\hline
\text{dice 1} & \frac{m}{m(2m - 2) - n + 2} & \frac{2m - 1}{m(2m - 2) - n + 2} & \cdots & \frac{2m - 1}{m(2m - 2) - n + 2} & \frac{m}{m(2m - 2) - n + 2} \\
\text{dice 2} & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\
\text{dice 3} & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{dice } m - 1 & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\
\text{dice } m & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\
\hline
\end{array}
\]

The minimum sum of squares for a set of \( m \) \( n \)-sided dice is

\[
\frac{(2m - 1)!}{2^{m-n-2}((m-1)!)^2(2m+(n-2)(2m-1))}
\]

(achieved by the above dice).

For these dice, the probability of rolling a sum of \( m \) is

\[
\frac{m!}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}
\]

The probability of rolling a sum of \( m + 1, \ldots, n + m - 3 \), or \( n + m - 2 \) is each

\[
\frac{m!}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}
\]

In general, the probability of rolling a sum of \( in + m - i \) is

\[
\frac{m!}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}
\]

and the probability of rolling a sum of \( in + m - i + 1, \ldots, (i+1)n + m - i - 2 \) is each

\[
\frac{(m-1)!}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}
\]

Therefore, the sum of squares for these dice is

\[
m^2 + \binom{m}{1}m^2 + \cdots + \binom{m}{m}m^2 + (n - 2)(2m - 1)^2 + (n - 2)(m - 1)^2 + \cdots + (n - 2)(m - 2)^2 + (m - 1)^2 + (m - 2)^2 + \cdots + (m - n - 2)^2
\]

\[
= \frac{m^2 \left( \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} \right) + (n - 2)(2m - 1)^2 + (n - 2)(m - 1)^2 + \cdots + (n - 2)(m - 2)^2 + (m - 1)^2}{2m-n-2(2m+(n-2)(2m-1))^2}
\]

\[
= \frac{m^2(2m) + (n - 2)(2m - 1)^2 + (m - 2)^2 + \cdots + (m - n - 2)^2}{2m-n-2(2m+(n-2)(2m-1))^2}
\]

\[
= \frac{m^2(2m) + (n - 2)(2m - 1)^2 + (m - 2)^2 + \cdots + (m - n - 2)^2}{2m-n-2(2m+(n-2)(2m-1))^2}
\]

\[
= \frac{(m - 1)!}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}
\]

\[
= \frac{1}{mn-m+1}
\]

Let \( \mu = \frac{1}{mn-m+1} \) be the mean. The coefficient of variance is

\[
\sqrt{\frac{x^2 + \cdots + x^2}{k} - \frac{1}{k^2} - \mu^2} = \sqrt{\frac{\mu \cdot 2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}{\mu} - \mu^2} = \sqrt{\frac{(2m - 1)!((mn - m + 1))}{2m-n-2((m-1)!)^2(2m+(n-2)(2m-1))}} - 1.
\]
For fixed $m$, we can see that the coefficient of variance is decreasing in terms of $n$ and that when $n$ goes to $\infty$, the coefficient of variance approaches

$$\sqrt{-1 + \frac{m}{2^{2m-2}} \left( \frac{2m-2}{m-1} \right)}.$$

Note that this is positive for any $m > 2$ and is 0 when $m = 2$. We can also see that for fixed $n$, the coefficient of variance is increasing in $m$ and approaches $\infty$ as $m \to \infty$ (although rather slowly). Therefore, for bigger $m$ or $n$, the sums become less equally likely. Also, we can only get equally likely sums when $m = 1$ or when $m = 2$ and $n \to \infty$. Our computations for the coefficient of variance for a few pairs of $m$ and $n$ support our observations:

<table>
<thead>
<tr>
<th>Coefficient of Variance Data</th>
<th>m number of dice</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$ sides</td>
<td></td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>0.3536</td>
<td>0.5000</td>
<td>0.6060</td>
<td>0.6903</td>
<td>0.7610</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.2673</td>
<td>0.4395</td>
<td>0.5590</td>
<td>0.6517</td>
<td>0.7281</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.2236</td>
<td>0.4146</td>
<td>0.5409</td>
<td>0.6374</td>
<td>0.7161</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.1961</td>
<td>0.4009</td>
<td>0.5313</td>
<td>0.6299</td>
<td>0.7099</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.1768</td>
<td>0.3922</td>
<td>0.5254</td>
<td>0.6253</td>
<td>0.7061</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>0.1622</td>
<td>0.3863</td>
<td>0.5213</td>
<td>0.6222</td>
<td>0.7035</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.1508</td>
<td>0.3819</td>
<td>0.5184</td>
<td>0.6199</td>
<td>0.7017</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>0.1414</td>
<td>0.3785</td>
<td>0.5162</td>
<td>0.6182</td>
<td>0.7003</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.1336</td>
<td>0.3759</td>
<td>0.5144</td>
<td>0.6169</td>
<td>0.6992</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.0410</td>
<td>0.3557</td>
<td>0.5013</td>
<td>0.6070</td>
<td>0.6912</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.0129</td>
<td>0.3538</td>
<td>0.5001</td>
<td>0.6061</td>
<td>0.6904</td>
<td></td>
</tr>
<tr>
<td>Limit</td>
<td>0</td>
<td>0.3536</td>
<td>0.5000</td>
<td>0.6060</td>
<td>0.6903</td>
<td></td>
</tr>
</tbody>
</table>

Another way to measure how equally likely the sums are is to minimize the range of our sums. That is, the smaller $\max_{1 \leq i < j \leq k} |x_i - x_j|$ is, the closer the sums are to uniformly likely. Again, we were able to find the dice which minimize the range for small pairs of $m$ and $n$ with mathematica, and we were able to use java programming to approximately verify this for bigger values of $m$ and $n$. Our data strongly supports the following conjecture.

**Conjecture 1.2.** The set of $m$ $n$-sided dice with the minimum range is

<table>
<thead>
<tr>
<th>dice</th>
<th>$P(1)$</th>
<th>$P(2)$</th>
<th>$P(n-1)$</th>
<th>$P(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{m+1}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{m+1}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m-1$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$\frac{m}{2^{\lceil \frac{m+1}{2} \rceil+m(n-2)}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
The minimum range for a set of \( m \) \( n \)-sided dice is

\[
\left( \frac{m-1}{m+1} \right) m - \left\lfloor \frac{m+1}{2} \right\rfloor \frac{2^{m-1}}{2^{\left\lfloor \frac{m+1}{2} \right\rfloor + m(n-2)}}.
\]

We can see that the probability of rolling a sum of \( m + in - i \) is

\[
\frac{\left( \frac{m}{m+1} \right) m - \left\lfloor \frac{m+1}{2} \right\rfloor}{2^{m-1}(2^{\left\lfloor \frac{m+1}{2} \right\rfloor + m(n-2))}}.
\]

and the probability of rolling a sum of \( m + in - i + 1, \ldots, m + (i+1)n - i - 2 \) is each

\[
\frac{\left( \frac{m-1}{m+1} \right) m - \left\lfloor \frac{m+1}{2} \right\rfloor}{2^{m-1}(2^{\left\lfloor \frac{m+1}{2} \right\rfloor + m(n-2))}}.
\]

It is not hard to see that the range of these sums is

\[
\left( \frac{m-1}{m+1} \right) m - \left\lfloor \frac{m+1}{2} \right\rfloor \frac{2^{m-1}}{2^{\left\lfloor \frac{m+1}{2} \right\rfloor + m(n-2)}}.
\]

For visual comparison of the distribution of sums for our dice for minimum variance and minimum range, see the graphs below.
4d4

5d3

5d4
2 Proof of two sides case of conjecture

First we prove the \( n = 2 \) special case of Conjecture 1.1.

**Theorem 2.1.** The following \( m \) two-sided dice (the \( n = 2 \) case of Conjecture 1.1)

<table>
<thead>
<tr>
<th>dice 1</th>
<th>( P(1) )</th>
<th>( P(2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>dice 2</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>dice ( m )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

minimizes the sum of squares (or variance).

**Proof.** Let our \( m \) dice be \( p_1(x) = a_1 + (1 - a_1)x, \ldots, p_m(x) = a_m + (1 - a_m)x \). It suffices to show that \( a_1 = \cdots = a_m = \frac{1}{2} \) is the global minimum for the sum of squares \( S \). Note that the sum of squares \( S \) can be represented by

\[
S = \frac{1}{2\pi i} \int_{|x|=1} \frac{(a_1 + (1 - a_1)x) \cdots (a_n + (1 - a_m)x)(a_1 + \frac{1-a_1}{x}) \cdots (a_n + \frac{1-a_m}{x})}{x} dx.
\]

We will first show that \( a_1 = \cdots = a_m = \frac{1}{2} \) is the only critical point of \( S \). Note that

\[
\frac{\partial S}{\partial a_1} = \frac{1}{2\pi i} \int_{|x|=1} \frac{[4a_1 - 2 + (1 - 2a_1)(\frac{1}{x} + x)] \prod_{j=2}^n [(a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x})]}{x} dx
\]

\[
= \frac{2a_1 - 1}{2\pi i} \int_{|x|=1} \frac{(2 - \frac{1}{x} - x) \prod_{j=2}^n [(a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x})]}{x} dx.
\]

This partial vanishes when \( a_1 = \frac{1}{2} \). Similarly \( \frac{\partial S}{\partial a_j} \) vanishes when \( a_j = \frac{1}{2} \). Therefore, \( a_1 = \cdots = a_m = \frac{1}{2} \) is a critical point.

Now we show that there are no other critical points. We show that \( \frac{\partial S}{\partial a_1} \neq 0 \) when \( a_1 \neq \frac{1}{2} \). Note that

\[
\frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^n \frac{(a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x})}{x} dx = \frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^n \frac{(a_j + (1 - a_j)x)(a_j + (1 - a_j)x)}{x} dx
\]

\[
= \frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^n \frac{(a_j + (1 - a_j)x)^2}{x} dx
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \prod_{j=2}^n \left| (a_j + (1 - a_j)e^{i\theta}) \right|^2 e^{i\theta} d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \prod_{j=2}^n \left| (a_j + (1 - a_j)e^{i\theta}) \right|^2 d\theta,
\]
and that

\[
\frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x}, \quad \frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x}
\]

are the linear and constant coefficients of \( \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \). Therefore both are real. Thus we have

\[
\frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x} = \text{Re} \left( \frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x} \right)
\]

\[
= \text{Re} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| \prod_{j=2}^{n} (a_j + (1 - a_j)e^{i\theta}) \right|^2 e^{i\theta} d\theta \right)
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \left| \prod_{j=2}^{n} (a_j + (1 - a_j)e^{i\theta}) \right|^2 \cos \theta d\theta
\]

and

\[
\frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x} = \text{Re} \left( \frac{1}{2\pi i} \int_{|x|=1} \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right] \frac{dx}{x} \right)
\]

\[
= \text{Re} \left( \frac{1}{2\pi} \int_{0}^{2\pi} \left| \prod_{j=2}^{n} (a_j + (1 - a_j)e^{i\theta}) \right|^2 e^{i\theta} d\theta \right)
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \left| \prod_{j=2}^{n} (a_j + (1 - a_j)e^{i\theta}) \right|^2 \cos \theta d\theta.
\]

Thus,

\[
\frac{\partial S}{\partial a_1} = (2a_1 - 1) \frac{1}{2\pi} \int_{|x|=1} \frac{(2 - \frac{1}{x} - x) \prod_{j=2}^{n} \left[ (a_j + (1 - a_j)x)(a_j + \frac{1-a_j}{x}) \right]}{x} \frac{dx}{x}
\]

\[
= 2(2a_1 - 1) \frac{1}{2\pi} \int_{0}^{2\pi} \left| \prod_{j=2}^{n} (a_j + (1 - a_j)e^{i\theta}) \right|^2 (1 - \cos \theta) d\theta
\]

Note that the integrand is positive when \( \theta = \frac{\pi}{2} \). Since the integrand is continuous and nonnegative, the integral is strictly positive. Therefore, \( \frac{\partial S}{\partial a_1} \neq 0 \) unless \( a_1 = \frac{1}{2} \). Thus \( a_i = \frac{1}{2} \) for all \( i \) is the only critical point.
Now we are ready to prove our theorem by induction. The base case of $m = 1$ is trivial. Suppose our theorem is true for some $m - 1 \geq 1$. For $m$ dice, since $a_i = \frac{1}{2}$ for all $i$ is the only critical point, we know that the global minimum occurs when $a_i = 0$ or $a_i = 1$ for some $i$ (the boundary) or when $a_i = \frac{1}{2}$ for all $i$. By our inductive hypothesis, our first case is minimized when $a_i = 0$, $a_j = \frac{1}{2}$ for all $j \neq i$. As we computed earlier, the sum of squares for our first case is

$$\frac{(2m-3)!}{2^{2m-4}((m-2)!)^2(2m-2)}$$

and the sum of squares for our second case is

$$\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m)}.$$ 

Since

$$\frac{(2m-1)!}{2^{2m-2}((m-1)!)^2(2m)} = \frac{(2m-1)(2m-2)}{2^2 (m-1)^2 \left( \frac{2m}{2m-2} \right)} = \frac{2m-1}{2m} < 1$$

we see that the global minimum for the $m$ dice two sides case is $a_j = \frac{1}{2}$ for all $j$. Our induction is complete.

$\square$

### 3 Local Minimum Proof for two dice

Next we partially prove a different special case of Conjecture 1.1. Specifically, we prove that the $m = 2$ case given in Conjecture 1.1 is a local minimizer for the sum of squares.

**Theorem 3.1.** The following two $n$-sided dice (which is the $m = 2$ case of Conjecture 1.1)

<table>
<thead>
<tr>
<th></th>
<th>$P(1)$</th>
<th>$P(2)$</th>
<th>$\cdots$</th>
<th>$P(n-1)$</th>
<th>$P(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dice 1</strong></td>
<td>$\frac{2}{3n-2}$</td>
<td>$\frac{3}{3n-2}$</td>
<td>$\cdots$</td>
<td>$\frac{3}{3n-2}$</td>
<td>$\frac{2}{3n-2}$</td>
</tr>
<tr>
<td><strong>dice 2</strong></td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
<td>$\cdots$</td>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

are a local minima for the sum of squares (and variance).

**Proof.** Our two dice are $p_1 + p_2 x + \cdots + p_n x^{n-1}$ and $q_1 + q_2 x + \cdots + q_n x^{n-1}$ and we will show that $p_1 = p_n = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, q_1 = q_n = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0$ are local minimizers of

$$S = (p_1 q_1)^2 + (p_1 q_2 + p_2 q_1)^2 + \cdots + (p_n q_n)^2$$

subject to the constraints

$$p_1 + \cdots + p_n = 1$$
$$q_1 + \cdots + q_n = 1$$
$$p_1, p_2, \ldots, p_n \geq 0$$
First note that
\[ S = \frac{1}{2\pi i} \int_{|x|=1} \frac{P(x)Q(x)P(\frac{1}{x})Q(\frac{1}{x})}{x} \, dx \]
and we have the Lagrangian
\[ L = S + \lambda(p_1 + \cdots + p_{n-1}) + \mu(q_1 + \cdots + q_{n-1}) + \lambda_1(-p_1) + \cdots + \lambda_n(-p_n) + \mu_1(-q_1) + \cdots + \mu_n(-q_n) \]
where \( \lambda \) and \( \mu \) are Lagrange multipliers and \( \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n \geq 0 \) are Karush-Kuhn-Tucker multipliers. We will first show that
\[ p_1 = p_n = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, q_1 = q_n = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0, \lambda = -\frac{3}{3n-2}, \mu = -\left(\frac{2}{3n-2}\right)^2 [3 + \left(\frac{3}{2}\right)^2 (n - 2)], \lambda_1 = \cdots = \lambda_n = 0, \mu_1 = \mu_n = 0, \mu_2 = \cdots = \mu_{n-1} = \frac{3}{(3n-2)^2} \]
is a solution of the Karush Kuhn Tucker equations
\[
\begin{align*}
\frac{\partial S}{\partial p_\ell} + \lambda - \lambda_\ell &= 0 \\
\frac{\partial S}{\partial q_\ell} + \mu - \mu_\ell &= 0 \\
\lambda_1(-p_1) + \cdots + \lambda_n(-p_n) + \mu_1(-q_1) + \cdots + \mu_n(-q_n) &= 0 \\
p_1 + p_2 + \cdots + p_{n-1} + 1 &= 0 \\
q_1 + q_2 + \cdots + q_{n-1} + 1 &= 0 \\
-p_1 \leq 0, \ldots, -p_n \leq 0 \\
-q_1 \leq 0, \ldots, -q_n \leq 0.
\end{align*}
\]
First observe that
\[
\frac{\partial S}{\partial p_\ell} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P\left(\frac{1}{x}\right) Q(x)Q\left(\frac{1}{x}\right) x^{\ell-2} + P(x)Q(x)Q\left(\frac{1}{x}\right) \frac{1}{x^{\ell}} \right] \, dx
\]
\[= \frac{1}{4} \left( \frac{6}{3n-2} \right) + \frac{1}{4} \left( \frac{6}{3n-2} \right) = \frac{3}{3n-2} = \lambda_\ell - \lambda \]
and that
\[
\frac{\partial S}{\partial q_1} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P(x)P \left( \frac{1}{x} \right) Q \left( \frac{1}{x} \right)\frac{1}{x} + P(x)Q \left( \frac{1}{x} \right)\frac{1}{x} \right] \, dx \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 1^2 + \left( \frac{3}{2} \right)^2 (n-2) + 1^2 \right] + \left( \frac{2}{3n-2} \right)^2 \left( \frac{1+1}{2} \right) \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 3 + \left( \frac{3}{2} \right)^2 (n-2) \right] \\
= -\mu,
\]

\[
\frac{\partial S}{\partial q_1} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P(x)P \left( \frac{1}{x} \right) Q \left( \frac{1}{x} \right) x^{n-2} + P(x)P \left( \frac{1}{x} \right) Q \left( \frac{1}{x} \right) x^{-n} \right] \, dx \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 1^2 + \left( \frac{3}{2} \right)^2 (n-2) + 1^2 \right] + \left( \frac{2}{3n-2} \right)^2 \left( \frac{1+1}{2} \right) \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 3 + \left( \frac{3}{2} \right)^2 (n-2) \right] \\
= -\mu,
\]

and

\[
\frac{\partial S}{\partial q_\ell} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P(x)P \left( \frac{1}{x} \right) Q \left( \frac{1}{x} \right) x^{\ell-1} + P(x)P \left( \frac{1}{x} \right) Q \left( \frac{1}{x} \right) x^{-\ell} \right] \, dx \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 3 + \left( \frac{3}{2} \right)^2 (n-3) + \frac{3}{2} \right] + \left( \frac{2}{3n-2} \right)^2 \left( \frac{3}{2} + \frac{3}{2} \right) \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 6 + \left( \frac{3}{2} \right)^2 (n-3) \right] \\
= \left( \frac{2}{3n-2} \right)^2 \left[ 3 + \left( \frac{3}{2} \right)^2 (n-2) + \frac{3}{4} \right] \\
= -\mu + \mu_\ell.
\]

Thus we have shown that our conjectured dice satisfy the Karush-Kuhn-Tucker conditions, which means that it is a critical point. Now we use the Hessian to show that it is indeed a strict local minimizer.

Since the constraint equations and constraint inequalities are all linear, we have \( \text{Hess}(L) = \text{Hess}(S) \). We consider \( y = (y_1 \cdots y_{2n})^T \) such that

\[
\text{grad}(p_1 + p_2 + \cdots + p_n - 1)y = 0,
\]
grad(q_1 + q_2 + \cdots + q_n - 1)y = 0

and

grad(-q_j)y = 0

for j = 2, \ldots, n - 1 because \mu_2, \ldots, \mu_{n-1} > 0. These equations imply y_{n+2} = \cdots = y_{2n-1} = 0, y_1 + \cdots + y_n = 0, and y_{n+1} + y_{2n} = 0.

\[ y^T \text{Hess}(L)y = y^T \text{Hess}(S)y \]

\[
= \sum_{k, \ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \sum_{k, \ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_\ell} y_k y_{n+\ell} + \sum_{k, \ell=1}^{n} \frac{\partial^2 S}{\partial q_k \partial q_\ell} y_{n+k}y_{n+\ell}
\]

\[
= \sum_{k, \ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_1} y_k y_{n+1} + \sum_{k=1}^{n} \frac{\partial^2 S}{\partial p_k \partial q_n} y_k y_{2n} + \frac{\partial^2 S}{\partial q_1^2} y_{n+1}^2 + 2 \frac{\partial^2 S}{\partial q_1 \partial q_n} y_{n+1}y_{2n} + \frac{\partial^2 S}{\partial q_n^2} y_{2n}^2
\]

where the last equality follows from the fact y_{n+2} = \cdots = y_{2n-1} = 0. We have

\[
\frac{\partial^2 S}{\partial p_k \partial p_\ell} = \frac{1}{2\pi i} \int_{|x|=1} \left[ Q(x)Q\left(\frac{1}{x}\right) x^{k-\ell-1} + Q(x)Q\left(\frac{1}{x}\right) x^{\ell-1-k} \right] dx
\]

\[
= \frac{1}{2\pi i} \int_{|x|=1} \frac{1}{4} \left( 2 + x^{n-1} + \frac{1}{x^{n-1}} \right) \left( x^{k-\ell-1} + x^{\ell-1-k} \right) dx
\]

\[
= \begin{cases} 
1 & \text{if } k = \ell \\
\frac{1}{2} & \text{if } k = n, \ell = 1 \\
\frac{1}{2} & \text{if } k = 1, \ell = n \\
0 & \text{otherwise}
\end{cases}
\]

The matrix \( \left( \frac{\partial^2 S}{\partial p_k \partial p_\ell} \right) \) is positive definite by checking principle minors.

We can also check that

\[
\frac{\partial^2 S}{\partial p_k \partial q_1} = \frac{\partial^2 S}{\partial p_k \partial q_n}
\]

at \( p_1 = \frac{2}{3n-2}, p_2 = \cdots = p_{n-1} = \frac{3}{3n-2}, p_n = \frac{2}{3n-2}, q_1 = \frac{1}{2}, q_2 = \cdots = q_{n-1} = 0, q_n = \frac{1}{2}. \)

Hence,

\[
y^T \text{Hess}(L)y = \sum_{k, \ell=1}^{n} \frac{\partial^2 S}{\partial p_k \partial p_\ell} y_k y_\ell + \frac{\partial^2 S}{\partial q_1^2} y_{n+1}^2 + 2 \frac{\partial^2 S}{\partial q_1 \partial q_n} y_{n+1}y_{2n} + \frac{\partial^2 S}{\partial q_n^2} y_{2n}^2.
\]

We also have

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\[
\frac{\partial^2 S}{\partial q_1^2} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P(x)P \left( \frac{1}{x} \right) + P(x)P \left( \frac{1}{x} \right) \right] \frac{1}{x} dx = 2 \left( \frac{2}{3n-2} \right)^2 \left[ 1^2 + \left( \frac{3}{2} \right)^2 (n-2) + 1^2 \right],
\]

\[
\frac{\partial^2 S}{\partial q_n^2} = \frac{1}{2\pi i} \int_{|x|=1} \left[ P(x)P \left( \frac{1}{x} \right) \frac{x^{n-2}}{x^{n-1}} + P(x)P \left( \frac{1}{x} \right) \frac{x^{n-1}}{x^n} \right] dx = 2 \left( \frac{2}{3n-2} \right)^2 \left[ 1^2 + \left( \frac{3}{2} \right)^2 (n-2) + 1^2 \right],
\]

\[
\frac{\partial^2 S}{\partial q_1 \partial q_n} = \frac{1}{2\pi i} \int_{|x|=1} \left( P(x)P \left( \frac{1}{x} \right) \frac{1}{x^n} + P(x)P \left( \frac{1}{x} \right) x^{n-2} \right) dx = \left( \frac{2}{3n-2} \right)^2 [1 + 1].
\]

Finally,

\[
y^T \text{Hess}(L)y = [y_1 \cdots y_n] \left( \frac{\partial^2 S}{\partial q_k \partial q_\ell} \right) y + \left( 2 \frac{\partial^2 S}{\partial q_1^2} - 2 \frac{\partial^2 S}{\partial q_1 \partial q_n} \right) y_{n+1}^2
\]

\[
= [y_1 \cdots y_n] \left( \frac{\partial^2 S}{\partial q_k \partial q_\ell} \right) y + 2 \left( \frac{2}{3n-2} \right)^2 \left[ \left( \frac{3}{2} \right)^2 (n-2) + 1 \right] y_{n+1}^2.
\]

Recall that \( \left( \frac{\partial^2 S}{\partial q_k \partial q_\ell} \right) \) is positive definite. Thus, if \( y_{n+1} \neq 0 \), we have \( y^T \text{Hess}(L)y \geq 2 \left( \frac{2}{3n-2} \right)^2 \left[ \left( \frac{3}{2} \right)^2 (n-2) + 1 \right] y_{n+1}^2 > 0 \). If \( y_{n+1} = 0 \), then since \( y_{n+1} + y_2 = 0 \) and \( (y_1 \cdots y_n, y_{n+1}, 0 \cdots 0, y_{2n})^T \neq \vec{0} \), we have \( y_{2n} = 0 \) and thus \( (y_1 \cdots y_n)^T \neq 0 \). Therefore, we have \( y^T \text{Hess}(L)y = y^T \left( \frac{\partial^2 S}{\partial q_k \partial q_\ell} \right) y > 0 \). Thus we have proved that our conjectured dice are indeed a strict local minimizer. \( \square \)